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The unique solution for periodic differential equations with upper and lower solutions in reverse order

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Abstract

In this paper, we obtain the unique solution for the following periodic problem

$$\begin{aligned}x''(t) &= g(t, x(t), x'(t)), \quad t \in (a, b), \\x(a) &= x(b), \quad x'(a) = x'(b),\end{aligned}$$

where $g : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function, by constructing an auxiliary system with bounded solutions. The upper and lower solution method and an anti-maximum principle are employed to establish the monotone iterative sequences and obtain the extremal solutions for the auxiliary system.

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1 Introduction

This paper is concerned with the unique solution to the following periodic problem:

$$\begin{cases}x''(t) = g(t, x(t), x'(t)), & t \in (a, b), \\x(a) = x(b), & x'(a) = x'(b),\end{cases} \quad (1.1)$$

where $g : [a, b] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous function.

It is well known that the upper and lower solution method together with the iterative technique is a powerful tool for proving the existence results for boundary value problems (see [1–12] and the references therein). Recently, the case when the upper solution and the lower solution are in the reversed order has received some attention (see [2–9] and the references therein). The monotone approximation method can be used in the case the lower and upper solutions are in the reversed order $\beta \leq \alpha$. This method works for any boundary value problem such that a uniform anti-maximum principle holds. This is the case for the Neumann and periodic problems.

For example, Cabada *et al.* in [3] used iteration schemes based on problems like

$$\begin{cases} \alpha_n'' - 2k|\alpha_n' - \alpha_{n-1}'| + \lambda\alpha_n = f(t, \alpha_{n-1}, \alpha_{n-1}') + \lambda\alpha_{n-1}, \\ \alpha_n'(a) = \alpha_n'(b) = 0, \end{cases}$$

discussed a Neumann boundary value problem

$$\begin{cases} u''(t) = f(t, u(t), u'(t)), & t \in [a, b], \\ x'(a) = x'(b) = 0. \end{cases}$$

In [4], Torres and Zhang considered a kind of 2π -periodic boundary value problem. The strategy of this paper was to exploit an anti-maximum principle for the linear equation in order to construct a monotone approximation scheme converging to the solution.

In [8], Zuo *et al.* further developed the monotone method and investigated the T -periodic solution of

$$y''(t) = f(t, y(t), y(w(t))), \quad t \in \mathbb{R}. \tag{1.2}$$

They used the monotone iterative technique with upper and lower solution in reversed order to define two sequences that converge uniformly to extremal solution of (1.2). However, they did not construct the explicit expression of the monotone iterative sequences.

We have investigated the periodic problems (1.1) in [13] by introducing the following auxiliary periodic system:

$$\begin{cases} u''(t) = f(t, u(t), u'(t), v(t)), \\ v''(t) = f(t, v(t), v'(t), u(t)), \\ u(a) = u(b), \quad u'(a) = u'(b), \\ v(a) = v(b), \quad v'(a) = v'(b), \end{cases}$$

where $f : [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a L^1 -Carathéodory function, and $f(t, u, w, u) \equiv g(t, u, w)$.

Motivated by the above works, we are mainly concerned with the periodic problem (1.1). One might have noticed that computing the iterative sequences $\{\alpha_n\}$ and $\{\beta_n\}$ can be a difficult work in those existed papers. In this paper, by adopting an auxiliary periodic system and an anti-maximum principle which are different from that in the reference [13], we also obtain the unique solution for the problem (1.1).

Here, we introduce the following auxiliary periodic system:

$$\begin{cases} u''(t) + Ku(t) = f(t, u(t), u'(t), v(t)) + Kv(t), & t \in (a, b), \\ v''(t) + Kv(t) = f(t, v(t), v'(t), u(t)) + Ku(t), & t \in (a, b), \\ u(a) = u(b), \quad u'(a) = u'(b), \\ v(a) = v(b), \quad v'(a) = v'(b), \end{cases} \tag{1.3}$$

where $f : [a, b] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuous function and $f(t, u, w, u) = g(t, u, w)$, the constant K is defined in Section 3.

The organization of this paper is as follows. We shall introduce some useful lemmas in Section 2. The main results and their proof about auxiliary periodic (1.3) are given in Section 3. Section 4 gives the unique solution of (1.1) and an example is introduced.

2 Related lemmas

For the convenience of the reader, we give some lemmas which will be used in the next section. Firstly, recall the continuation theorem of Mawhin.

Lemma 2.1 [12] *Let X and Y be two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, and $\Omega \subset X$ an open and bounded set. Suppose $L : X \cap \text{dom} L \rightarrow Y$ is a Fredholm operator of index zero and $N_\lambda : \overline{\Omega} \rightarrow Y, \lambda \in [0,1]$ is L -compact. In addition, if*

- (A1) $Lx \neq \lambda Nx$ for $\lambda \in (0,1), x \in (\text{dom} L \setminus \ker L) \cap \partial\Omega$;
- (A2) $Nx \notin \text{Im} L$ for $x \in \ker L \cap \partial\Omega$;
- (A3) $\text{deg}\{JQN|_{\overline{\Omega} \cap \ker L}, \Omega \cap \ker L, 0\} \neq 0$, where $Q : Y \rightarrow Y$ is a projection such that $\text{Im} L = \ker Q$ and $J : \text{Im} Q \rightarrow \ker L$ is a homeomorphism.

Then the abstract equation $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

Secondly, to prove the validity of the monotone iterative technique, we present the anti-maximum comparison principle as follows.

Lemma 2.2 [1] *Let $p \in \mathbb{R}, q > 0, \sigma \in L^1(a,b)$ and $A \in \mathbb{R}$. Suppose u is a solution of*

$$\begin{cases} x''(t) + 2p|x'(t)| + qx(t) = \sigma(t), & t \in (a,b), \\ x(a) - x(b) = 0, & x'(a) - x'(b) = A \end{cases} \tag{2.1}$$

for some $A \geq 0$ and $\sigma \geq 0$. Then $u \geq 0$ provided that $p \geq 0$ and $\theta(q,p) \geq \frac{b-a}{2}$, where

$$\theta(q,p) = \begin{cases} \frac{1}{\sqrt{p^2-q}} \operatorname{arctanh} \frac{\sqrt{p^2-q}}{p}, & 0 < q < p^2, \\ \frac{1}{p}, & 0 < q = p^2, \\ \frac{1}{\sqrt{q-p^2}} \left(\frac{\pi}{2} - \operatorname{arctan} \frac{p}{\sqrt{q-p^2}} \right), & 0 \leq p^2 < q. \end{cases} \tag{2.2}$$

Finally, recall some classical integral inequalities.

Lemma 2.3 [14] *If $u \in C^1[a,b]$ is such that $\bar{u} = 0$, where $\bar{u} = \frac{1}{b-a} \int_a^b u(s) ds$. Then*

$$\|u\|_2 \leq \frac{b-a}{\pi} \|u'\|_2, \quad \|u\|_\infty \leq \sqrt{b-a} \|u'\|_2. \tag{2.3}$$

3 Extremal solution for (1.3)

Functions $\alpha, \beta \in C^2[a,b]$ are said to be a pair of lower and upper solutions to (1.3) if they satisfy

$$\begin{cases} \alpha''(t) + K\alpha(t) \geq f(t, \alpha(t), \alpha'(t), \beta(t)) + K\beta(t), & t \in (a,b), \\ \alpha(a) = \alpha(b), & \alpha'(a) \geq \alpha'(b), \\ \beta''(t) + K\beta(t) \leq f(t, \beta(t), \beta'(t), \alpha(t)) + K\alpha(t), & t \in (a,b), \\ \beta(a) = \beta(b), & \beta'(a) \leq \beta'(b). \end{cases} \tag{3.1}$$

For given $\alpha, \beta \in C[a, b]$, we shall write $\beta \leq \alpha$ if $\beta(t) \leq \alpha(t)$ for all $t \in [a, b]$. In such a case, we shall denote

$$[\beta, \alpha] = \{u \in C[a, b] : \beta(t) \leq u(t) \leq \alpha(t), t \in [a, b]\}.$$

Assume that there exist constants $N > 0, M, L \geq 0$ and $K > 0$ such that $L - K \geq 0$, we need the following hypotheses:

(H1) For given $\alpha, \beta \in C[a, b]$ with $\beta \leq \alpha$,

$$|f(t, x, y_1, z) - f(t, x, y_2, z)| \leq N|y_1 - y_2|, \quad \forall x, z \in [\beta, \alpha], y_1, y_2 \in \mathbb{R};$$

(H2) For $\beta \leq x_2 \leq x_1 \leq \alpha, \beta \leq z_1 \leq z_2 \leq \alpha$,

$$f(t, x_1, y, z_1) - f(t, x_2, y, z_2) \geq M(x_1 - x_2) - L(z_1 - z_2);$$

(H3) $1 - \frac{b-a}{\pi}N - (\frac{b-a}{\pi})^2K > 0$;

(H4) $b - a \leq 2\theta(K, \frac{N}{2})$, where θ is defined in (2.2).

In order to develop the monotone iterative technique for (1.3), we shall first consider the existence of solutions for the following periodic problems:

$$\begin{cases} u''(t) + Ku(t) = F_{\xi, \eta}(t, u'(t)), & t \in (a, b), \\ u(a) = u(b), \quad u'(a) = u'(b) \end{cases} \quad (3.2)$$

and

$$\begin{cases} v''(t) + Kv(t) = F_{\eta, \xi}(t, v'(t)), & t \in (a, b), \\ v(a) = v(b), \quad v'(a) = v'(b) \end{cases} \quad (3.3)$$

for each fixed $\xi, \eta \in [\beta, \alpha]$, here

$$F_{\xi, \eta}(t, w) = f(t, \xi, w, \eta) + K\eta, \quad F_{\eta, \xi}(t, w) = f(t, \eta, w, \xi) + K\xi.$$

Lemma 3.1 *Assume that (H1), (H3) and (H4) hold. Then the problems (3.2) and (3.3) have unique solutions.*

Proof In order to apply Lemma 2.1 to (3.2), we choose the Banach spaces $X = C^1[a, b]$ with the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty = \max_{t \in [a, b]} |u(t)|$, and $Y = C[a, b]$. Define the linear operator

$$L : \text{dom}L \subset X \rightarrow Y, \quad (Lu)(t) = u''(t)$$

and the nonlinear operator

$$N : X \rightarrow Y, \quad (Nu)(t) = F_{\xi, \eta}(t, u'(t)) - Ku(t),$$

here,

$$\text{dom } L = \{u \in X : u \in C^2[a, b], u(a) = u(b), u'(a) = u'(b)\}.$$

Let

$$(Pu)(t) = \bar{u}, \quad (Qu)(t) = \bar{u},$$

where $\bar{u} = \frac{1}{b-a} \int_a^b u(s) ds$. Then, $\ker L = \{c \in \text{dom } L : c \in \mathbb{R}\}$ and

$$\text{Im } L = \left\{ u \in Y : \int_a^b u(s) ds = 0 \right\}.$$

Now, $\text{Im } L$ is closed in Y . It is easy to see that $\text{Im } L = \ker Q$ and $\dim \ker L = 1 = \dim \text{Im } Q = \text{codim Im } L$. Thus, L is a linear Fredholm operator with index zero.

Furthermore, let $L_p = L|_{\text{dom } L \cap \ker P}$ and $K_p : \text{Im } L \rightarrow \text{dom } L \cap \ker P$ denoting the inverse of L_p be given by

$$(K_p u)(t) = \int_a^b k(t, s) u(s) ds,$$

where

$$k(t, s) := \frac{1}{2(b-a)} \begin{cases} (s-a)(2t-b-s), & a \leq s < t \leq b, \\ (b-s)(a+s-2t), & a \leq t \leq s \leq b. \end{cases}$$

Obviously, QN and $K_p(I-Q)N$ are continuous. By Arzelà-Ascoli theorem, we can show that $K_p(I-Q)N(\bar{\Omega})$ is relative compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Thus, N is L -compact on $\bar{\Omega}$.

In the following, we complete the remainder proof by four steps.

Step 1. Define $\Omega_1 = \{u \in \text{dom } L \setminus \ker L : Lu = \lambda Nu, \lambda \in (0, 1)\}$. For any $u \in \Omega_1$,

$$u''(t) = \lambda F_{\xi, \eta}(t, u'(t)) - \lambda Ku(t). \tag{3.4}$$

Integrating (3.4) on $[a, b]$ to obtain

$$K\bar{u} = \frac{1}{b-a} \int_a^b F_{\xi, \eta}(s, u'(s)) ds. \tag{3.5}$$

Set $x = u - \bar{u}$, then $x \in \text{dom } L$, $\bar{x} = 0$, $x' = u'$. It follows from (3.4) and (3.5) that

$$x''(t) = \lambda F_{\xi, \eta}(t, x'(t)) - \lambda Kx(t) - \frac{\lambda}{b-a} \int_a^b F_{\xi, \eta}(s, x'(s)) ds. \tag{3.6}$$

Notice that $\bar{x} = 0$, by Lemma 2.3, we have

$$\|x\|_2 \leq \frac{b-a}{\pi} \|x'\|_2, \quad \|x\|_\infty \leq \sqrt{b-a} \|x'\|_2.$$

In view of (H1), we have

$$\begin{aligned} |F_{\xi,\eta}(t, w)| &= |f(t, \xi, w, \eta) + K\eta| \\ &\leq |f(t, \xi, w, \eta) - f(t, \xi, 0, \eta)| + |f(t, \xi, 0, \eta)| + K|\eta| \\ &\leq N|w| + C \end{aligned}$$

for some constant C . Multiplying (3.6) by $x(t)$ and integrating it on $[a, b]$, we can obtain

$$\begin{aligned} \int_a^b |x'(t)|^2 dt &= -\lambda \int_a^b x(t)F_{\xi,\eta}(t, x'(t)) dt + \lambda K \int_a^b |x(t)|^2 dt \\ &\leq K \int_a^b |x(t)|^2 dt + N \int_a^b |x(t)||x'(t)| dt + C \int_a^b |x(t)| dt \\ &\leq K \left(\frac{b-a}{\pi}\right)^2 \int_a^b |x'(t)|^2 dt + N \left(\int_a^b |x(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_a^b |x'(t)|^2 dt\right)^{\frac{1}{2}} \\ &\quad + C\sqrt{b-a} \left(\int_a^b |x(t)|^2 dt\right)^{\frac{1}{2}} \\ &\leq \left(K \left(\frac{b-a}{\pi}\right)^2 + N \frac{b-a}{\pi}\right) \|x'\|_2^2 + C \frac{(b-a)^{\frac{3}{2}}}{\pi} \|x'\|_2. \end{aligned}$$

Notice the condition (H3), we can find a constant $M_1 > 0$ such that $\|x'\|_2 \leq M_1$ and also $\|x\|_\infty \leq \sqrt{b-a}M_1$. Now,

$$|\bar{u}| \leq \frac{1}{K(b-a)} \int_a^b |F_{\xi,\eta}(s, u'(s))| ds \leq \frac{N}{K(b-a)} \int_a^b |u'(t)| dt + \frac{C}{K} \leq \frac{NM_1}{K\sqrt{b-a}} + \frac{C}{K}$$

implies

$$\|u\|_\infty \leq \|x\|_\infty + |\bar{u}| \leq \sqrt{b-a}M_1 + \frac{NM_1}{K\sqrt{b-a}} + \frac{C}{K} =: M_2.$$

Since $u(a) = u(b)$, there exists $t_0 \in (a, b)$ such that $u'(t_0) = 0$. Then

$$\begin{aligned} |u'(t)| &\leq \int_a^b |u''(t)| dt \\ &\leq \int_a^b |F_{\xi,\eta}(t, u'(t))| dt + K \int_a^b |u(t)| dt \\ &\leq N\sqrt{b-a}M_1 + C(b-a) + K(b-a)M_2 \\ &=: M_3, \end{aligned}$$

i.e. $\|u'\|_\infty \leq M_3$. Hence, Ω_1 is bounded.

Step 2. Let $\Omega_2 = \{u \in \ker L : QNx = 0\}$. For $u \in \Omega_2$, we have $u = c \in \mathbb{R}$, and

$$QNc = \frac{1}{b-a} \int_a^b [F_{\xi,\eta}(t, 0) - Kc] dt = 0.$$

Then

$$c = \frac{1}{K(b-a)} \int_a^b F_{\xi,\eta}(t,0) dt \quad \text{and so} \quad |c| \leq \frac{C}{K} \leq M_2,$$

that is, Ω_2 is bounded.

Step 3. Let $\Omega \supset \Omega_1 \cup \Omega_2$ with Ω open and bounded. Clearly, conditions (A1) and (A2) in Lemma 2.1 are satisfied. The remainder is to verify (A3). To this end, we define an isomorphism $J : \text{Im } Q \rightarrow \ker L$ by $J = I$. Let

$$H(u, \mu) = \mu u + (1 - \mu)JQN u, \quad (u, \mu) \in \overline{\Omega} \times [0, 1],$$

i.e.

$$H(u, \mu) = \mu u + \frac{1 - \mu}{b - a} \int_a^b (F_{\xi,\eta}(s, u'(s)) - Ku(s)) ds.$$

It is easy to see that $H(u, \mu) \neq 0$ for $(u, \mu) \in (\partial\Omega \cap \ker L) \times [0, 1]$. Hence,

$$\begin{aligned} & \deg\{JQN|_{\overline{\Omega} \cap \ker L}, \Omega \cap \ker L, 0\} \\ &= \deg\{H(\cdot, 0), \Omega \cap \ker L, 0\} \\ &= \deg\{H(\cdot, 1), \Omega \cap \ker L, 0\} \\ &= \deg\{I, \Omega \cap \ker L, 0\} \\ &\neq 0. \end{aligned}$$

Lemma 2.1 yields that $Lu = Nu$ has at least one solution.

Step 4. We claim that the problem (3.2) has a unique solution. Suppose to the contrary, that there exist two solutions u_1 and u_2 of (3.2).

Let $u = u_2 - u_1$, by (H1) and (H4),

$$\begin{aligned} u''(t) + Ku(t) &= F_{\xi,\eta}(t, u'_2(t)) - F_{\xi,\eta}(t, u'_1(t)) \\ &= f(t, \xi(t), u'_2(t), \eta(t)) - f(t, \xi(t), u'_1(t), \eta(t)) \\ &\geq -N|u'_2(t) - u'_1(t)| \\ &= -N|u'(t)|. \end{aligned}$$

Applying Lemma 2.2, we have $u(t) \geq 0$ on $[a, b]$. That is, $u_2 \geq u_1$. Also, we can obtain $u_1 \geq u_2$ by the same method. Thus, $u_1 \equiv u_2$.

By a similar argument, we can obtain the existence of unique solution for the problem (3.3). \square

Now, we investigate the extremal solution of the periodic system (1.3). For given $\alpha, \beta \in X$, we consider the approximation schemes

$$\begin{cases} \alpha_0 = \alpha, & \beta_0 = \beta, \\ \alpha_n'' + K\alpha_n = f(t, \alpha_{n-1}, \alpha_n', \beta_{n-1}) + K\beta_{n-1}, \\ \beta_n'' + K\beta_n = f(t, \beta_{n-1}, \beta_n', \alpha_{n-1}) + K\alpha_{n-1}, \\ \alpha_n(a) = \alpha_n(b), & \alpha_n'(a) = \alpha_n'(b), \\ \beta_n(a) = \beta_n(b), & \beta_n'(a) = \beta_n'(b), \end{cases} \quad (3.7)$$

Now, we give the main result for (1.3).

Theorem 3.1 *Assume that α, β are a pair of lower and upper solutions of (1.3) defined by (3.1) such that $\beta \leq \alpha$. Suppose that (H1)-(H4) hold.*

Then there exist two monotone iterative sequences $\{\alpha_n\}, \{\beta_n\} \subset [\beta, \alpha]$, non-increasing and nondecreasing, respectively, defined by (3.7) such that $\alpha_n \rightarrow u^, \beta_n \rightarrow v^*$ ($n \rightarrow \infty$) uniformly on $t \in [a, b]$, and the pair (u^*, v^*) is a solution of (1.3) satisfying*

$$\beta \leq v^* \leq u^* \leq \alpha.$$

Moreover, any solution (u, v) of (1.3) with $\beta \leq u \leq \alpha, \beta \leq v \leq \alpha$ is such that

$$v^* \leq u \leq u^*, \quad v^* \leq v \leq u^*.$$

Proof Let $R = \{(u, v) \in C[a, b] \times C[a, b] : u \geq 0, v \leq 0\}$ be an ordered normal cone, the order is defined by

$$(u_1, v_1) \leq (u_2, v_2) \iff u_1 \leq u_2, \quad v_1 \geq v_2$$

for any $(u_1, v_1), (u_2, v_2) \in R$. We define the operator $T : [\beta, \alpha] \times [\beta, \alpha] \rightarrow X \times X$ by $T(\xi, \eta) = (u, v)$, where u and v are the unique solutions of problems (3.2) and (3.3), respectively, with given $\xi, \eta \in [\beta, \alpha]$. Firstly, we claim that the mapping T has the following properties:

- (i) $(\beta, \alpha) \leq T(\alpha, \beta) \leq (\alpha, \beta)$;
- (ii) $T(\xi_1, \eta_1) \leq T(\xi_2, \eta_2)$, when $(\xi_1, \eta_1) \leq (\xi_2, \eta_2), \xi_i, \eta_i \in [\beta, \alpha], i = 1, 2$.

We start with (i). Set $(\alpha_1, \beta_1) = T(\alpha, \beta), w = \alpha - \alpha_1$. From (H1) and (H2), we have

$$\begin{aligned} w''(t) + Kw(t) &\geq f(t, \alpha(t), \alpha'(t), \beta(t)) + K\beta(t) - f(t, \alpha(t), \alpha_1'(t), \beta(t)) - K\beta(t) \\ &\geq -N|\alpha'(t) - \alpha_1'(t)| = -N|w'(t)|, \quad t \in (a, b). \end{aligned}$$

Obviously, $w(a) = w(b)$ and $w'(a) \geq w'(b)$. By Lemma 2.2, we have $w \geq 0$, and so $\alpha \geq \alpha_1$. A similar argument shows that $\beta_1 \geq \beta$. Thus, $T(\alpha, \beta) \leq (\alpha, \beta)$. Set $w = \alpha_1 - \beta$, then

$$\begin{aligned} w''(t) + Kw(t) &\geq f(t, \alpha(t), \alpha_1'(t), \beta(t)) + K\beta(t) - f(t, \beta(t), \beta'(t), \alpha(t)) - K\alpha(t) \\ &= [f(t, \alpha(t), \alpha_1'(t), \beta(t)) - f(t, \alpha(t), \beta'(t), \beta(t))] \\ &\quad + [f(t, \alpha(t), \beta'(t), \beta(t)) - f(t, \beta(t), \beta'(t), \alpha(t))] - K(\alpha(t) - \beta(t)) \end{aligned}$$

$$\begin{aligned} &\geq -N|\alpha'_1(t) - \beta(t)| + (M + L - K)(\alpha(t) - \beta(t)) \\ &\geq -N|w'(t)|, \quad t \in (a, b). \end{aligned}$$

Lemma 2.2 implies that $\alpha_1 \geq \beta$. Similarly, we can check that $\alpha \geq \beta_1$. Hence, $(\beta, \alpha) \leq T(\alpha, \beta)$.

Now, we prove (ii). Let $T(\xi_i, \eta_i) = (u_i, v_i)$, $i = 1, 2$. Set $w = u_2 - u_1$,

$$\begin{aligned} w''(t) + Kw(t) &= F_{\xi_2, \eta_2}(t, u'_2(t)) - F_{\xi_1, \eta_1}(t, u'_1(t)) \\ &= f(t, \xi_2(t), u'_2(t), \eta_2(t)) - f(t, \xi_1(t), u'_1(t), \eta_1(t)) - K(\eta_1(t) - \eta_2(t)) \\ &= [f(t, \xi_2(t), u'_2(t), \eta_2(t)) - f(t, \xi_2(t), u'_1(t), \eta_2(t))] \\ &\quad + [f(t, \xi_2(t), u'_1(t), \eta_2(t)) - f(t, \xi_1(t), u'_1(t), \eta_1(t))] - K(\eta_1(t) - \eta_2(t)) \\ &\geq -N|u'_2(t) - u'_1(t)| + M(\xi_2(t) - \xi_1(t)) + (L - K)(\eta_1(t) - \eta_2(t)) \\ &\geq -N|w'(t)|, \quad t \in (a, b). \end{aligned}$$

Applying Lemma 2.2, we get $u_2 \geq u_1$. A similar argument shows that $v_1 \geq v_2$. Thus, $T(\xi_2, \eta_2) \geq T(\xi_1, \eta_1)$.

To prove the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded, we define $\{(\alpha_n, \beta_n)\}$ by $(\alpha_{n+1}, \beta_{n+1}) = T(\alpha_n, \beta_n)$. Notice that $\{(\alpha_n, \beta_n)\}$ is non-increasing and $\{(\beta_n, \alpha_n)\}$ is non-decreasing. It is clear that $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded. Hence, there exists a constant $C_1 > 0$ such that $\max\{\|\alpha_n\|_\infty, \|\beta_n\|_\infty\} \leq C_1$ for all $n \in \mathbb{N}$, and

$$\begin{aligned} \alpha''_{n+1}(t) + K\alpha_{n+1}(t) &= F_{\alpha_n, \beta_n}(t, \alpha'_{n+1}(t)), \\ \beta''_{n+1}(t) + K\beta_{n+1}(t) &= F_{\beta_n, \alpha_n}(t, \beta'_{n+1}(t)). \end{aligned}$$

Essentially, the same as in the proof of Lemma 3.1 guarantees that there exists a constant $C_2 > 0$ such that $\max\{\|\alpha'_n\|_\infty, \|\beta'_n\|_\infty\} \leq C_2$. This shows that $\{(\alpha_n, \beta_n)\}$ converges in X , i.e. there exists $(u^*, v^*) \in X$ so that $(\alpha_n, \beta_n) \rightarrow (u^*, v^*)$, $\beta_0 \leq u^* \leq \alpha_0$, $\beta_0 \leq v^* \leq \alpha_0$, and also $\beta_0 \leq v^* \leq u^* \leq \alpha_0$ from (i).

Finally, we will prove that any other solution $(u, v) \in X$ of (1.3) such that $(\beta_0, \alpha_0) \leq (u, v) \leq (\alpha_0, \beta_0)$ satisfies

$$v^* \leq v \leq u^*, \quad v^* \leq u \leq u^*.$$

We can proceed as in the proof of (i) to show that $(\beta_n, \alpha_n) \leq (u, v) \leq (\alpha_n, \beta_n)$, i.e. $\beta_n \leq u \leq \alpha_n$ and $\beta_n \leq v \leq \alpha_n$, $n = 0, 1, 2, \dots$. Hence, $v^* \leq u \leq u^*$ and $v^* \leq v \leq u^*$, i.e. $(v^*, u^*) \leq (u, v) \leq (u^*, v^*)$. \square

Remark 3.1 Comparing with the main result (Theorem 3.2) in [8], we construct the explicit expression of the monotone iterative sequences $\{\alpha_n\}$ and $\{\beta_n\}$ by (3.7) in Theorem 3.1, while reference [8] didn't do so.

Remark 3.2 Notice that if (u, u) is a solution of the auxiliary system (1.3), then u is a solution of the given problem (1.1) under the assumption $f(t, u, w, u) \equiv g(t, u, w)$. Thus, u^* and v^* are bounds on solutions of (1.1).

If the bounds u^* and v^* obtained in Theorem 3.1 are equal, then $u^* = v^*$ is a unique solution of the problem (1.1). In particular, under appropriate assumptions, this theorem provides an approximation scheme to the unique solution of (1.1).

4 Unique solution for (1.1)

Theorem 4.1 *Suppose that the conditions (H1)-(H4) hold. Then the periodic problem (1.1) has a unique solution provided that $M + L > 2K + \frac{N^2}{4}$.*

Proof If $u^* > v^*$, we compute

$$\begin{aligned} & \int_a^b [((v^*(t))'' - (u^*(t))'') + K(v^*(t) - u^*(t))](u^*(t) - v^*(t)) dt \\ &= \int_a^b [(f(t, v^*(t), (v^*(t))', u^*(t)) - f(t, u^*(t), (u^*(t))', v^*(t))) \\ &\quad + K(u^*(t) - v^*(t))](u^*(t) - v^*(t)) dt \\ &\leq -(M + L - K) \int_a^b |u^*(t) - v^*(t)|^2 dt \\ &\quad + N \left(\int_a^b |u^*(t) - v^*(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_a^b |(v^*(t))' - (u^*(t))'|^2 dt \right)^{\frac{1}{2}} \\ &=: Q_1. \end{aligned}$$

On the other hand,

$$\begin{aligned} & - \int_a^b [((v^*(t))'' - (u^*(t))'') + K(v^*(t) - u^*(t))](v^*(t) - u^*(t)) dt \\ &= \int_a^b |(v^*(t))' - (u^*(t))'|^2 dt - K \int_a^b |v^*(t) - u^*(t)|^2 dt \\ &=: Q_2. \end{aligned}$$

If we denote $\gamma_1 = (\int_a^b |u^*(t) - v^*(t)|^2 dt)^{\frac{1}{2}}$ and $\gamma_2 = (\int_a^b |(u^*(t))' - (v^*(t))'|^2 dt)^{\frac{1}{2}}$, then

$$\begin{aligned} Q_2 - Q_1 &= \gamma_2^2 - K\gamma_1^2 + (M + L - K)\gamma_1^2 - N\gamma_1\gamma_2 \\ &= \left(\gamma_2 - \frac{N}{2}\gamma_1 \right)^2 + \left(M + L - 2K - \frac{N^2}{4} \right) \gamma_1^2 \\ &> 0, \end{aligned}$$

which is a contradiction. □

Example 4.1 Consider the problem

$$\begin{cases} u''(t) = g(t, u(t), u'(t)), & t \in (0, 2\pi), \\ u(0) = u(2\pi), & u'(0) = u'(2\pi). \end{cases} \tag{4.1}$$

Corresponding to the problem (1.1), we take $a = 0, b = 2\pi$ and

$$g(t, x, y) = \frac{t}{2\pi \times 10^2} + \frac{2\pi^2}{500}x - \frac{1}{4} \left(\frac{\pi^2}{400} \right)^4 x^4 + \frac{\pi}{10\sqrt{2}} \arctan y. \tag{4.2}$$

Let

$$f(t, x, y, z) = \frac{t}{2\pi \times 10^2} + \frac{2\pi^2}{500}x - \frac{1}{4} \left(\frac{\pi^2}{400} \right)^4 z^4 + \frac{\pi}{10\sqrt{2}} \arctan y.$$

Consider the following system:

$$\begin{cases} u''(t) + \frac{\pi^2}{400}u(t) = f(t, u(t), u'(t), v(t)) + \frac{\pi^2}{400}v(t), & t \in (0, 2\pi), \\ v''(t) + \frac{\pi^2}{400}v(t) = f(t, v(t), v'(t), u(t)) + \frac{\pi^2}{400}u(t), & t \in (0, 2\pi), \\ u(0) = u(2\pi), & u'(0) = u'(2\pi), \\ v(0) = v(2\pi), & v'(0) = v'(2\pi) \end{cases} \tag{4.3}$$

with $K = \frac{\pi^2}{400}$ corresponding to (1.3). Clearly, the system (4.3) has $\alpha = \frac{400}{\pi^2}$ and $\beta = -\frac{400}{\pi^2}$ as the lower and upper solutions respectively according to the definition by (3.1). Take $M = \frac{2\pi^2}{500}, N = \frac{\pi}{10\sqrt{2}}$ and $L = K = \frac{\pi^2}{400}$. It is easy to check that the conditions (H1)-(H3) hold. Since $K > \frac{N^2}{4}$,

$$\theta \left(K, \frac{N^2}{4} \right) = \frac{1}{\sqrt{K - \frac{N^2}{4}}} \left(\frac{\pi}{2} - \arctan \frac{\frac{N}{2}}{\sqrt{K - \frac{N^2}{4}}} \right) = 5\sqrt{2} \geq \pi.$$

Thus, (H4) is satisfied. Thanks to Theorems 3.1, the system (4.3) has a solution (u^*, v^*) . In view of the values of N, K, M, L , we have $M + L > 2K + \frac{N^2}{4}$. Hence, Theorem 4.1 implies that $u^* = v^*$, that is, the problem (4.1) has a unique solution.

Remark 4.1 When $\alpha = \frac{400}{\pi^2}, \beta = -\frac{400}{\pi^2}$, and g is given by (4.3), Example 4.1 does not satisfy the conditions imposed in [1, Theorem 3.3] because α, β are not a pair of lower and upper solutions of Eq. (3.7) in [1]. In fact, our definitions of the lower and upper α, β , and the monotone iterative sequences $\{\alpha_n\}, \{\beta_n\}$ are all different from [1, Theorem 3.3]. In addition, the ultimate goal of this paper is the unique solution of (1.1) by constructing an auxiliary system (1.3), which is one of the highlights of this article.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript and read and approved the final manuscript.

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