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Existence of solutions for a class of degenerate quasilinear elliptic equation in R^N with vanishing potentials

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Abstract

We establish the existence of positive solution for the following class of degenerate quasilinear elliptic problem

$$(P) \quad \begin{cases} -\mathcal{L}u_{ap} + V(x)|x|^{-ap^*}|u|^{p-2}u = f(u) & \text{in } R^N, \\ u > 0 \quad \text{in } R^N; & u \in \mathcal{D}_a^{1,p}(R^N), \end{cases}$$

where $-\mathcal{L}u_{ap} = -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $a \le e \le a+1$, $d = 1 + a - e$, and $p^* := p^*(a,e) = \frac{Np}{N-dp}$ denote the Hardy-Sobolev's critical exponent, V is a bounded nonnegative vanishing potential and f has a subcritical growth at infinity. The technique used here is a truncation argument together with the variational approach.

MSC: 35B09; 35J10; 35J20; 35J70

1 Introduction

Consider the following degenerate quasilinear elliptic problem in R^N:

$$(P) \quad \begin{cases} -\mathcal{L}u_{ap} + V(x)|x|^{-ap^*}|u|^{p-2}u = f(u) & \text{in } R^N, \\ u > 0 \quad \text{in } R^N; & u \in \mathcal{D}_a^{1,p}(R^N), \end{cases}$$

where $-\mathcal{L}u_{ap} = -\operatorname{div}(|x|^{-ap}|\nabla u|^{p-2}\nabla u)$, $1 < p < N$, $-\infty < a < \frac{N-p}{p}$, $a \le e \le a+1$, $d = 1+a-e$, and $p^* := p^*(a,e) = \frac{Np}{N-dp}$ denote the Hardy-Sobolev's critical exponent, $V : R^N \to R$ is a bounded, nonnegative and vanishing potential and $f : R \to R$ a continuous function with a subcritical growth at infinity. Here, $\mathcal{D}_a^{1,p}(R^N)$ is the completion of the $C_0^\infty(R^N)$ with the norm $|u| = (\int_{R^N} |x|^{-ap}|\nabla u|^p\,dx)^{\frac{1}{p}}$. We impose the following hypotheses on f and V:

$f : R \to R$ is a continuous function verifying

(f_1) $\limsup_{s\to 0^+} \frac{sf(s)}{|x|^{-ap^*}s^{p^*}} < \infty$, uniformly in x.

(f_2) There exists $\alpha \in (p, p^*)$ such that $\limsup_{s\to\infty} \frac{sf(s)}{|x|^{-ap^*}s^{\alpha}} < \infty$, uniformly in x.

(f_3) There exists $\theta > p$ such that $\theta F(s) \le sf(s)$, for all $s > 0$.

$V : R^N \to R$ is a continuous function verifying

(V_1) $V(x) \ge 0$, for all $x \in R^N$.

(V_2) There are $\Lambda > 0$ and $\bar{r} > 1$ such that $\inf_{|x|>\bar{r}} V(x)|x|^{\frac{p^2[N-p(a+1)]}{(p-1)(N-p)}} \geq \Lambda$.

Remark 1.1 The conditions (f_1) and (f_2) imply the following:

$$\left|sf(s)\right| \leq C_0|x|^{-ap^*}|s|^{\alpha} \quad \text{with } \alpha \in \left(p,p^*\right), \text{ for all } s \in R. \tag{1}$$

Example 1.2 An example for the function f is given by

$$f(t) = \begin{cases} |x|^{-ap^*}t^{\lambda-1}, & \text{if } 0 \leq t \leq 1, \\ |x|^{-ap^*}t^{\alpha-1}, & \text{if } t \geq 1, \end{cases}$$

with $\lambda > p^*$ and α given by (f_2).

By vanishing potential we mean a potential that vanish on some bounded domain or become very close to zero at infinity. An important example for a such potential is given by

$$V(x) = \begin{cases} 0, & \text{if } |x| \leq \bar{r} - 1, \\ \Lambda \bar{r}^{-\frac{p^2[N-p(a+1)]}{(p-1)(N-p)}}(|x| - \bar{r} + 1), & \text{if } \bar{r} - 1 < |x| \leq \bar{r}, \\ \Lambda |x|^{-\frac{p^2[N-p(a+1)]}{(p-1)(N-p)}}, & \text{if } |x| \geq \bar{r}, \end{cases}$$

with $\Lambda > 0$.

Consider first the case $a = 0$, that is $-\mathcal{L}u_{ap}$ is the p-Laplacian operator, and the potential is bounded from below by a positive constant $V_0 > 0$.

Equations involving the p-Laplacian operator appear in many problems of nonlinear diffusion. Just to mention, in nonlinear optics, plasma physics, condensed matter physics and in modeling problems in non-Newtonian fluids. For more information on the physical background, we refer to [1].

For the case $p = 2$, we cite [2–12], and references therein. In [13], in addition to the above assumptions, the authors consider a local condition, namely,

$$\min_{x\in\overline{\Omega}} V < \min_{x\in\partial\Omega} V,$$

where $\Omega \subset R^N$ is a open bounded set, instead of the global condition imposed by Rabinowitz in [12]. For $p \neq 2$, see [14–17].

Now, consider that V is the zero mass case, that is $\lim_{|x|\to\infty} V(x) = 0$. When $p = 2$, we cite [18–20] and the recent paper [21] by Alves and Souto.

Let us now consider the case $a \neq 0$ and the potential bounded from below by a positive constant $V_0 > 0$.

In this case, the equations arise in problems of existence of stationary waves for anisotropic Schrödinger equation (see [22]) and others problems (for example, see [6, 23]). We cite [22] for $p = 2$; and [24, 25] for $p \neq 2$. For the case $V \equiv 0$, we cite [26], for $p = 2$ and $a \neq 0$; and [27], for $p \neq 2$ and $a = 0$.

The result presented here for $1 < p < N$ and $a \neq 0$ extends that one in [21] for $p = 2$ and $a = 0$. In [21], the presence of Hilbertian structure and some compact embeddings provide the convergence of the gradient. In the case studied here, with the absence of this structure, we do not obtain the convergence so directly. To overcome this problem, we use a result found in [28, 29], whose ideas come from [30, 31], when the domain is a smooth and bounded. In addition to this difficulty, there are others. For instance, in the present situation, our space is no longer Hilbert, which forces us to obtain new estimates. Since the problem involves singular terms, the estimates are more refined and for which the principal ingredient is the Caffarelli-Kohn-Nirenberg's inequality (see [32]). Now we state the main result of this work.

Theorem 1.3 *Suppose that V and f satisfy, respectively, (V_1) and (V_2) and (f_1) to (f_3). Then there is a constant $\Lambda^* = \Lambda^*(V_\infty, \theta, p, c_0) > 0$ such that the problem (P) has a positive solution, for all $\Lambda \geq \Lambda^*$, being V_∞ the maximum of the f in the ball of R^N centered in the origin with radius* 1.

In order to prove this theorem, we first build an auxiliary problem (AP), and then we solve the problem (AP) using variational methods. To finish, we show that the solution of (AP) is also a solution of (P). These steps are the content of the next three sections.

Hereafter, C is a positive constant which can change value in a sequence of inequalities. We denote $B_R = B_R(0)$ the ball in R^N centered in the origin with radius R. The weak $(\rightharpoonup)$ and strong $(\rightarrow)$ convergences are always taken as $n \to \infty$ and $\int_A f$ means $\int_A f(x)\,dx$. The weighted L^p spaces are denoted by $L^p_\alpha(A) = \{u : R^N \to R : \int_A |x|^{-\alpha}|u|^p < \infty\}$. When $\alpha = 0$, we denote $\|\cdot\|_{L^p(A)}$ the usual norm in $L^p(A)$, with $1 \leq p \leq \infty$. For $A = R^N$, we use $\|\cdot\|_p$.

2 The auxiliary problem

As usual, since we are looking for positive solutions of problem (P), we set $f(t) = 0$, for all $t \leq 0$. The hypothesis (V_1) allows us to consider the space

$$E = \left\{ u \in \mathcal{D}^{1,p}_a\left(R^N\right) : \int_{R^N} V|x|^{-ap^*}|u|^p < \infty \right\},$$

with norm

$$\|u\| = \left(\int_{R^N} |x|^{-ap}|\nabla u|^p + V|x|^{-ap^*}|u|^p \right)^{\frac{1}{p}}.$$

Associated to the problem (P), we define on E, the Euler-Lagrange functional

$$I(u) = \int_{R^N} |x|^{-ap}|\nabla u|^p + V|x|^{-ap^*}|u|^p - \int_{R^N} F(u),$$

being $F(s) = \int_0^s f(t)\,dt$. From the assumptions on f, it follows that I is C^1 with Gâteaux derivative

$$I'(u)v = \int_{R^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u \nabla v + V|x|^{-ap^*}|u|^{p-2}uv - \int_{R^N} f(u)v, \quad v \in E.$$

To obtain solutions of problem (P), we introduce some truncation of the function f. Consider $k = \frac{p\theta}{\theta-p} > p$, $\overline{r} > 1$ and define

$$g(x,t) = \begin{cases} f(t), & |x| \leq \overline{r}, \\ f(t), & |x| > \overline{r} \text{ and } f(t) \leq \frac{V}{k}|x|^{-ap^*}|t|^{p-2}t, \\ \frac{V}{k}|x|^{-ap^*}|t|^{p-2}t, & |x| > \overline{r} \text{ and } f(t) > \frac{V}{k}|x|^{-ap^*}|t|^{p-2}t. \end{cases} \tag{2}$$

Now we define the auxiliary problem:

$$(AP) \quad \begin{cases} -\mathcal{L}u_{ap} + V(x)|x|^{-ap^*}|u|^{p-2}u = g(x,u) & \text{in } R^N, \\ u > 0 \quad \text{in } R^N; \qquad u \in \mathcal{D}_a^{1,p}(R^N). \end{cases} \tag{3}$$

Associated to the problem (AP), we define, on E, the Euler-Lagrange functional

$$J(u) = \frac{1}{p}\int_{R^N} |x|^{-ap}|\nabla u|^p + V|x|^{-ap^*}|u|^p - \int_{R^N} G(x,u) = \frac{1}{p}\|u\|^p - \int_{R^N} G(x,u),$$

being $G(x,s) = \int_0^s g(x,t)\,dt$. From the assumptions on f, it follows that J is C^1 with Gâteaux derivative

$$J'(u)v = \int_{R^N} |x|^{-ap}|\nabla u|^{p-2}\nabla u\nabla v + V|x|^{-ap^*}|u|^{p-2}uv - \int_{R^N} g(x,u)v, \qquad v \in E.$$

3 Solving the problem (*AP*)

In this section, we show that the problem (AP) has a least energy solution, but first we define some minimax levels. To begin with we set in the space $\mathcal{D}_a^{1,p}(B_1)$, the norm $|||u||| = (\int_{B_1} |x|^{-ap}|\nabla u|^p + V_\infty|x|^{-ap^*}|u|^p)^{\frac{1}{p}}$ and we define the functional I_0 given by $I_0(u) = \frac{1}{p}\int_{B_1} |x|^{-ap}|\nabla u|^p + V_\infty|x|^{-ap^*}|u|^p - \int_{B_1} F(u)$. Here, $\mathcal{D}_a^{1,p}(B_1)$ is the completion of the $C_0^\infty(B_1)$ with the norm $|u| = (\int_{B_1} |x|^{-ap}|\nabla u|^p)^{\frac{1}{p}}$.

Lemma 3.1 *The functional I_0 has the mountain pass geometry, namely,*

1. $\exists r_0, \rho_0 > 0$ *such that* $I_0(u) \geq \rho_0$ *for* $|||u||| = r_0$.
2. $\exists e_0 \in \mathcal{D}_a^{1,p}(B_1)$ *such that* $|||e_0||| \geq r_0$ *and* $J(e_0) \leq 0$.

Proof By using the growth of f given in Remark 1.1 and the Caffarelli-Kohn-Nirenberg's inequality (see [32]), we get $\int_{B_1} F(u) \leq \int_{B_1} c_0|x|^{-ap^*}|u|^{p^*} \leq c_0|||u|||^{p^*}$, and hence $I_0(u) \geq \frac{1}{p}|||u|||^p - c_0|||u|||^{p^*}$. Since $p^* > p$, there exists r_0 such that $\rho_0 := \frac{1}{p}r_0^p - cr_0^{p^*} > 0$. Thus, we have $I_0(u) \geq \rho_0$ for $|||u||| = r_0$. By (f_3), it follows that there exist $\theta > p$ and $C > 0$ such that $F(s) \geq C|s|^\theta$. Now $u_0 \in \mathcal{D}_a^{1,p}(B_1)$ implies $I_0(tu_0) \leq \frac{t^p}{p}|||u_0|||^p - t^\theta C\int_{B_1} |u_0|^\theta$. Since $\theta > p$, there exists a t_0 large enough such that, taking $e_0 = t_0u_0$, we have $I_0(e_0) < 0$ and $|||e_0||| \geq r_0$. □

Lemma 3.2 *The functional J has the mountain pass geometry, namely,*

1. $\exists r_1, \rho_1 > 0$ *such that* $J(u) \geq \rho_1$ *for* $\|u\| = r_1$.
2. $\exists e_1 \in E$ *such that* $\|e_1\| \geq r_1$ *and* $J(e_1) \leq 0$.

Proof From the definition of G, we have $\int_{R^N} G(x,u) \leq \int_{R^N} F(u)$. Thus, like the previous lemma, we have $J(u) \geq \rho_1 := \frac{1}{p}r_1^p - cr_1^{p^*} > 0$ for $\|u\| = r_1$. Take the same $u_0 \in \mathcal{D}_a^{1,p}(B_1)$ of the proof of the previous lemma. Thus, $u_0 \in E$ and $G(x,u) = F(u)$. With the same argument, we have $J(tu_0) \leq \frac{t^p}{p}\|u_0\|^p - t^\theta C\int_{R^N}|u_0|^\theta$. Since $\theta > p$, there exists a t_1 large enough such that, taking $e_1 = t_1u_0$, we have $J(e_1) < 0$ and $\|e_1\| \geq r_1$. □

Next, we are going to define two minimax levels, which will play an important role in our arguments. Note that is possible to take t such that $e = tu_0$ satisfies two previous lemmas. This allows to define the minimax levels c and $\overline{c}$ by

$$c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J\big(\gamma(t)\big) \quad \text{with } \Gamma = \big\{\gamma \in C\big([0,1],E\big) : \gamma(0) = 0 \text{ and } \gamma(1) = e\big\}$$

and

$$\overline{c} = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_0\big(\gamma(t)\big) \quad \text{with } \Gamma = \big\{\gamma \in C\big([0,1],\mathcal{D}_a^{1,p}(B_1)\big) : \gamma(0) = 0 \text{ and } \gamma(1) = e\big\}$$

respectively. Since $J(u_0) \leq I_0(u_0)$ in $\mathcal{D}_a^{1,p}(B_1)$, we have $c \leq \overline{c}$, by their definitions. Now using the above lemma together with the mountain pass theorem [33, Theorem 2.2], we conclude that there exists a Palais-Smale sequence ((PS) sequence for short) $(u_n) \subset E$ for J, *i.e.*, (u_n) satisfies $J(u_n) \to c$ and $J'(u_n) \to 0$.

Lemma 3.3 *Suppose* (V_1) *and* (f_1) *to* (f_3) *and let* $(u_n) \subset E$ *be a* (PS) *sequence for the functional* J. *Then* (u_n) *is bounded in* E.

Proof Define the set $A = \{x \in R^N : |x| \leq R \text{ or } f(u(x)) \leq \frac{V(x)}{k}|x|^{-ap^*}|u(x)|^{p-2}u(x)\}$. In A, we have $G(x,u) = F(u)$. By using (f_3), we conclude that there is $\theta > p$ such that $-G(x,u) + \frac{1}{\theta}ug(x,u) \geq 0$. So, we have

$$\begin{aligned}
&\frac{1}{p}\int_A |x|^{-ap}|\nabla u|^p + V|x|^{-ap^*}|u|^p - \int_A G(x,u) \\
&\quad - \frac{1}{\theta}\left(\int_A |x|^{-ap}|\nabla u|^p + V|x|^{-ap^*}|u|^p - \int_A ug(x,u)\right) \\
&\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\int_A |x|^{-ap}|\nabla u|^p + V|x|^{-ap^*}|u|^p \geq \frac{(p-1)}{pk}\int_A |x|^{-ap}|\nabla u|^p + V|x|^{-ap^*}|u|^p.
\end{aligned}$$

Now consider the set $B = A^c = \{x \in R^N : |x| > R \text{ and } f(u(x)) > \frac{V(x)}{k}|x|^{-ap^*}|u(x)|^{p-2}u(x)\}$. In B, we have $\int_B ug(x,u) > 0$ and $G(x,u) = \frac{V}{pk}|x|^{-ap^*}|u|^p$. Then

$$\begin{aligned}
&\frac{1}{p}\int_B |x|^{-ap}|\nabla u|^p + V|x|^{-ap^*}|u|^p - \int_B G(x,u) \\
&\quad - \frac{1}{\theta}\left(\int_B |x|^{-ap}|\nabla u|^p + V|x|^{-ap^*}|u|^p - \int_B ug(x,u)\right) \\
&\geq \frac{1}{k}\int_B |x|^{-ap}|\nabla u|^p + V|x|^{-ap^*}|u|^p - \int_B \frac{V}{pk}|x|^{-ap^*}|u|^p \\
&\geq \frac{(p-1)}{pk}\int_B |x|^{-ap}|\nabla u|^p + V|x|^{-ap^*}|u|^p.
\end{aligned}$$

Therefore, we get $J(u) - \frac{1}{\theta}J'(u)u \geq \frac{(p-1)}{pk}\|u\|^p$. In particular, the above equation holds for the (PS) sequence (u_n) for J, and we have $J(u_n) - \frac{1}{\theta}J'(u_n)u_n \geq \frac{(p-1)}{pk}\|u_n\|^p$. On the other hand, we have $J(u_n) \to c$, $\theta > p > 1$ and $J'(u_n)\frac{u_n}{\|u_n\|} \to 0$, since $J'(u_n) \to 0$. Thus, we get $J(u_n) - \frac{1}{\theta}J'(u_n)u_n \leq M + \frac{1}{\theta}\|u_n\| \leq M + \|u_n\|$, for some constant $M > 0$. Then we have $\frac{(p-1)}{pk}\|u_n\|^p \leq M + \|u_n\|$, which can be rewritten as

$$\|u_n\|\big((p-1)\|u_n\|^{p-1} - pk\big) \leq pkM. \tag{4}$$

Assuming $\|u_n\| \to \infty$, equation (4) implies that $(p-1)\|u_n\|^{p-1} - pk \to 0$. So $\|u_n\| \to (\frac{pk}{p-1})^{\frac{1}{p-1}}$, which is a contradiction. Therefore, $\|u_n\|$ is bounded in E. □

Lemma 3.4 *Suppose* (V_1) *and* (f_1) *to* (f_3). *Then the functional* J *satisfies the Palais-Smale condition*, *i.e.*, *every* (PS) *sequence has a convergent subsequence*.

Proof Observe that, given the (PS) sequence (u_n), by Lemma 3.3, there exists $u \in E$ such that $u_n \rightharpoonup u$, because E is reflexive space.[a] Thus, it is enough to show that $\|u_n\| \to \|u\|$. We divide this task in the four claims below.

1. Claim 1 $\int_{R^N} u_n g(x,u_n) \to \int_{R^N} ug(x,u)$.
2. Claim 2 $\int_{R^N} ug(x,u_n) \to \int_{R^N} ug(x,u)$.
3. Claim 3 $\int_{R^N} V|x|^{-ap^*}|u_n|^{p-2}u_n u \to \int_{R^N} V|x|^{-ap^*}|u|^p$.
4. Claim 4 $\int_{R^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla u \to \int_{R^N} |x|^{-ap}|\nabla u|^p$.

Assuming Claims 1 to 4 for now, we proceed with the proof of lemma.

Since $J'(u_n)u_n \to 0$, we have $\int_{R^N} |x|^{-ap}|\nabla u_n|^p + V|x|^{-ap^*}|u_n|^p - \int_{R^N} u_n g(x,u_n) = o_n(1)$, and by Claim 1, we get

$$\limsup_{n\to\infty} \|u_n\|^p = \int_{R^N} ug(x,u). \tag{5}$$

As $J'(u_n)u \to 0$, we get $\int_{R^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla u + V|x|^{-ap^*}|u_n|^{p-2}u_n u - \int_{R^N} ug(x,u_n) = o_n(1)$. Passing the limit in the above equation and using Claims 2, 3 and 4, we get

$$\|u\|^p = \int_{R^N} |x|^{-ap}|\nabla u|^p + V|x|^{-ap^*}|u|^p = \int_{R^N} ug(x,u). \tag{6}$$

Using equations (5) and (6), we have $\|u_n\| \to \|u\|$. □

In order to prove the claims, for a given $\epsilon > 0$, we choose r satisfying, the following two conditions:

1. $\max\{\int_{B_{2r}\setminus B_r} |x|^{-ap^*}|u|^{p^*}, \int_{B_{2r}^c} V|x|^{-ap^*}|u|^p\} \leq \epsilon$.
2. $\eta = \eta_r \in C_0^\infty(B_r^c)$ is such that $\eta \equiv 1$ in B_{2r}^c and $0 \leq \eta \leq 1$, $|\nabla\eta| \leq \frac{2}{r^d}$, for all $x \in R^N$.

Observe that condition 1 follows by integrability of $|x|^{-ap^*}|u|^{p^*}$ and $V|x|^{-ap^*}|u|^p$. Note that, when one of these conditions holds for some r_0, it also verifies for every $r \geq r_0$. Thus, we can choose an r that satisfies both conditions.

Remark 3.5 From now on, we consider the function g defined in (2) with $\bar{r} = r$ satisfying conditions 1 and 2 above. From the growth of g and the choice of r, we conclude:

1. $g(x,t) = f(t)$, $G(x,t) = F(t)$ in B_r;
2. $g(x,t) \leq \frac{V}{k}|x|^{-ap^*}|t|^{p-2}t$, $G(x,t) \leq \frac{V}{pk}|x|^{-ap^*}|t|^p$ in B_r^c;
3. $\int_{B_{2r}^c} |ug(x,u)| \leq \int_{B_{2r}^c} |u|\frac{V}{k}|x|^{-ap^*}|u|^{p-1} \leq \frac{1}{k}\int_{B_{2r}^c} V|x|^{-ap^*}|u|^p < \epsilon$.

Verification of claims

In all proofs, except for Claim 4, we consider separately integration in B_{2r} and B_{2r}^c.

Claim 1: In B_{2r}, by combining the dominated convergence theorem with the compact embedding of E in $L_\alpha^r(B_{2r})$, $1 \le r < p^*$ and $\alpha < (a+1)r + N(1-\frac{r}{p})$ (see [34, Theorem 2.1]), we obtain $\int_{B_{2r}} u_n g(x,u_n) \to \int_{B_{2r}} ug(x,u)$. On the other hand, when integrating in B_{2r}^c we do not have the compact embedding of E in $L_\alpha^r(B_{2r}^c)$. In this case, we first estimate $\int_{B_{2r}^c} |x|^{-ap}|\nabla u_n|^p + V|x|^{-ap^*}|u_n|^p$, by using the cut-off function η defined before. As (ηu_n) is also bounded we have $J'(u_n)(\eta u_n) \to 0$, namely,

$$\int_{R^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla(\eta u_n) + V|x|^{-ap^*}|u_n|^{p-2}u_n \eta u_n = o_n(1) + \int_{R^N} \eta g(x,u_n)u_n. \tag{7}$$

Since $\eta \equiv 0$ in B_r, equation (7) holds in B_r^c. Adding to this, the fact that $g(x,t) \le \frac{V}{k}|x|^{-ap^*}|t|^{p-2}t$ in B_r^c we have

$$\begin{aligned}
&\int_{B_r^c} \eta\bigl(|x|^{-ap}|\nabla u_n|^p + V|x|^{-ap^*}|u_n|^p\bigr) + |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n(\nabla\eta)u_n \\
&\quad = \int_{B_r^c} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n\nabla(\eta u_n) + V|x|^{-ap^*}|u_n|^{p-2}u_n\eta u_n \\
&\quad = o_n(1) + \int_{B_r^c} \eta g(x,u_n)u_n \le o_n(1) + \int_{B_r^c} \eta\frac{V}{k}|x|^{-ap^*}|u_n|^p.
\end{aligned}$$

From this, we obtain

$$\begin{aligned}
&\int_{B_r^c} \eta\bigl(|x|^{-ap}|\nabla u_n|^p + V|x|^{-ap^*}|u_n|^p\bigr) \\
&\quad \le o_n(1) + \int_{B_r^c} \eta\frac{V}{k}|x|^{-ap^*}|u_n|^p + \int_{B_r^c} |x|^{-ap}|u_n||\nabla u_n|^{p-1}|\nabla\eta| \\
&\quad \le o_n(1) + \int_{B_r^c} \eta\frac{V}{k}|x|^{-ap^*}|u_n|^p + \int_{B_{2r}\setminus B_r} |x|^{-ap}|u_n||\nabla u_n|^{p-1}|\nabla\eta| \\
&\qquad + \int_{B_{2r}^c} |x|^{-ap}|u_n||\nabla u_n|^{p-1}|\nabla\eta| \\
&\quad \le o_n(1) + \frac{1}{k}\int_{B_r^c} \eta\bigl(|x|^{-ap}|\nabla u_n|^p + V|x|^{-ap^*}|u_n|^p\bigr) + \frac{2}{r^d}\int_{B_{2r}\setminus B_r} |x|^{-ap}|u_n||\nabla u_n|^{p-1}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_{B_r^c} \eta\bigl(|x|^{-ap}|\nabla u_n|^p + V|x|^{-ap^*}|u_n|^p\bigr) \\
&\quad \le o_n(1) + \left(1-\frac{1}{k}\right)^{-1}\frac{2}{r^d}\int_{B_{2r}\setminus B_r} |x|^{-ap}|u_n||\nabla u_n|^{p-1}.
\end{aligned} \tag{8}$$

Using the boundedness of (u_n) in E, *i.e.*, $\|u_n\| \le C$, strong convergence of (u_n) in $L_{ap}^p(B_{2r} \setminus B_r)$ and Hölder's inequality we conclude that

$$\begin{aligned}
\limsup_{n\to\infty} \int_{B_{2r}\setminus B_r} |x|^{-ap}|u_n||\nabla u_n|^{p-1} &\le C\left(\int_{B_{2r}\setminus B_r} |x|^{-ap}|u|^p\right)^{\frac{1}{p}} \\
&\le C\omega_N^{\frac{d}{N}}(2r)^d\left(\int_{B_{2r}\setminus B_r} |x|^{-ap^*}|u|^{p^*}\right)^{\frac{1}{p^*}},
\end{aligned} \tag{9}$$

where ω_N is the volume of the unit sphere in R^N. Therefore, given $\epsilon > 0$, by the choice of r and by equations (9) and (8), we infer that

$$\begin{aligned}
&\limsup_{n\to\infty} \int_{B_{2r}^c} |x|^{-ap}|\nabla u_n|^p + V|x|^{-ap^*}|u_n|^p \\
&\quad\le \limsup_{n\to\infty} \int_{B_r^c} \eta\bigl(|x|^{-ap}|\nabla u_n|^p + V|x|^{-ap^*}|u_n|^p\bigr) \\
&\quad\le \limsup_{n\to\infty}\left[o_n(1) + \left(1-\frac{1}{k}\right)^{-1}\frac{2}{r^d}\int_{B_{2r}\setminus B_r} |x|^{-ap}|u_n||\nabla u_n|^{p-1}\right] \\
&\quad\le C\omega_N^{\frac{d}{N}}\left(\int_{B_{2r}\setminus B_r} |x|^{-ap^*}|u|^{p^*}\right)^{\frac{1}{p^*}} \le \epsilon.
\end{aligned}$$

Hence,

$$\limsup_{n\to\infty} \int_{B_{2r}^c} |x|^{-ap}|\nabla u_n|^p + V|x|^{-ap^*}|u_n|^p = 0. \tag{10}$$

Using this fact, we conclude that

$$\begin{aligned}
\limsup_{n\to\infty} \int_{B_{2r}^c} u_n g(x,u_n) &\le \limsup_{n\to\infty} \frac{1}{k}\int_{B_{2r}^c} V|x|^{-ap^*}|u_n|^p \\
&\le \limsup_{n\to\infty} \frac{1}{k}\int_{B_{2r}^c} |x|^{-ap}|\nabla u_n|^p + V|x|^{-ap^*}|u_n|^p = 0.
\end{aligned}$$

Now, using Remark 3.5, we have $\limsup_{n\to\infty}\int_{B_{2r}^c} |u_n g(x,u_n) - ug(x,u)| = 0$.

Therefore, $\int_{B_{2r}^c} u_n g(x,u_n) \to \int_{B_{2r}^c} ug(x,u)$ and the proof of Claim 1 is completed.

Claim 2: The proof of this fact is made as in the proof of Claim 1.

Claim 3: In B_{2r} the proof proceeds like in Claim 1. For integration in B_{2r}^c, we estimate the value of $\int_{B_{2r}} V|x|^{-ap^*}|u_n|^{p-2}u_n u$ by using the Hölder's inequality

$$\begin{aligned}
\int_{B_{2r}^c} V|x|^{-ap^*}|u_n|^{p-2}u_n u &\le \left(\int_{B_{2r}^c} V|x|^{-ap^*}|u|^p\right)^{\frac{1}{p}}\left(\int_{B_{2r}^c} V|x|^{-ap^*}|u_n|^p\right)^{\frac{1}{p'}} \\
&\le C\left(\int_{B_{2r}^c} V|x|^{-ap^*}|u_n|^p\right)^{\frac{1}{p'}} \\
&\le C\left(\int_{B_{2r}^c} |x|^{-ap}|\nabla u_n|^p + V|x|^{-ap^*}|u_n|^p\right)^{\frac{1}{p'}}.
\end{aligned} \tag{11}$$

Using equations (10) and (11), we have

$$\limsup_{n\to\infty} \int_{B_{2r}^c} V|x|^{-ap^*}|u_n|^{p-2}u_n u \le \limsup_{n\to\infty} C\left(\int_{B_{2r}^c} |x|^{-ap}|\nabla u_n|^p + V|x|^{-ap^*}|u_n|^p\right)^{\frac{1}{p'}} = 0.$$

From this fact, we get $\int_{B_{2r}^c} V|x|^{-ap^*}|u_n|^{p-2}u_n u \to \int_{B_{2r}^c} V|x|^{-ap^*}|u|^p$ and the proof of this claim is completed.

Claim 4: Let p' be dual of p. Since $u \in \mathcal{D}_a^{1,p}(R^N) = \{u : R^N \to R^N : |x|^{-a}u \in L^{p^*}(R^N)$ and $|x|^{-a}\nabla u \in L^p(R^N)\}$,[b] we see that $h = |x|^{-a}|\nabla u| \in L^p(R^N)$ and $w_n = |x|^{-a(p-1)}|\nabla u_n|^{p-2}\nabla u_n \in$

$L^{p'}(R^N)$. As $\|u_n\|$ is bounded, we conclude that $\|w_n\|_{L^{p'}(R^N)}$ is also bounded. Moreover, (u_n) satisfies the hypothesis of Lemma 3.6, below. So that we have $\nabla u_n \to \nabla u$ a.e. $x \in R^N$, which give us $w_n \to w = |x|^{-a(p-1)}|\nabla u|^{p-2}\nabla u$ a.e. $x \in R^N$. Using a theorem [35, Theorem 13.44], we conclude that $w_n \rightharpoonup w$ in $L^{p'}(R^N)$. Thus, we have $\int_{R^N} w_n h \to \int_{R^N} wh$, and hence $\int_{R^N} |x|^{-ap}|\nabla u_n|^{p-2}\nabla u_n \nabla u \to \int_{R^N} |x|^{-ap}|\nabla u|^p$.

Lemma 3.6 *Let E and J be respectively the space and functional defined in Section 2. Let $(u_n) \subset E$ a bounded sequence such that $u_n \rightharpoonup u$ in E and $J'(u_n) \to 0$. Then, passing to a subsequence if necessary, we have $\nabla u_n \to \nabla u$, a.e. $x \in R^N$.*

The proof of this lemma follows using the same ideas made in [30] and [31], for a bounded domain. It can be found in [28, Claim 1] and [29, Lemma 1].

Using Lemmas 3.2, 3.3, 3.4 and the mountain pass theorem, in [36, Theorem 2.4], we conclude that there exists $u \in E$ which is a critical point for the functional J, in the minimax level c. Moreover, u is the least energy solution to the problem (AP).

4 The solution of (AP) is solution of (P)

Now, our aim is to show that the solution found in the previous section is also a solution of the problem (P). It is sufficient to verify $f(u) \le \frac{V}{k}|x|^{-ap^*}|u|^{p-2}u$, for all $x \in B_r^c$.

Lemma 4.1 *Any least energy solution u of (AP) satisfies the estimate $\|u\|^p \le \frac{pk\bar{c}}{p-1}$.*

Proof Since u is a critical point in the minimax level $c \le \bar{c}$, by Lemma 3.3, we have $\frac{(p-1)}{pk}\|u\|^p \le J(u) - \frac{1}{\theta}J'(u)u = c \le \bar{c}$. So that $\|u\|^p \le \frac{pk\bar{c}}{p-1}$. □

Remark 4.2 The constant $\frac{pk\bar{c}}{p-1}$ depends only on V_∞, θ and f.

Lemma 4.3 *Let h be such that $|x|^{-ap^*}|h|^q$ is integrable, with $pq > N$. Consider $H : R^N \times R \to R$, and $b : R^N \to R$ nonnegative and continuous functions such that $|H(x,s)| \le h(x)|x|^{-ap^*}|s|^{p-1}$, for all $s > 0$. Let $v \in E$ be a weak solution of $(AP2)$:*

$$-\mathcal{L}v_{ap} + b|x|^{-ap^*}|v|^{p-2}v = H(x,v) \quad \text{in } R^N.$$

Then there exists a constant $M = M(q, \|h\|_{L^q_{ap^*}(R^N)}) > 0$ such that $\|v\|_\infty \le M\||x|^{-a}v\|_{p^*}$.

Proof Given $m \in N$ and $\beta > 1$, set $A_m = \{x \in R^N : |v|^{\beta-1} \le m\}$, $B_m = R^N \setminus A_m$ and

$$v_m = \begin{cases} v|v|^{p(\beta-1)} & \text{in } A_m, \\ m^p v & \text{in } B_m. \end{cases}$$

Thus

$$\nabla v_m = \begin{cases} (p\beta - p + 1)|v|^{p(\beta-1)}\nabla v & \text{in } A_m, \\ m^p \nabla v & \text{in } B_m. \end{cases}$$

Then $v_m \in E$ and using it as the test function in $(AP2)$. we have

$$\int_{R^N} |x|^{-ap}|\nabla v|^{p-2}\nabla v \nabla v_m + b|x|^{-ap^*}|v|^{p-2}v v_m = \int_{R^N} H(x,v)v_m. \tag{12}$$

By definition of v_m, we get

$$\begin{aligned}\int_{R^N} & |x|^{-ap}|\nabla v|^{p-2}\nabla v\nabla v_m \\ &= (p\beta - p + 1)\int_{A_m} |x|^{-ap}|v|^{p(\beta-1)}|\nabla v|^p + m^p\int_{B_m}|x|^{-ap}|\nabla v|^p \end{aligned} \tag{13}$$

and

$$\int_{R^N} b|x|^{-ap^*}|v|^{p-2}vv_m = \int_{A_m} b|x|^{-ap^*}|v|^{p\beta} + m^p\int_{B_m} b|x|^{-ap^*}|v|^p > 0. \tag{14}$$

Using equations (13) and (14), we have

$$\begin{aligned}\int_{A_m} & |x|^{-ap}|v|^{p(\beta-1)}|\nabla v|^p \\ &\le (p\beta - p + 1)^{-1}\int_{R^N} |x|^{-ap}|\nabla v|^{p-2}\nabla v\nabla v_m + b|x|^{-ap^*}|v|^{p-2}vv_m. \end{aligned} \tag{15}$$

Putting

$$w_m = \begin{cases} v|v|^{\beta-1} & \text{in } A_m, \\ mv & \text{in } B_m, \end{cases}$$

we have

$$\nabla w_m = \begin{cases} \beta|v|^{\beta-1}\nabla v & \text{in } A_m, \\ m\nabla v & \text{in } B_m. \end{cases}$$

Thus,

$$\int_{R^N} |x|^{-ap}|\nabla w_m|^p = \beta^p\int_{A_m} |x|^{-ap}|v|^{p(\beta-1)}|\nabla v|^p + m^p\int_{B_m}|x|^{-ap}|\nabla v|^p. \tag{16}$$

Now, taking into account (14), we get

$$\begin{aligned}\int_{R^N} b|x|^{-ap^*}|w_m|^p &= \int_{A_m} b|x|^{-ap^*}|v|^{p\beta} + m^p\int_{B_m} b|x|^{-ap^*}|v|^p \\ &= \int_{R^N} b|x|^{-ap^*}|v|^{p-2}vv_m. \end{aligned} \tag{17}$$

Using (17), (13) and (16), we have

$$\begin{aligned}\int_{R^N} & |x|^{-ap}|\nabla w_m|^p + b|x|^{-ap^*}|w_m|^p - \int_{R^N} |x|^{-ap}|\nabla v|^{p-2}\nabla v\nabla v_m + b|x|^{-ap^*}|v|^{p-2}vv_m \\ &= \left(\beta^p - p\beta + p - 1\right)\int_{A_m} |x|^{-ap}|v|^{p(\beta-1)}|\nabla v|^p. \end{aligned} \tag{18}$$

Combining (18), (15) and (12), we obtain

$$\int_{R^N} |x|^{-ap}|\nabla w_m|^p + b|x|^{-ap^*}|w_m|^p \le \beta^p\int_{R^N} H(x,v)v_m. \tag{19}$$

Let S be the best constant for inequality $(\int_{R^N}|x|^{-ap^*}|u|^{p^*})^{\frac{p}{p^*}} \leq S\int_{R^N}|x|^{-ap}|\nabla u|^p$, for all $u \in \mathcal{D}_a^{1,p}(R^N)$. (See [32].) By the inequality (19), the boundedness of H and the definitions of v_m and S, we get $(\int_{A_m}|x|^{-ap^*}|w_m|^{p^*})^{\frac{p}{p^*}} \leq S\beta^p\int_{R^N}h(x)|x|^{-ap^*}|v|^{p\beta}$. Since $|w_m| = |v|^\beta$ in A_m, we have

$$\left(\int_{A_m}|x|^{-ap^*}|v|^{p^*\beta}\right)^{\frac{p}{p^*}} \leq S\beta^p\int_{R^N}h(x)|x|^{-ap^*}|v|^{p\beta}.$$

Making $m \to \infty$ and using the monotone convergence theorem, we get

$$\left(\int_{R^N}|x|^{-ap^*}|v|^{p^*\beta}\right)^{\frac{p}{p^*}} \leq S\beta^p\int_{R^N}h(x)|x|^{-ap^*}|v|^{p\beta}.$$

Since $pq > N$, $\sigma = \frac{N}{q_1(N-p)} > 1$. Thus, we can consider $\beta = \sigma^j$ for $j = 1, 2, 3, \ldots$ and, using the Hölder's inequality, we have

$$\begin{aligned}
\left\||x|^{-\frac{a}{\sigma^j}}v\right\|_{p^*\sigma^j}^{p\sigma^j} &\leq S\sigma^{jp}\int_{R^N}h(x)|x|^{-ap^*}|v|^{p\sigma^j}\\
&\leq S\sigma^{jp}\left[\int_{R^N}|x|^{-ap^*}\left|h(x)\right|^q\right]^{\frac{1}{q}}\left[\int_{R^N}\left||x|^{-\frac{a}{\sigma^{j-1}}}|v|\right|^{p\sigma^j q_1}\right]^{\frac{1}{q_1}}\\
&\leq M_0\sigma^{jp}\left[\int_{R^N}\left||x|^{-\frac{a}{\sigma^{j-1}}}|v|\right|^{p\sigma^j q_1}\right]^{\frac{1}{q_1}} \leq M_0\sigma^{jp}\left\||x|^{-\frac{a}{\sigma^{j-1}}}v\right\|_{p\sigma^j q_1}^{p\sigma^j},
\end{aligned}$$

being $M_0 = S[\int_{R^N}|x|^{-ap^*}|h(x)|^q]^{\frac{1}{q}}$. Note that M_0 is independent of j. Thus, we get

$$\left\||x|^{-\frac{a}{\sigma^j}}v\right\|_{p^*\sigma^j} \leq M_0^{\frac{1}{p\sigma^j}}\sigma^{\frac{j}{\sigma^j}}\left\||x|^{-\frac{a}{\sigma^{j-1}}}v\right\|_{p\sigma^j q_1} \quad \text{for all } j = 1, 2, 3, \ldots. \tag{20}$$

For $j = 1$ and $j = 2$, we have $p\sigma q_1 = p^*$ and $p\sigma^2 q_1 = p^*\sigma$. Applying this in (20), we have

$$\left\||x|^{-\frac{a}{\sigma}}v\right\|_{p^*\sigma} \leq M_0^{\frac{1}{p\sigma}}\sigma^{\frac{1}{\sigma}}\left\||x|^{-a}v\right\|_{p^*} \quad \text{and}$$

$$\left\||x|^{-\frac{a}{\sigma^2}}v\right\|_{p^*\sigma^2} \leq M_0^{\frac{1}{p\sigma^2}}\sigma^{\frac{2}{\sigma^2}}\left\||x|^{-\frac{a}{\sigma}}v\right\|_{p^*\sigma}.$$

By iterating, we have

$$\left\||x|^{-\frac{a}{\sigma^2}}v\right\|_{p^*\sigma^2} \leq \sigma^{\frac{1}{\sigma}+\frac{2}{\sigma^2}}M_0^{\frac{1}{p}(\frac{1}{\sigma}+\frac{1}{\sigma^2})}\left\||x|^{-a}v\right\|_{p^*}.$$

Thus, we get

$$\left\||x|^{-\frac{a}{\sigma^j}}v\right\|_{p^*\sigma^j} \leq \sigma^{\frac{1}{\sigma}+\frac{2}{\sigma^2}+\cdots+\frac{j}{\sigma^j}}M_0^{\frac{1}{p}(\frac{1}{\sigma}+\frac{1}{\sigma^2}+\cdots+\frac{1}{\sigma^j})}\left\||x|^{-a}v\right\|_{p^*} \quad \text{for all } j = 1, 2, 3, \ldots.$$

Then we can say that, for all $t \geq p^*$, we have

$$\|v\|_t \leq \sigma^{\frac{\sigma}{(\sigma-1)^2}}M_0^{\frac{1}{p(\sigma-1)}}\left\||x|^{-a}v\right\|_{p^*}.$$

Putting $M := \sigma^{\frac{\sigma}{(\sigma-1)^2}} M_0^{\frac{1}{p(\sigma-1)}}$, we get

$$\|v\|_\infty = \lim_{t\to\infty} \|v\|_t \le \lim_{t\to\infty} M \big\| |x|^{-a} v \big\|_{p^*} = M \big\| |x|^{-a} v \big\|_{p^*}.$$

As σ depends on q, we have, by definition of M, that $M = M(q, \|h\|_{L^q_{ap^*}(R^N)}) > 0$. □

Remark 4.4 In the previous lemma, the constant M does not depend on the potential b of the problem $(AP2)$.

Lemma 4.5 *There exists a constant $M_1 > 0$ such that $\|u\|_\infty \le M_1$, for all u positive solution of (AP).*

Proof Take r from the definition of g and define $A = \{x \in R^N : |x| \le r \text{ or } f(u(x)) \le \frac{V(x)}{k} |x|^{-ap^*} |u(x)|^{p-2} u(x)\}$ and $B = R^N \setminus A = \{x \in R^N : |x| > r \text{ and } f(u(x)) > \frac{V(x)}{k} |x|^{-ap^*} \times |u(x)|^{p-2} u(x)\}$. Now define H and b by

$$H(x,t) = \begin{cases} f(t) & \text{in } A, \\ 0 & \text{in } B \end{cases} \quad \text{and} \quad b(x) = \begin{cases} V(x) & \text{in } A, \\ (1 - \frac{1}{k}) V(x) & \text{in } B. \end{cases}$$

We will show that, if u is solution of (AP), then u is weak solution of $(AP2)$.

In particular, we recall that $H(x,u) = f(u) = g(x,u)$, in A, and $g(x,u) = \frac{V}{k} |x|^{-ap^*} |u|^{p-2} u$, in B. Using the fact that R^N is the disjoint union of A and B and the definitions above, for a given $\phi \in E$, we write

$$\begin{aligned} &\int_{R^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \phi + b |x|^{-ap^*} |u|^{p-2} u \phi - \int_{R^N} H(x,u) \phi \\ &\quad = \int_{R^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \phi + V |x|^{-ap^*} |u|^{p-2} u \phi - \int_{R^N} g(x,u) \phi = 0. \end{aligned}$$

Hence, u verifies $\int_{R^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla \phi + b |x|^{-ap^*} |u|^{p-2} u \phi = \int_{R^N} H(x,u) \phi$, for all $\phi \in E$. From (f_1) and (f_2), we obtain $|f(s)| \le c_0 |x|^{-ap^*} |s|^{\alpha-1}$, for all $s > 0$, with $\alpha \in (p, p^*)$. From the definition of H, it follows that $|H(x,u)| \le |f(u)| \le h(x) |x|^{-ap^*} |u|^{p-1}$, with $h(x) = c_0 |u|^{\alpha-p}$. Taking $q = \frac{p^*}{\alpha-p}$, we have $|x|^{-ap^*} |h|^q = |x|^{-ap^*} |u|^{p^*}$ that is integrable. Then u satisfies the hypotheses of Lemma 4.3, that is,

$$\|u\|_\infty \le M \big\| |x|^{-a} u \big\|_{p^*}. \tag{21}$$

Using the definition of the constant S and Lemma 4.1, we have

$$\begin{aligned} \big\| |x|^{-a} u \big\|_{p^*} &= \left(\int_{R^N} |x|^{-ap^*} |u|^{p^*} \right)^{\frac{1}{p^*}} \le \left(S \int_{R^N} |x|^{-ap} |\nabla u|^p \right)^{\frac{1}{p}} \\ &\le \left(S \|u\|^p \right)^{\frac{1}{p}} \le \left(S \frac{pk\bar{c}}{p-1} \right)^{\frac{1}{p}}. \end{aligned} \tag{22}$$

Combining equations (21) and (22), we obtain $\|u\|_\infty \le M (S \frac{pk\bar{c}}{p-1})^{\frac{1}{p}} := M_1$. □

Lemma 4.6 *Let r be as in the definition of g and consider $R_0 \geq r$. Let u a be any positive solution of (AP). Then we have*

$$u(x) \leq \|u\|_\infty R_0^{\frac{N-p}{p-1}} |x|^{-\frac{N-p(a+1)}{p-1}} \leq M_1 R_0^{\frac{N-p}{p-1}} |x|^{-\frac{N-p(a+1)}{p-1}} \quad \textit{for all } x \in B^c_{R_0}.$$

Proof Consider $v(x) = M_1 R_0^{\frac{N-p}{p-1}} |x|^{-\frac{N-p(a+1)}{p-1}}$. By Lemma 4.5, we have $\|u\|_\infty \leq M_1$. So $u \leq v$, for $|x| = R_0$. It follows that $(u-v)^+ = 0$, in $|x| = R_0$, and the function given by

$$w = \begin{cases} 0, & |x| < R_0, \\ (u-v)^+, & |x| \geq R_0 \end{cases}$$

is so that $w \in \mathcal{D}^{1,p}_a(R^N)$. Moreover, $w \in E$, because $u, v \in E$. Let us show now that $(u-v)^+ = 0$, in $|x| \geq R_0$. Taking w as the test function, using the hypotheses on g and V and the fact that u is positive solution of (AP), we have

$$\begin{aligned} \int_{R^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla w &= \int_{R^N} g(x,u) w - \int_{R^N} V |x|^{-ap^*} |u|^{p-2} u w \\ &= \int_{B^c_{R_0}} g(x,u) w - \int_{B^c_{R_0}} V |x|^{-ap^*} |u|^{p-2} u w \\ &= \left(\frac{1}{k} - 1\right) \int_{B^c_{R_0}} V |x|^{-ap^*} |u|^{p-2} u w \leq 0. \end{aligned} \tag{23}$$

Here, we considered that $k > 1$. Using the radially symmetric form of the operator $-\mathcal{L}v_{ap}$ (see [37, 38]), we have that $-\operatorname{div}(|x|^{-ap} |\nabla v|^{p-2} \nabla v) = 0$ in $B^c_{R_0}$. In the weak form, it is

$$\int_{B^c_{R_0}} |x|^{-ap} |\nabla v|^{p-2} \nabla v \nabla \phi = 0 \quad \text{for all } \phi \in E.$$

Hence,

$$\int_{R^N} |x|^{-ap} |\nabla v|^{p-2} \nabla v \nabla w = \int_{B^c_{R_0}} |x|^{-ap} |\nabla v|^{p-2} \nabla v \nabla w = 0. \tag{24}$$

Putting $A = \{x \in R^N : |x| \geq R_0 \text{ and } u(x) > v(x)\}$ and $B = R^N \setminus A$, we have

$$w = \begin{cases} u - v & \text{in } A, \\ 0 & \text{in } B. \end{cases}$$

By (23) and (24), we obtain, for all $1 < p < N$, that

$$\begin{aligned} &\int_A |x|^{-ap} \big[|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v\big][\nabla u - \nabla v] \\ &\quad = \int_A \big[|x|^{-ap} |\nabla u|^{p-2} \nabla u - |x|^{-ap} |\nabla v|^{p-2} \nabla v\big] \nabla w \\ &\quad = \int_{R^N} |x|^{-ap} |\nabla u|^{p-2} \nabla u \nabla w - \int_{R^N} |x|^{-ap} |\nabla v|^{p-2} \nabla v \nabla w \leq 0. \end{aligned} \tag{25}$$

Consider $2 \leq p < N$. Using the Tolksdorf's inequality (see [39, Lemma 2.1] or [31, Lemma 4.1]) and equation (25), we obtain

$$\begin{aligned}\int_{R^N} |x|^{-ap}|\nabla w|^p &= \int_A |x|^{-ap}|\nabla u - \nabla v|^p \\ &\leq c\int_A |x|^{-ap}\bigl[|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr][\nabla u - \nabla v] \leq 0.\end{aligned}$$

In the case $1 < p < 2$, in addition to the above arguments used, we also use the Hölder's inequality. Then

$$\begin{aligned}&\int_{R^N} |x|^{-ap}|\nabla w|^p \\ &\quad = \int_A |x|^{-ap}|\nabla w|^p = \int_A |x|^{-ap}\bigl(|\nabla u - \nabla v|^2\bigr)^{\frac{p}{2}} \\ &\quad \leq c\int_A |x|^{-ap}\bigl(\bigl[|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr][\nabla u - \nabla v]\bigr)^{\frac{p}{2}}\bigl(|\nabla u| + |\nabla v|\bigr)^{\frac{(2-p)p}{2}} \\ &\quad \leq c\biggl\{\int_A |x|^{-ap}\bigl[|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr][\nabla u - \nabla v]\biggr\}^{\frac{p}{2}}\biggl\{\int_A |x|^{-ap}\bigl[|\nabla u| + |\nabla v|\bigr]^p\biggr\}^{\frac{2-p}{2}} \\ &\quad \leq 0.\end{aligned}$$

Then $\int_{R^N} |x|^{-ap}|\nabla w|^p \leq 0$ for all $1 < p < N$. Thus, we have $w = 0$, in R^N, which implies that $(u - v)^+ = 0$, in $|x| \geq R_0$. From this, we conclude that $u \leq v$ in R^N, and lemma is proved. □

Proof of Theorem 1.3 We will show that $f(u) \leq \frac{V}{k}|x|^{-ap^*}|u|^{p-2}u$ in B_r^c, for all solution u of (AP). By Remark 1 we have $|sf(s)| \leq C_0|x|^{-ap^*}|s|^{p^*}$, which gives us $\frac{f(u)}{|x|^{-ap^*}|u|^{p-2}u} \leq C_0|u|^{\frac{p^2}{N-p}}$. Now, note that the hypothesis (V_3) holds for all $R_0 > R$. Hence, for $R_0 = r > R$, we can use (V_3) and Lemma 4.6. Thus, for each x in B_r^c, and $\Lambda^* = kC_0M_1^{\frac{p^2}{N-p}}R_0^{\frac{p^2}{p-1}}$, we have

$$\begin{aligned}\frac{f(u)}{|x|^{-ap^*}|u|^{p-2}u} &\leq C_0|u|^{\frac{p^2}{N-p}} \leq C_0\bigl|M_1R_0^{\frac{N-p}{p-1}}|x|^{-\frac{N-p(a+1)}{p-1}}\bigr|^{\frac{p^2}{N-p}} \\ &= C_0M_1^{\frac{p^2}{N-p}}R_0^{\frac{p^2}{p-1}}\frac{V}{V|x|^{\frac{p^2[N-p(a+1)]}{(p-1)(N-p)}}} = \frac{\Lambda^*}{k}\frac{V}{V|x|^{\frac{p^2[N-p(a+1)]}{(p-1)(N-p)}}} \leq \frac{\Lambda^*}{\Lambda}\frac{V}{k}.\end{aligned}$$

Now, taking $\Lambda^* \leq \Lambda$ it follows that $\frac{f(u)}{|x|^{-ap^*}|u|^{p-2}u} \leq \frac{V}{k}$, for every x in B_r^c, which give us $f(u) \leq \frac{V}{k}|x|^{-ap^*}|u|^{p-2}u$, in B_r^c. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that all authors collaborated and dedicated the same amount of time in order to perform this article.

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Endnotes

[a] The proof of this fact follows by using the same ideas made in the usual Sobolev spaces; see [40].

[b] The proof of this fact follows by using the same ideas made in the case $\mathcal{D}^{1,p}(R^N)$; see [41].

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