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Inverse eigenvalue problem for a class of Dirac operators with discontinuous coefficient

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Abstract

In this paper, the inverse problem of recovering the coefficient of a Dirac operator is studied from the sequences of eigenvalues and normalizing numbers. The theorem on the necessary and sufficient conditions for the solvability of this inverse problem is proved and a solution algorithm of the inverse problem is given.

MSC: 34A55; 34L40

Keywords: Dirac operator; inverse problem; necessary and sufficient condition

1 Introduction

In this paper, we consider the boundary value problem generated by the system of Dirac equations on the finite interval $0 < x < \pi$:

$$By' + \Omega(x)y = \lambda\rho(x)y \tag{1}$$

with boundary conditions

$$y_1(0) = y_1(\pi) = 0, \tag{2}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \qquad y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

$p(x)$, $q(x)$ are real valued functions, $p(x) \in L_2(0,\pi)$, $q(x) \in L_2(0,\pi)$, λ is a spectral parameter,

$$\rho(x) = \begin{cases} 1, & 0 \le x \le a, \\ \alpha, & a < x \le \pi, \end{cases}$$

and $1 \neq \alpha > 0$.

The inverse problem for the Dirac operator with separable boundary conditions was completely solved by two spectra in [1, 2]. The reconstruction of the potential from one spectrum and norming constants was investigated in [3]. For the Dirac operator, the inverse periodic and antiperiodic boundary value problems were given in [4–6]. Using the Weyl-Titschmarsh function, the direct and inverse problems for a Dirac type-system were developed in [7, 8]. Uniqueness of the inverse problem for the Dirac operator with a discontinuous coefficient by the Weyl function was studied in [9] and discontinuity conditions inside an interval were worked out in [10, 11]. The inverse problem for weighted Dirac equations was obtained in [12]. The reconstruction of the potential by the spectral function was given in [13]. For the Dirac operator with peculiarity, the inverse problem was found in [14]. Inverse nodal problems for the Dirac operator were examined in [15, 16]. In the case of potentials that belong entrywise to $L_p(0,1)$, for some $p \in [1,\infty)$, the inverse spectral problem for the Dirac operator was studied in [17], and in this work, not only the Gelfand-Levitan-Marchenko method but also the Krein method [18] was used. In the positive half line, the inverse scattering problem for the Dirac operator with discontinuous coefficient was analyzed in [19]. Besides, in a finite interval, for Sturm-Liouville operator inverse problem has widely been developed (see [20–22]). The inverse problem of the Sturm-Liouville operator with discontinuous coefficient was worked out in [23, 24] and discontinuous conditions inside an interval were obtained in [25]. In the mathematical and physical literature, the direct and inverse problems for the Dirac operator are widespread, so there are numerous investigations as regards the Dirac operator. Therefore, we can mention the studies concerned with a discontinuity, which is close to our topic, in the references list.

In this paper, our aim is to solve the inverse problem for the Dirac operator with a piecewise continuous coefficient on a finite interval. Let λ_n and α_n ($n \in \mathbb{Z}$) be, respectively, eigenvalues and normalizing numbers of the boundary value problem (1), (2). The quantities $\{\lambda_n, \alpha_n\}$ ($n \in \mathbb{Z}$) are called spectral data. We can state the inverse problem for a system of Dirac equations in the following way: knowing the spectral data $\{\lambda_n, \alpha_n\}$ ($n \in \mathbb{Z}$) to indicate a method of determining the potential $\Omega(x)$ and to find necessary and sufficient conditions for $\{\lambda_n, \alpha_n\}$ ($n \in \mathbb{Z}$) to be the spectral data of a problem (1), (2). In this paper, this problem is completely solved.

We give a brief account of the contents of this paper in the following section.

2 Preliminaries

Let $S(x,\lambda)$ be solution of the system (1) satisfying the initial conditions

$$S_1(0,\lambda) = 0, \qquad S_2(0,\lambda) = -1.$$

The solution $S(x,\lambda)$ has an integral representation [26] as follows:

$$S(x,\lambda) = S_0(x,\lambda) + \int_0^{\mu(x)} A(x,t) \begin{pmatrix} \sin\lambda t \\ -\cos\lambda t \end{pmatrix} dt, \tag{3}$$

where

$$S_0(x,\lambda) = \begin{pmatrix} \sin\lambda\mu(x) \\ -\cos\lambda\mu(x) \end{pmatrix}, \qquad \mu(x) = \begin{cases} x, & 0 \le x \le a, \\ \alpha x - \alpha a + a, & a < x \le \pi, \end{cases}$$

$A = (A_{ij})_{i,j=1}^2$ is a quadratic matrix function $A_{ij}(x,\cdot) \in L_2(0,\pi)$ and $A(x,t)$ is the solution of the problem

$$\begin{aligned} &BA'_x(x,t) + \rho(x)A'_t(x,t)B = -\Omega(x)A(x,t), \\ &\Omega(x) = \rho(x)A\bigl(x,\mu(x)\bigr)B - BA\bigl(x,\mu(x)\bigr), \\ &A(x,0)B = 0. \end{aligned} \tag{4}$$

Equation (4) gives the relation between the kernel $A(x,t)$ and the coefficient $\Omega(x)$ of (1). Let $\psi(x,\lambda)$ be solutions of the system (1) satisfying the initial conditions

$$\psi_1(\pi,\lambda) = 0, \qquad \psi_2(\pi,\lambda) = -1.$$

The characteristic function $\Delta(\lambda)$ of the problem (1), (2) is

$$\Delta(\lambda) := W\bigl[S(x,\lambda),\psi(x,\lambda)\bigr] = S_2(x,\lambda)\psi_1(x,\lambda) - S_1(x,\lambda)\psi_2(x,\lambda), \tag{5}$$

where $W[S(x,\lambda),\psi(x,\lambda)]$ is the Wronskian of the solutions $S(x,\lambda)$ and $\psi(x,\lambda)$ and independent of $x \in [0,\pi]$. The zeros λ_n of the characteristic function coincide with the eigenvalues of the boundary value problem (1), (2). The functions $S(x,\lambda)$ and $\psi(x,\lambda)$ are eigenfunctions and there exists a sequence β_n such that

$$\psi(x,\lambda_n) = \beta_n S(x,\lambda_n), \quad \beta_n \neq 0. \tag{6}$$

Denote the normalizing numbers by

$$\alpha_n := \int_0^\pi \bigl(\bigl|S_1(x,\lambda_n)\bigr|^2 + \bigl|S_2(x,\lambda_n)\bigr|^2\bigr)\rho(x)\,dx.$$

The following relation is valid:

$$\dot{\Delta}(\lambda_n) = \alpha_n\beta_n, \tag{7}$$

where $\dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$. In fact, since $S(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of the problem (1), (2), we get

$$\begin{aligned} &\psi_2'(x,\lambda) + p(x)\psi_1(x,\lambda) + q(x)\psi_2(x,\lambda) = \lambda\rho(x)\psi_1(x,\lambda), \\ &-\psi_1'(x,\lambda) + q(x)\psi_1(x,\lambda) - p(x)\psi_2(x,\lambda) = \lambda\rho(x)\psi_2(x,\lambda), \\ &S_2'(x,\lambda_n) + p(x)S_1(x,\lambda_n) + q(x)S_2(x,\lambda_n) = \lambda_n\rho(x)S_1(x,\lambda_n), \\ &-S_1'(x,\lambda_n) + q(x)S_1(x,\lambda_n) - p(x)S_2(x,\lambda_n) = \lambda_n\rho(x)S_2(x,\lambda_n). \end{aligned}$$

Multiplying the equations by $S_1'(x,\lambda_n)$, $S_2'(x,\lambda_n)$, $-\psi_1'(x,\lambda)$, $-\psi_2'(x,\lambda)$, respectively, adding them together, integrating from 0 to π and using the condition (2),

$$\int_0^\pi \bigl\{S_1(x,\lambda_n)\psi_1(x,\lambda) + S_2(x,\lambda_n)\psi_2(x,\lambda)\bigr\}\rho(x)\,dx = \frac{\Delta(\lambda) - \Delta(\lambda_n)}{\lambda - \lambda_n}$$

is found. From (6) as $\lambda \to \lambda_n$, we obtain

$$\beta_n \alpha_n = \dot{\Delta}(\lambda_n).$$

The following two theorems are obtained by Huseynov and Latifova in [27].

Theorem 1 (i) *The boundary value problem* (1), (2) *has a countable set of simple eigenvalues* λ_n $(n \in \mathbb{Z})$ *where*

$$\lambda_n = \frac{n\pi}{\alpha\pi - \alpha a + a} + \epsilon_n, \quad \{\epsilon_n\} \in l_2. \tag{8}$$

(ii) *The eigen vector-functions of problem* (1), (2) *can be represented in the form*

$$S(x, \lambda_n) = \begin{pmatrix} \sin \frac{n\pi\mu(x)}{\alpha\pi - \alpha a + a} \\ -\cos \frac{n\pi\mu(x)}{\alpha\pi - \alpha a + a} \end{pmatrix} + \begin{pmatrix} \xi_n^{(1)}(x) \\ \xi_n^{(2)}(x) \end{pmatrix},$$

$$\sum_{n=-\infty}^{\infty} \{ |\xi_n^{(1)}(x)|^2 + |\xi_n^{(2)}(x)|^2 \} \le C; \qquad \mu(x) = \begin{cases} x, & 0 \le x \le a, \\ \alpha x - \alpha a + a, & a < x \le \pi. \end{cases}$$

(iii) *The normalizing numbers of problem* (1), (2) *have the form*

$$\alpha_n = \alpha\pi - \alpha a + a + \delta_n, \quad \{\delta_n\} \in l_2. \tag{9}$$

Theorem 2 (i) *The system of eigen vector-functions* $\{S(x, \lambda_n)\}$ $(n \in \mathbb{Z})$ *of problem* (1), (2) *is complete in space* $L_{2,\rho}(0, \pi; \mathbb{C}^2)$.

(ii) *Let* $f(x)$ *be an absolutely continuous vector-function on the segment* $[0, \pi]$ *and* $f_1(0) = f_1(\pi) = 0$. *Then*

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n S(x, \lambda_n), \tag{10}$$

$$c_n = \frac{1}{\alpha_n} \langle f(x), S(x, \lambda_n) \rangle,$$

moreover, the series converges uniformly with respect to $x \in [0, \pi]$.

(iii) *For* $f(x) \in L_{2,\rho}(0, \pi; \mathbb{C}^2)$ *series* (10) *converges in* $L_{2,\rho}(0, \pi; \mathbb{C}^2)$; *moreover, the Parseval equality holds*:

$$\|f\|^2 = \sum_{n=-\infty}^{+\infty} \alpha_n |c_n|^2. \tag{11}$$

From [27], the following inequality holds:

$$|\Delta(\lambda)| \ge C_\delta \exp(|\operatorname{Im} \lambda| \mu(\pi)), \tag{12}$$

where C_δ is a positive number and this inequality is valid in the domain

$$G_\delta = \{ \lambda : |\lambda - \lambda_n^0| \ge \delta, n = 0, \pm 1, \pm 2, \dots \},$$

where $\lambda_n^0 = \frac{n\pi}{\mu(\pi)}$ ($n \in \mathbb{Z}$) are zeros of the function $\Delta_0(\lambda) = \sin\lambda\mu(\pi)$ and δ is a sufficiently small number.

In Section 3, the fundamental equation

$$A\bigl(x,\mu(t)\bigr) + F(x,t) + \int_0^{\mu(x)} A(x,\xi)F_0(\xi,t)\,d\xi = 0, \quad 0 < t < x,$$

is derived by using the method by Gelfand-Levitan-Marchenko, where

$$F_0(x,t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{\alpha_n}\begin{pmatrix}\sin\lambda_n x\\ -\cos\lambda_n x\end{pmatrix}\tilde{S}_0(t,\lambda_n) - \frac{1}{\alpha_n^0}\begin{pmatrix}\sin\lambda_n^0 x\\ -\cos\lambda_n^0 x\end{pmatrix}\tilde{S}_0\bigl(t,\lambda_n^0\bigr)\right]$$

and

$$F(x,t) = F_0\bigl(\mu(x),t\bigr).$$

In Section 4, we show that the fundamental equation has a unique solution $A(x,t)$ and the boundary value problem (1), (2) can be uniquely determined from the spectral data. In Section 5, the result is obtained from Lemma 6 that the function $S(x,\lambda)$ defined by (3) satisfies the equation

$$BS'(x,\lambda) + \Omega(x)S(x,\lambda) = \lambda\rho(x)S(x,\lambda),$$

where

$$\Omega(x) = \rho(x)A\bigl(x,\mu(x)\bigr)B - BA\bigl(x,\mu(x)\bigr),$$

where $A(x,t)$ is the solution of the fundamental equation. In Lemma 7, using the fundamental equation, the Parseval equality

$$\int_0^{\pi}\bigl(g_1^2(x) + g_2^2(x)\bigr)\rho(x)\,dx = \sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}\left(\int_0^{\pi}\tilde{S}(t,\lambda_n)g(t)\rho(t)\,dt\right)^2$$

is found. We demonstrate by using Lemma 6, Lemma 9, and Lemma 10 that $\{\lambda_n,\alpha_n\}$ ($n \in \mathbb{Z}$) are spectral data of the boundary value problem (1), (2). Then necessary and sufficient conditions for the solvability of problem (1), (2) are obtained in Theorem 11. Finally, we give an algorithm of the construction of the function $\Omega(x)$ by the spectral data $\{\lambda_n,\alpha_n\}$ ($n \in \mathbb{Z}$).

Note that throughout this paper, $\tilde{\phi}$ denotes the transposed matrix of ϕ.

3 Fundamental equation

Theorem 3 *For each fixed $x \in (0,\pi]$ the kernel $A(x,t)$ from the representation* (3) *satisfies the following equation*:

$$A\bigl(x,\mu(t)\bigr) + F(x,t) + \int_0^{\mu(x)} A(x,\xi)F_0(\xi,t)\,d\xi = 0, \quad 0 < t < x, \tag{13}$$

where

$$F_0(x,t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{\alpha_n} \begin{pmatrix} \sin\lambda_n x \\ -\cos\lambda_n x \end{pmatrix} \tilde{S}_0(t,\lambda_n) - \frac{1}{\alpha_n^0} \begin{pmatrix} \sin\lambda_n^0 x \\ -\cos\lambda_n^0 x \end{pmatrix} \tilde{S}_0\bigl(t,\lambda_n^0\bigr) \right] \tag{14}$$

and

$$F(x,t) = F_0\bigl(\mu(x),t\bigr), \tag{15}$$

where λ_n^0 *and* α_n^0 *are, respectively, eigenvalues and normalizing numbers of the boundary value problem* (1), (2) *when* $\Omega(x) \equiv 0$.

Proof According to (3) we have

$$S_0(x,\lambda) = S(x,\lambda) - \int_0^{\mu(x)} A(x,t) \begin{pmatrix} \sin\lambda t \\ -\cos\lambda t \end{pmatrix} dt. \tag{16}$$

It follows from (3) and (16) that

$$\begin{aligned} \sum_{n=-N}^{N} \frac{1}{\alpha_n} S(x,\lambda_n)\tilde{S}_0(t,\lambda_n) &= \sum_{n=-N}^{N} \frac{1}{\alpha_n} S_0(x,\lambda_n)\tilde{S}_0(t,\lambda_n) \\ &\quad + \int_0^{\mu(x)} A(x,\xi) \left(\sum_{n=-N}^{N} \frac{1}{\alpha_n} \begin{pmatrix} \sin\lambda_n\xi \\ -\cos\lambda_n\xi \end{pmatrix} \tilde{S}_0(t,\lambda_n) \right) d\xi \end{aligned}$$

and

$$\begin{aligned} \sum_{n=-N}^{N} \frac{1}{\alpha_n} S(x,\lambda_n)\tilde{S}_0(t,\lambda_n) &= \sum_{n=-N}^{N} \frac{1}{\alpha_n} S(x,\lambda_n)\tilde{S}(t,\lambda_n) \\ &\quad - \sum_{n=-N}^{N} \frac{1}{\alpha_n} S(x,\lambda_n) \int_0^{\mu(t)} (\sin\lambda_n\xi, -\cos\lambda_n\xi)\tilde{A}(t,\xi)\, d\xi. \end{aligned}$$

Using the last two equalities, we obtain

$$\begin{aligned} &\sum_{n=-N}^{N} \left[\frac{1}{\alpha_n} S(x,\lambda_n)\tilde{S}(t,\lambda_n) - \frac{1}{\alpha_n^0} S\bigl(x,\lambda_n^0\bigr)\tilde{S}\bigl(t,\lambda_n^0\bigr) \right] \\ &\quad = \sum_{n=-N}^{N} \left[\frac{1}{\alpha_n} S_0(x,\lambda_n)\tilde{S}_0(t,\lambda_n) - \frac{1}{\alpha_n^0} S_0\bigl(x,\lambda_n^0\bigr)\tilde{S}_0\bigl(t,\lambda_n^0\bigr) \right] \\ &\qquad + \int_0^{\mu(x)} A(x,\xi) \sum_{n=-N}^{N} \left[\frac{1}{\alpha_n^0} \begin{pmatrix} \sin\lambda_n^0\xi \\ -\cos\lambda_n^0\xi \end{pmatrix} \tilde{S}_0\bigl(t,\lambda_n^0\bigr) \right] d\xi \\ &\qquad + \int_0^{\mu(x)} A(x,\xi) \sum_{n=-N}^{N} \left[\frac{1}{\alpha_n} \begin{pmatrix} \sin\lambda_n\xi \\ -\cos\lambda_n\xi \end{pmatrix} \tilde{S}_0(t,\lambda_n) - \frac{1}{\alpha_n^0} \begin{pmatrix} \sin\lambda_n^0\xi \\ -\cos\lambda_n^0\xi \end{pmatrix} \tilde{S}_0\bigl(t,\lambda_n^0\bigr) \right] d\xi \\ &\qquad + \sum_{n=-N}^{N} \frac{1}{\alpha_n} S(x,\lambda_n) \int_0^{\mu(t)} (\sin\lambda_n\xi, -\cos\lambda_n\xi)\tilde{A}(t,\xi)\, d\xi \end{aligned}$$

or

$$\Phi_N(x,t) = I_{N1}(x,t) + I_{N2}(x,t) + I_{N3}(x,t) + I_{N4}(x,t), \tag{17}$$

where

$$\Phi_N(x,t) = \sum_{n=-N}^{N} \left[\frac{1}{\alpha_n} S(x,\lambda_n)\tilde{S}(t,\lambda_n) - \frac{1}{\alpha_n^0} S(x,\lambda_n^0)\tilde{S}(t,\lambda_n^0) \right],$$

$$I_{N1}(x,t) = \sum_{n=-N}^{N} \left[\frac{1}{\alpha_n} S_0(x,\lambda_n)\tilde{S}_0(t,\lambda_n) - \frac{1}{\alpha_n^0} S_0(x,\lambda_n^0)\tilde{S}_0(t,\lambda_n^0) \right],$$

$$I_{N2}(x,t) = \int_0^{\mu(x)} A(x,\xi) \sum_{n=-N}^{N} \left[\frac{1}{\alpha_n^0} \begin{pmatrix} \sin\lambda_n^0\xi \\ -\cos\lambda_n^0\xi \end{pmatrix} \tilde{S}_0(t,\lambda_n^0) \right] d\xi,$$

$$I_{N3}(x,t) = \int_0^{\mu(x)} A(x,\xi) \sum_{n=-N}^{N} \left[\frac{1}{\alpha_n} \begin{pmatrix} \sin\lambda_n\xi \\ -\cos\lambda_n\xi \end{pmatrix} \tilde{S}_0(t,\lambda_n) - \frac{1}{\alpha_n^0} \begin{pmatrix} \sin\lambda_n^0\xi \\ -\cos\lambda_n^0\xi \end{pmatrix} \tilde{S}_0(t,\lambda_n^0) \right] d\xi,$$

$$I_{N4}(x,t) = \sum_{n=-N}^{N} \frac{1}{\alpha_n} S(x,\lambda_n) \int_0^{\mu(t)} (\sin\lambda_n\xi, -\cos\lambda_n\xi)\tilde{A}(t,\xi)\, d\xi.$$

It is easily found by using (14) and (15) that

$$F(x,t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{\alpha_n} S_0(x,\lambda_n)\tilde{S}_0(t,\lambda_n) - \frac{1}{\alpha_n^0} S_0(x,\lambda_n^0)\tilde{S}_0(t,\lambda_n^0) \right]. \tag{18}$$

Let $f(x) \in AC[0,\pi]$. Then according to the expansion formula (10) in Theorem 2, we obtain uniformly on $x \in [0,\pi]$

$$\lim_{N\to\infty} \int_0^{\pi} \Phi_N(x,t)f(t)\rho(t)\, dt = \sum_{n=-\infty}^{\infty} c_n S(x,\lambda_n) - \sum_{n=-\infty}^{\infty} c_n^0 S(x,\lambda_n^0) = 0. \tag{19}$$

From (18), we find

$$\begin{aligned} &\lim_{N\to\infty} \int_0^{\pi} I_{N1}(x,t)f(t)\rho(t)\, dt \\ &\quad = \lim_{N\to\infty} \int_0^{\pi} \sum_{n=-N}^{N} \left[\frac{1}{\alpha_n} S_0(x,\lambda_n)\tilde{S}_0(t,\lambda_n) - \frac{1}{\alpha_n^0} S_0(x,\lambda_n^0)\tilde{S}_0(t,\lambda_n^0) \right] f(t)\rho(t)\, dt \\ &\quad = \int_0^{\pi} F(x,t)f(t)\rho(t)\, dt. \end{aligned} \tag{20}$$

It follows from (3) that

$$\begin{pmatrix} \sin\lambda\xi \\ -\cos\lambda\xi \end{pmatrix} = \begin{cases} S_0(\xi,\lambda), & \xi < a, \\ S_0(\frac{\xi}{\alpha} + a - \frac{a}{\alpha},\lambda), & \xi > a. \end{cases} \tag{21}$$

Taking into account (21) and expansion formula (10) in Theorem 2, we get

$$\begin{aligned}
&\lim_{N\to\infty}\int_0^\pi I_{N2}(x,t)f(t)\rho(t)\,dt \\
&= \int_0^\pi \left[\int_0^{\mu(x)} A(x,\xi)\sum_{n=-N}^{N}\left[\frac{1}{\alpha_n^0}\begin{pmatrix}\sin\lambda_n^0\xi\\ -\cos\lambda_n^0\xi\end{pmatrix}\tilde{S}_0\big(t,\lambda_n^0\big)\right]d\xi\right]f(t)\rho(t)\,dt \\
&= \int_0^\pi \left[\int_0^a A(x,\xi)\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n^0}S_0\big(\xi,\lambda_n^0\big)\tilde{S}_0\big(x,\lambda_n^0\big)\,d\xi\right]f(t)\rho(t)\,dt \\
&\quad+ \int_0^\pi \left[\int_a^{\alpha x-\alpha a+a} A(x,\xi)\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n^0}S_0\left(\frac{\xi}{\alpha}+a-\frac{a}{\alpha},\lambda_n^0\right)\tilde{S}_0\big(x,\lambda_n^0\big)\,d\xi\right]f(t)\rho(t)\,dt \\
&= \int_0^a A(x,\xi)f(\xi)\,d\xi + \int_a^{\alpha x-\alpha a+a} A(x,\xi)f\left(\frac{\xi}{\alpha}+a-\frac{a}{\alpha}\right)d\xi.
\end{aligned}$$

Substituting $\frac{\xi}{\alpha}+a-\frac{a}{\alpha}\to\xi'$, we obtain

$$\begin{aligned}
&\lim_{N\to\infty}\int_0^\pi I_{N2}(x,t)f(t)\rho(t)\,dt \\
&= \int_0^a A(x,\xi)f(\xi)\,d\xi + \alpha\int_a^x A\big(x,\alpha\xi'-\alpha a+a\big)f\big(\xi'\big)\,d\xi' \\
&= \int_0^a A(x,t)f(t)\,dt + \alpha\int_a^x A(x,\alpha t-\alpha a+a)f(t)\,dt \\
&= \int_0^x A\big(x,\mu(t)\big)f(t)\rho(t)\,dt. \qquad (22)
\end{aligned}$$

Now, we calculate

$$\begin{aligned}
&\lim_{N\to\infty}\int_0^\pi I_{N3}(x,t)f(t)\rho(t)\,dt \\
&= \lim_{N\to\infty}\int_0^\pi\int_0^{\mu(x)} A(x,\xi)\sum_{n=-N}^{N}\left[\frac{1}{\alpha_n}\begin{pmatrix}\sin\lambda_n\xi\\ -\cos\lambda_n\xi\end{pmatrix}\tilde{S}_0(t,\lambda_n)\right. \\
&\quad\left.-\frac{1}{\alpha_n^0}\begin{pmatrix}\sin\lambda_n^0\xi\\ -\cos\lambda_n^0\xi\end{pmatrix}\tilde{S}_0\big(t,\lambda_n^0\big)\right]f(t)\rho(t)\,d\xi\,dt \\
&= \int_0^\pi\left[\int_0^{\mu(x)} A(x,\xi)F_0(\xi,t)\,d\xi\right]f(t)\rho(t)\,dt. \qquad (23)
\end{aligned}$$

Using (7) and the residue theorem, we get

$$\begin{aligned}
&\lim_{N\to\infty}\int_0^\pi I_{N4}(x,t)f(t)\rho(t)\,dt \\
&= \lim_{N\to\infty}\int_0^\pi\left[\sum_{n=-N}^{N}\frac{1}{\alpha_n}S(x,\lambda_n)\int_0^{\mu(t)}(\sin\lambda_n\xi,-\cos\lambda_n\xi)\tilde{A}(t,\xi)\,d\xi\right]f(t)\rho(t)\,dt \\
&= \lim_{N\to\infty}\int_0^\pi\left[\sum_{n=-N}^{N}\frac{\psi(x,\lambda_n)}{\dot{\Delta}(\lambda_n)}\int_0^{\mu(t)}(\sin\lambda_n\xi,-\cos\lambda_n\xi)\tilde{A}(t,\xi)\,d\xi\right]f(t)\rho(t)\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{N\to\infty} \int_0^\pi \left[\sum_{n=-N}^{N} \operatorname*{Res}_{\lambda=\lambda_n} \frac{\psi(x,\lambda)}{\Delta(\lambda)} \int_0^{\mu(t)} (\sin\lambda\xi, -\cos\lambda\xi)\tilde{A}(t,\xi)\,d\xi \right] f(t)\rho(t)\,dt \\
&= \lim_{N\to\infty} \int_0^\pi \left[\frac{1}{2\pi i} \int_{\Gamma_N} \frac{\psi(x,\lambda)}{\Delta(\lambda)} \int_0^{\mu(t)} (\sin\lambda\xi, -\cos\lambda\xi)\tilde{A}(t,\xi)\,d\xi\,d\lambda \right] f(t)\rho(t)\,dt \\
&= \lim_{N\to\infty} \int_0^\pi \left[\frac{1}{2\pi i} \int_{\Gamma_N} \frac{\psi(x,\lambda)}{\Delta(\lambda)} e^{|\operatorname{Im}\lambda|\mu(t)} \right. \\
&\quad \left. \times e^{-|\operatorname{Im}\lambda|\mu(t)} \int_0^{\mu(t)} (\sin\lambda\xi, -\cos\lambda\xi)\tilde{A}(t,\xi)\,d\xi\,d\lambda \right] f(t)\rho(t)\,dt,
\end{aligned}
\tag{24}
$$

where $\Gamma_N = \{\lambda : |\lambda| = \lambda_N^0 + \frac{\pi}{2\mu(\pi)}\}$ is oriented counter-clockwise, N is a sufficiently large number. Taking into account the asymptotic formulas as $|\lambda| \to \infty$

$$
\begin{aligned}
\psi_1(x,\lambda) &= -\sin\lambda\bigl(\mu(\pi)-\mu(x)\bigr) + O\left(\frac{1}{|\lambda|} e^{|\operatorname{Im}\lambda|(\mu(\pi)-\mu(x))}\right), \\
\psi_2(x,\lambda) &= -\cos\lambda\bigl(\mu(\pi)-\mu(x)\bigr) + O\left(\frac{1}{|\lambda|} e^{|\operatorname{Im}\lambda|(\mu(\pi)-\mu(x))}\right)
\end{aligned}
$$

and the relations ([20], Lemma 1.3.1)

$$
\begin{aligned}
&\lim_{|\lambda|\to\infty} \max_{0\le t\le\pi} e^{-|\operatorname{Im}\lambda|(\mu(t))} \left| \int_0^{\mu(t)} A_{i,1}(t,\xi)\sin\lambda\xi\,d\xi \right| = 0, \\
&\lim_{|\lambda|\to\infty} \max_{0\le t\le\pi} e^{-|\operatorname{Im}\lambda|(\mu(t))} \left| \int_0^{\mu(t)} A_{i,2}(t,\xi)\cos\lambda\xi\,d\xi \right| = 0, \quad i = 1,2,
\end{aligned}
$$

it follows from (12) and (24) that

$$
\lim_{N\to\infty} \int_0^\pi I_{N4}(x,t) f(t)\rho(t)\,dt = 0. \tag{25}
$$

Thus, using (17), (19), (20), (22) (23), and (25), we find

$$
\begin{aligned}
&\int_0^x A\bigl(x,\mu(t)\bigr) f(t)\rho(t)\,dt + \int_0^\pi F(x,t) f(t)\rho(t)\,dt \\
&\quad + \int_0^\pi \left[\int_0^{\mu(x)} A(x,\xi)F_0(\xi,t)\,d\xi \right] f(t)\rho(t)\,dt = 0.
\end{aligned}
$$

Since $f(x)$ can be chosen arbitrarily,

$$
A\bigl(x,\mu(t)\bigr) + F(x,t) + \int_0^{\mu(x)} A(x,\xi)F_0(\xi,t)\,d\xi = 0, \quad 0 < t < x
$$

is obtained. □

4 Uniqueness

Lemma 4 *For each fixed* $x \in (0,\pi]$, (13) *has a unique solution* $A(x,\cdot) \in L_2(0,\mu(x))$.

Proof When $a < x$, (13) can be rewritten as

$$
L_x A(x,\cdot) + K_x A(x,\cdot) = -F(x,\cdot),
$$

where

$$(L_x f)(t) = \begin{cases} f(t), & t \le a < x, \\ f(\alpha t - \alpha a + a), & a < t \le x, \end{cases} \tag{26}$$

$$(K_x f) = \int_0^{\alpha x - \alpha a + a} f(\xi) F_0(\xi, t)\, d\xi, \quad 0 < t < x.$$

Now, we shall prove that L_x is invertible, *i.e.* has a bounded inverse in $L_2(0, \pi)$.

Consider the equation $(L_x f)(t) = \varphi(t)$, $\varphi(t) \in L_2(0, \pi; \mathbb{C}^2)$. From this and (24),

$$f(t) = \left(L_x^{-1}\varphi\right)(t) = \begin{cases} \varphi(t), & t \le a, \\ \varphi(\frac{t + \alpha a - a}{\alpha}), & a < t. \end{cases}$$

We show that

$$\|f\|_{L_2} = \left\|L_x^{-1}\varphi\right\| \le C\|\varphi\|_{L_2}.$$

In fact,

$$\begin{aligned} &\int_0^\pi \left(\left|f_1(t)\right|^2 + \left|f_2(t)\right|^2\right) dt \\ &\quad = \int_0^a \left(\left|\varphi_1(t)\right|^2 + \left|\varphi_2(t)\right|^2\right) dt \\ &\qquad + \int_a^\pi \left(\left|\varphi_1\left(\frac{t + \alpha a - a}{\alpha}\right)\right|^2 + \left|\varphi_2\left(\frac{t + \alpha a - a}{\alpha}\right)\right|^2\right) dt \\ &\quad = \int_0^a \left(\left|\varphi_1(t)\right|^2 + \left|\varphi_2(t)\right|^2\right) dt + \alpha \int_a^{(\frac{\pi + \alpha a - a}{\alpha})} \left(\left|\varphi_1(t)\right|^2 + \left|\varphi_2(t)\right|^2\right) dt \\ &\quad \le C \int_0^\pi \left(\left|\varphi_1(t)\right|^2 + \left|\varphi_2(t)\right|^2\right) dt. \end{aligned}$$

Thus, the operator L_x is invertible in $L_2(0, \pi)$. Therefore the fundamental equation (13) is equivalent to

$$A(x, \cdot) + L_x^{-1} K_x A(x, \cdot) = -L_x^{-1} F(x, \cdot)$$

and $L_x^{-1} K_x$ is completely continuous in $L_2(0, \pi)$. Then it is sufficient to prove that the equation

$$g\left(\mu(t)\right) + \int_0^{\mu(x)} g(\xi) F_0(\xi, t)\, d\xi = 0 \tag{27}$$

has only the trivial solution $g(t) = 0$. Let $g(t)$ be a non-trivial solution of (27). Then

$$\begin{aligned} &\int_0^x \left(g_1^2\left(\mu(t)\right) + g_2^2\left(\mu(t)\right)\right) \rho(t)\, dt \\ &\qquad + \int_0^x \int_0^{\mu(x)} \left(g(\xi) F_0(\xi, t), g\left(\mu(t)\right)\right) \rho(t)\, d\xi\, dt = 0. \end{aligned}$$

It follows from (14) that

$$\begin{aligned}&\int_0^x \big(g_1^2(\mu(t)) + g_2^2(\mu(t))\big)\rho(t)\,dt \\ &\quad+ \int_0^x \tilde{g}(\mu(t))\rho(t)\int_0^{\mu(x)} g(\xi)\left(\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}\begin{pmatrix}\sin\lambda_n\xi\\ -\cos\lambda_n\xi\end{pmatrix}\tilde{S}_0(t,\lambda_n)\right. \\ &\quad\left.-\frac{1}{\alpha_n^0}\begin{pmatrix}\sin\lambda_n^0\xi\\ -\cos\lambda_n^0\xi\end{pmatrix}\tilde{S}_0(t,\lambda_n^0)\right)d\xi\,dt = 0.\end{aligned}$$

Using (21), we get

$$\begin{aligned}&\int_0^x \big(g_1^2(\mu(t)) + g_2^2(\mu(t))\big)\rho(t)\,dt \\ &\quad+ \int_0^x \tilde{g}(\mu(t))\rho(t)\int_0^a g(\xi)\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}S_0(\xi,\lambda_n)\tilde{S}_0(t,\lambda_n)\,d\xi\,dt \\ &\quad- \int_0^x \tilde{g}(\mu(t))\rho(t)\int_0^a g(\xi)\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n^0}S_0(\xi,\lambda_n^0)\tilde{S}_0(t,\lambda_n^0)\,d\xi\,dt \\ &\quad+ \int_0^x \tilde{g}(\mu(t))\rho(t)\int_0^{\alpha x-\alpha a+a} g(\xi)\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}S_0\left(\frac{\xi}{\alpha}+a-\frac{a}{\alpha},\lambda_n\right)\tilde{S}_0(t,\lambda_n)\,d\xi\,dt \\ &\quad- \int_0^x \tilde{g}(\mu(t))\rho(t)\int_0^{\alpha x-\alpha a+a} g(\xi)\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n^0}S_0\left(\frac{\xi}{\alpha}+a-\frac{a}{\alpha},\lambda_n^0\right)\tilde{S}_0(t,\lambda_n^0)\,d\xi\,dt = 0.\end{aligned}$$

Substituting $\frac{\xi}{\alpha} + a - \frac{a}{\alpha} \to \xi$ into the last two integrals, we obtain

$$\begin{aligned}&\int_0^x \big(g_1^2(\mu(t)) + g_2^2(\mu(t))\big)\rho(t)\,dt \\ &\quad+ \int_0^x \tilde{g}(\mu(t))\rho(t)\int_0^a g(\xi)\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}S_0(\xi,\lambda_n)\tilde{S}_0(t,\lambda_n)\,d\xi\,dt \\ &\quad- \int_0^x \tilde{g}(\mu(t))\rho(t)\int_0^a g(\xi)\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n^0}S_0(\xi,\lambda_n^0)\tilde{S}_0(t,\lambda_n^0)\,d\xi\,dt \\ &\quad+ \alpha\int_0^x \tilde{g}(\mu(t))\rho(t)\int_a^x g(\alpha\xi-\alpha a+a)\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}S_0(\xi,\lambda_n)\tilde{S}_0(t,\lambda_n)\,d\xi\,dt \\ &\quad- \alpha\int_0^x \tilde{g}(\mu(t))\rho(t)\int_a^x g(\alpha\xi-\alpha a+a)\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n^0}S_0(\xi,\lambda_n^0)\tilde{S}_0(t,\lambda_n^0)\,d\xi\,dt \\ &= \int_0^x \big(g_1^2(\mu(t)) + g_2^2(\mu(t))\big)\rho(t)\,dt \\ &\quad+ \int_0^x \tilde{g}(\mu(t))\rho(t)\int_0^x g(\mu(\xi))\rho(\xi)\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}S_0(\xi,\lambda_n)\tilde{S}_0(t,\lambda_n)\,d\xi\,dt \\ &\quad- \int_0^x \tilde{g}(\mu(t))\rho(t)\int_0^x g(\mu(\xi))\rho(\xi)\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n^0}S_0(\xi,\lambda_n^0)\tilde{S}_0(t,\lambda_n^0)\,d\xi\,dt = 0. \qquad (28)\end{aligned}$$

Using the Parseval equality,

$$g(\mu(t)) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{\alpha_n^0} \int_0^x g(\mu(t)) \tilde{S}_0(t, \lambda_n^0) \rho(t)\, dt \right) S_0(t, \lambda_n^0),$$

it follows from (28) that

$$\int_0^x \tilde{g}(\mu(t)) \rho(t) \int_0^x g(\mu(\xi)) \rho(\xi) \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n} S_0(\xi, \lambda_n) \tilde{S}_0(t, \lambda_n)\, d\xi\, dt = 0.$$

Since the system $\{S_0(t, \lambda_n)\}$ $(n \in \mathbb{Z})$ is complete in $L_{2,\rho}(0, \pi; \mathbb{C}^2)$, we have $g(\mu(t)) \equiv 0$, *i.e.* $(L_x g)(t) = 0$. For L_x invertible in $L_2(0, \pi)$, $A(x, \cdot) = 0$ is obtained. □

Theorem 5 *Let* $L(\Omega(x))$ *and* $\hat{L}(\hat{\Omega}(x))$ *be two boundary value problems and*

$$\lambda_n = \hat{\lambda}_n, \qquad \alpha_n = \hat{\alpha}_n \quad (n \in \mathbb{Z}).$$

Then

$$\Omega(x) = \hat{\Omega}(x) \quad \textit{a.e. on } (0, \pi).$$

Proof According to (14) and (15), $F_0(x,t) = \hat{F}_0(x,t)$ and $F(x,t) = \hat{F}(x,t)$. Then, from the fundamental equation (13), we have $A(x,t) = \hat{A}(x,t)$. It follows from (4) that $\Omega(x) = \hat{\Omega}(x)$ a.e. on $(0, \pi)$. □

5 Reconstruction by spectral data

Let the real numbers $\{\lambda_n, \alpha_n\}$ $(n \in \mathbb{Z})$ of the form (8) and (9) be given. Using these numbers, we construct the functions $F_0(x,t)$ and $F(x,t)$ by (14) and (15) and determine $A(x,t)$ from the fundamental equation (13).

Now, let us construct the function $S(x, \lambda)$ by (3) and the function $\Omega(x)$ by (4). From [2], $F_0(x,t)$ and $F(x,t)$ have a derivative in both variables and these derivatives belong to $L_{2,\rho}(0, \pi)$.

Lemma 6 *The following relations hold*:

$$BS'(x, \lambda) + \Omega(x) S(x, \lambda) = \lambda \rho(x) S(x, \lambda), \tag{29}$$

$$S_1(0, \lambda) = 0, \qquad S_2(0, \lambda) = -1. \tag{30}$$

Proof Differentiating to x and y, (13), respectively, we get

$$A'_x(x, \mu(t)) + F'_x(x,t) + A(x, \mu(x)) F_0(\mu(x), t) + \int_0^{\mu(x)} A'_x(x, \xi) F_0(\xi, t)\, d\xi = 0, \tag{31}$$

$$\rho(t) A'_t(x, \mu(t)) + F'_t(x,t) + \int_0^{\mu(x)} A(x, \xi) F'_{0_t}(\xi, t)\, d\xi = 0. \tag{32}$$

It follows from (14) and (15) that

$$\frac{\partial}{\partial t} F_0(x,t) B + \rho(t) B \frac{\partial}{\partial x} F_0(x,t) = 0, \tag{33}$$

$$\rho(x)\frac{\partial}{\partial t}F(x,t)B+\rho(t)B\frac{\partial}{\partial x}F(x,t)=0, \tag{34}$$

and using the fundamental equation (13), we obtain

$$A(x,0)B=0. \tag{35}$$

Multiplying (31) on the left by B and $\rho(t)$ we get

$$\begin{aligned}&\rho(t)BF'_x(x,t)+\rho(t)BA'_x\big(x,\mu(t)\big)+\rho(t)BA\big(x,\mu(x)\big)F_0\big(\mu(x),t\big)\\&\quad+\rho(t)\int_0^{\mu(x)}BA'_x(x,\xi)F_0(\xi,t)\,d\xi=0\end{aligned} \tag{36}$$

and multiplying (32) on the right by B and $\rho(x)$ we have

$$\rho(x)F'_t(x,t)B+\rho(x)\rho(t)A'_t\big(x,\mu(t)\big)B+\rho(x)\int_0^{\mu(x)}A(x,\xi)F'_{0_t}(\xi,t)B\,d\xi=0. \tag{37}$$

Adding (36) and (37) and using (34), we find

$$\begin{aligned}&\rho(t)BA'_x\big(x,\mu(t)\big)+\rho(t)BA\big(x,\mu(x)\big)F_0\big(\mu(x),t\big)+\rho(t)\int_0^{\mu(x)}BA'_x(x,\xi)F_0(\xi,t)\,d\xi\\&\quad=-\rho(x)\rho(t)A'_t\big(x,\mu(t)\big)B-\rho(x)\int_0^{\mu(x)}A(x,\xi)F'_{0_t}(\xi,t)B\,d\xi\equiv I(x,t).\end{aligned} \tag{38}$$

From (33), we get

$$I(x,t)=-\rho(x)\rho(t)A'_t\big(x,\mu(t)\big)B+\rho(x)\rho(t)\int_0^{\mu(x)}A(x,\xi)BF'_{0_\xi}(\xi,t)\,d\xi. \tag{39}$$

Integrating by parts and from (35)

$$\begin{aligned}I(x,t)&=-\rho(x)\rho(t)A'_t\big(x,\mu(t)\big)B+\rho(t)\rho(x)A\big(x,\mu(x)\big)BF_0\big(\mu(x),t\big)\\&\quad-\rho(x)\rho(t)\int_0^{\mu(x)}A'_\xi(x,\xi)BF_0(\xi,t)\,d\xi\end{aligned} \tag{40}$$

is obtained. Substituting (40) into (38) and dividing by $\rho(t)\neq 0$, we have

$$\begin{aligned}&BA'_x\big(x,\mu(t)\big)+BA\big(x,\mu(x)\big)F_0\big(\mu(x),t\big)+\rho(x)A'_t\big(x,\mu(t)\big)B\\&\quad-\rho(x)A\big(x,\mu(x)\big)BF_0\big(\mu(x),t\big)\\&\quad+\int_0^{\mu(x)}\big[BA'_x(x,\xi)+\rho(x)A'_\xi(x,\xi)B\big]F_0(\xi,t)\,d\xi=0.\end{aligned} \tag{41}$$

Multiplying (13) on the left by $\Omega(x)$ in the form of (4) and adding to (41)

$$\begin{aligned}&BA'_x\big(x,\mu(x)\big)+\rho(x)A'_t\big(x,\mu(t)\big)B+\Omega(x)A\big(x,\mu(t)\big)\\&\quad+\int_0^{\mu(x)}\big[BA'_x(x,\xi)+\rho(x)A'_\xi(x,\xi)B+\Omega(x)A(x,\xi)\big]F_0(\xi,t)\,dt=0\end{aligned} \tag{42}$$

is obtained. Setting

$$J(x,t) := BA'_x(x,t) + \rho(x)A'_t(x,t)B + \Omega(x)A(x,t),$$

we can rewrite (42) as follows:

$$J\bigl(x,\mu(t)\bigr) + \int_0^{\mu(x)} J(x,\xi)F_0(\xi,t)\,d\xi = 0. \tag{43}$$

According to Lemma 4, the homogeneous equation (43) has only the trivial solution, *i.e.*

$$BA'_x(x,t) + \rho(x)A'_t(x,t)B + \Omega(x)A(x,t) = 0, \quad 0 < t < x. \tag{44}$$

Differentiating (3) and multiplying on the left by B, we have

$$\begin{aligned} BS'(x,\lambda) &= \lambda\rho(x)B\begin{pmatrix}\cos\lambda\mu(x)\\ \sin\lambda\mu(x)\end{pmatrix} + BA\bigl(x,\mu(x)\bigr)\begin{pmatrix}\sin\lambda\mu(x)\\ -\cos\lambda\mu(x)\end{pmatrix} \\ &\quad + \int_0^{\mu(x)} BA'_x(x,t)\begin{pmatrix}\sin\lambda t\\ -\cos\lambda t\end{pmatrix}dt. \end{aligned} \tag{45}$$

On the other hand, multiplying (3) on the left by $\lambda\rho(x)$ and then integrating by parts and using (35), we find

$$\begin{aligned} \lambda\rho(x)S(x,\lambda) &= -\lambda\rho(x)\begin{pmatrix}\sin\lambda\mu(x)\\ -\cos\lambda\mu(x)\end{pmatrix} + \rho(x)A\bigl(x,\mu(x)\bigr)B\begin{pmatrix}\sin\lambda\mu(x)\\ -\cos\lambda\mu(x)\end{pmatrix} \\ &\quad - \rho(x)\int_0^{\mu(x)} A'_t(x,t)B\begin{pmatrix}\sin\lambda t\\ -\cos\lambda t\end{pmatrix}dt. \end{aligned} \tag{46}$$

It follows from (45) and (46) that

$$\begin{aligned} \lambda\rho(x)S(x,\lambda) &= BS'(x,\lambda) - \bigl[BA\bigl(x,\mu(x)\bigr) - \rho(x)A\bigl(x,\mu(x)\bigr)B\bigr]\begin{pmatrix}\sin\lambda\mu(x)\\ -\cos\lambda\mu(x)\end{pmatrix} \\ &\quad - \int_0^{\mu(x)} \bigl[BA'_x(x,t) + \rho(x)A'_t(x,t)B\bigr]\begin{pmatrix}\sin\lambda t\\ -\cos\lambda t\end{pmatrix}dt. \end{aligned}$$

Taking into account (4) and (44),

$$BS'(x,\lambda) + \Omega(x)S(x,\lambda) = \lambda\rho(x)S(x,\lambda)$$

is obtained. For $x = 0$, from (3) we get (30). □

Lemma 7 *For each function $g(x) \in L_{2,\rho}(0,\pi;\mathbb{C}^2)$, the following relation holds*:

$$\int_0^{\pi} \bigl(g_1^2(x) + g_2^2(x)\bigr)\rho(x)\,dx = \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n}\left(\int_0^{\pi} \tilde{S}(t,\lambda_n)g(t)\rho(t)\,dt\right)^2. \tag{47}$$

Proof It follows from (3) and (21) that

$$S(x,\lambda) = S_0(x,\lambda) + \int_0^x A\bigl(x,\mu(t)\bigr)S_0(t,\lambda)\rho(t)\,dt. \tag{48}$$

Using the expression

$$F_0(x,t) = \begin{cases} F(x,t), & x < a, \\ F(\frac{x}{\alpha} + a - \frac{a}{\alpha}, t), & x > a, \end{cases}$$

the fundamental equation (13) is transformed into the following form:

$$A\bigl(x,\mu(t)\bigr) + F(x,t) + \int_0^x A\bigl(x,\mu(\xi)\bigr)F(\xi,t)\rho(\xi)\,d\xi = 0. \tag{49}$$

From (48), we get

$$S_0(x,\lambda) = S(x,\lambda) + \int_0^x H\bigl(x,\mu(t)\bigr)S(x,\lambda)\rho(t)\,dt \tag{50}$$

and for the kernel $H(x,\mu(t))$ we have the identity

$$\tilde{H}\bigl(x,\mu(t)\bigr) = F(x,t) + \int_0^x A\bigl(x,\mu(\xi)\bigr)F(\xi,t)\rho(\xi)\,d\xi, \quad 0 < t < x. \tag{51}$$

Denote

$$Q(\lambda) := \int_0^\pi \tilde{S}(t,\lambda)g(t)\rho(t)\,dt$$

and using (48) it is transformed into the following form:

$$Q(\lambda) = \int_0^\pi \tilde{S}_0(t,\lambda)h(t)\,dt,$$

where

$$h(t) = g(t) + \int_t^\pi \tilde{A}\bigl(s,\mu(t)\bigr)g(s)\rho(s)\,ds. \tag{52}$$

Similarly, in view of (50), we have

$$g(t) = h(t) + \int_t^\pi \tilde{H}\bigl(s,\mu(t)\bigr)h(s)\rho(s)\,ds. \tag{53}$$

According to (52),

$$\begin{aligned}\int_0^\pi F(x,t)h(t)\rho(t)\,dt &= \int_0^\pi F(x,t)\left[g(t) + \int_t^\pi \tilde{A}\bigl(s,\mu(t)\bigr)g(s)\rho(s)\,ds\right]\rho(t)\,dt \\ &= \int_0^\pi \left[F(x,t) + \int_0^t F(x,s)\tilde{A}\bigl(t,\mu(s)\bigr)\rho(s)\,ds\right]g(t)\rho(t)\,dt\end{aligned}$$

$$
\begin{aligned}
&= \int_0^x \left[F(x,t) + \int_0^t F(x,s)\tilde{A}\bigl(t,\mu(s)\bigr)\rho(s)\,ds\right]g(t)\rho(t)\,dt \\
&\quad + \int_x^\pi \left[F(x,t) + \int_0^t F(x,s)\tilde{A}\bigl(t,\mu(s)\bigr)\rho(s)\,ds\right]g(t)\rho(t)\,dt.
\end{aligned}
$$

It follows from (49) and (51) that

$$
\int_0^\pi F(x,t)h(t)\rho(t)\,dt = \int_0^x H\bigl(x,\mu(t)\bigr)g(t)\rho(t)\,dt - \int_x^\pi \tilde{A}\bigl(t,\mu(x)\bigr)g(t)\rho(t)\,dt. \tag{54}
$$

From (18) and the Parseval equality we obtain

$$
\begin{aligned}
&\int_0^\pi \bigl(h_1^2(t) + h_2^2(t)\bigr)\rho(t)\,dt + \int_0^\pi \tilde{h}(x)F(x,t)h(t)\rho(t)\rho(x)\,dx\,dt \\
&\quad = \int_0^\pi \bigl(h_1^2(t) + h_2^2(t)\bigr)\rho(t)\,dt + \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n}\left(\int_0^\pi \tilde{S}_0(t,\lambda_n)h(t)\rho(t)\,dt\right)^2 \\
&\qquad - \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n^0}\left(\int_0^\pi \tilde{S}_0\bigl(t,\lambda_n^0\bigr)h(t)\rho(t)\,dt\right)^2 \\
&\quad = \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n}\left(\int_0^\pi \tilde{S}_0(t,\lambda_n)h(t)\rho(t)\,dt\right)^2 = \sum_{n=-\infty}^{\infty} \frac{Q^2(\lambda_n)}{\alpha_n}.
\end{aligned}
$$

Taking into account (54), we have

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{Q^2(\lambda_n)}{\alpha_n} &= \int_0^\pi \bigl(h_1^2(t) + h_2^2(t)\bigr)\rho(t)\,dt \\
&\quad + \int_0^\pi \tilde{h}(x)\left(\int_0^x H\bigl(x,\mu(t)\bigr)g(t)\rho(t)\,dt\right)\rho(x)\,dx \\
&\quad - \int_0^\pi \tilde{h}(x)\left(\int_x^\pi \tilde{A}\bigl(t,\mu(x)\bigr)g(t)\rho(t)\,dt\right)\rho(x)\,dx \\
&= \int_0^\pi \bigl(h_1^2(t) + h_2^2(t)\bigr)\rho(t)\,dt \\
&\quad + \int_0^\pi \left(\int_t^\pi \tilde{h}(x)H\bigl(x,\mu(t)\bigr)\rho(x)\,dx\right)g(t)\rho(t)\,dt \\
&\quad - \int_0^\pi \tilde{h}(x)\left(\int_x^\pi \tilde{A}\bigl(t,\mu(x)\bigr)g(t)\rho(t)\,dt\right)\rho(x)\,dx,
\end{aligned}
$$

whence, by (52) and (53),

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \frac{Q^2(\lambda_n)}{\alpha_n} &= \int_0^\pi \bigl(h_1^2(t) + h_2^2(t)\bigr)\rho(t)\,dt \\
&\quad + \int_0^\pi \bigl(\tilde{g}(t) - \tilde{h}(t)\bigr)g(t)\rho(t)\,dt - \int_0^\pi \tilde{h}(x)\bigl(h(x) - g(x)\bigr)\rho(x)\,dx \\
&= \int_0^\pi \bigl(g_1^2(t) + g_2^2(t)\bigr)\rho(t)\,dt
\end{aligned}
$$

is obtained, *i.e.*, (47) is valid. □

Corollary 8 *For any function* $f(x)$ *and* $g(x) \in L_{2,\rho}(0,\pi;\mathbb{C}^2)$, *the following relation holds*:

$$\int_0^\pi \tilde{g}(x)f(x)\rho(x)\,dx = \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n}\left(\int_0^\pi \tilde{g}(t)S(t,\lambda_n)\rho(t)\,dt\right)\left(\int_0^\pi \tilde{S}(t,\lambda_n)f(t)\rho(t)\,dt\right). \quad (55)$$

Lemma 9 *The relation*

$$\int_0^\pi \tilde{S}(t,\lambda_n)S(t,\lambda_k)\rho(t)\,dt = \begin{cases} 0, & n \neq k, \\ \alpha_n, & n = k \end{cases} \quad (56)$$

is valid.

Proof (1) Let $f(x) \in W_2^1[0,\pi]$. Consider the series

$$f^*(x) = \sum_{n=-\infty}^{\infty} c_n S(x,\lambda_n), \quad (57)$$

where

$$c_n := \frac{1}{\alpha_n}\int_0^\pi \tilde{S}(x,\lambda_n)f(x)\rho(x)\,dx. \quad (58)$$

Using Lemma 6 and integrating by parts, we get

$$\begin{aligned} c_n &= \frac{1}{\lambda_n\alpha_n}\int_0^\pi \left[-\frac{\partial}{\partial x}\tilde{S}(x,\lambda_n)B + \tilde{S}(x,\lambda_n)\Omega(x)\right]f(x)\,dx \\ &= -\frac{1}{\lambda_n\alpha_n}\left\{\tilde{S}(\pi,\lambda_n)Bf(\pi) - \tilde{S}(0,\lambda_n)Bf(0)\right\} \\ &\quad + \frac{1}{\lambda_n\alpha_n}\int_0^\pi \tilde{S}(x,\lambda_n)\left[Bf'(x) + \Omega(x)f(x)\right]dx. \end{aligned}$$

Applying the asymptotic formulas in Theorem 1, $\{c_n\} \in l_2$ is found. Consequently the series (57) converges absolutely and uniformly on $[0,\pi]$. According to (55) and (58), we have

$$\begin{aligned} \int_0^\pi \tilde{g}(x)f(x)\rho(x)\,dx &= \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n}\left(\int_0^\pi \tilde{g}(t)S(t,\lambda_n)\rho(t)\,dt\right)\left(\int_0^\pi \tilde{S}(t,\lambda_n)f(t)\rho(t)\,dt\right) \\ &= \sum_{n=-\infty}^{\infty} c_n\left(\int_0^\pi \tilde{g}(t)S(t,\lambda_n)\rho(t)\,dt\right) \\ &= \int_0^\pi \tilde{g}(t)\sum_{n=-\infty}^{\infty} c_n S(t,\lambda_n)\rho(t)\,dt \\ &= \int_0^\pi \tilde{g}(t)f^*(t)\,dt. \end{aligned}$$

Since $g(x)$ is arbitrary, $f(x) = f^*(x)$ is obtained, *i.e.*

$$f(x) = \sum_{n=-\infty}^{\infty} c_n S(x,\lambda_n). \quad (59)$$

(2) Fix $k \in \mathbb{Z}$ and assume $f(x) = S(x, \lambda_k)$. Then, by virtue of (59),

$$S(x, \lambda_k) = \sum_{n=-\infty}^{\infty} c_{nk} S(x, \lambda_n),$$

where

$$c_{nk} = \frac{1}{\alpha_n} \int_0^{\pi} \tilde{S}(x, \lambda_n) S(x, \lambda_k)\, dx.$$

The system $S_0(x, \lambda_n)$ is minimal in $L_{2,\rho}(0, \pi; \mathbb{C}^2)$ and consequently by (3), the system $S(x, \lambda_n)$ is minimal in $L_{2,\rho}(0, \pi; \mathbb{C}^2)$. Hence $c_{nk} = \delta_{nk}$ and we obtain (56). □

Lemma 10 *For all $n \in \mathbb{Z}$ the equality*

$$S_1(\pi, \lambda_n) = 0$$

is valid.

Proof It is easily found that

$$\begin{aligned}(\lambda_n - \lambda_m) &\int_0^{\pi} \big[S_1(x, \lambda_n) S_1(x, \lambda_m) - S_2(x, \lambda_n) S_2(x, \lambda_m)\big] \rho(x)\, dx \\ &= \big[S_2(x, \lambda_n) S_1(x, \lambda_m) - S_1(x, \lambda_n) S_2(x, \lambda_m)\big]\big|_0^{\pi}.\end{aligned}$$

According to (56), we get

$$S_2(\pi, \lambda_n) S_1(\pi, \lambda_m) - S_1(\pi, \lambda_n) S_2(\pi, \lambda_m) = 0. \tag{60}$$

We shall prove that for any n, $S_2(\pi, \lambda_n) \neq 0$. Assume the contrary, *i.e.* there exists m such that $S_2(\pi, \lambda_m) = 0$. Then for $n \neq m$, it follows from (60) that $S_2(\pi, \lambda_n) = 0$. On the other hand, since as $n \to \infty$

$$S_2(\pi, \lambda_n) = (-1)^{n+1} + O\left(\frac{1}{n}\right),$$

$S_2(\pi, \lambda_n) \neq 0$. This contradicts the condition $S_2(\pi, \lambda_n) = 0$, $n \neq m$. Hence, $S_2(\pi, \lambda_n) \neq 0$ for any n. From (60), we have

$$\frac{S_1(\pi, \lambda_n)}{S_2(\pi, \lambda_n)} = \frac{S_1(\pi, \lambda_m)}{S_2(\pi, \lambda_m)} = H.$$

Thus, we get $S_1(\pi, \lambda_n) = H S_2(\pi, \lambda_n)$, for any n. Since

$$S_1(\pi, \lambda_n) = O\left(\frac{1}{n}\right), \quad \text{as } n \to \infty,$$

we find $H = 0$, and then $S_1(\pi, \lambda_n) = 0$ is obtained. □

Theorem 11 *For the sequences* $\{\lambda_n, \alpha_n\}$ $(n \in \mathbb{Z})$ *to be the spectral data for a certain boundary value problem* $L(\Omega(x))$ *of the form* (1), (2) *with* $\Omega(x) \in L_2(0,\pi)$, *it is necessary and sufficient that the relations* (8) *and* (9) *hold*.

Proof Necessity of the problem is proved in [27]. Let us prove the sufficiency. Let the real numbers $\{\lambda_n, \alpha_n\}$ $(n \in \mathbb{Z})$ of the form (8) and (9) be given. It follows from Lemma 6, Lemma 9, and Lemma 10 that the numbers $\{\lambda_n, \alpha_n\}$ $(n \in \mathbb{Z})$ are spectral data for the constructed boundary value problem $L(\Omega(x))$. The theorem is proved. □

The algorithm of the construction of the function $\Omega(x)$ by the spectral data $\{\lambda_n, \alpha_n\}$ $(n \in \mathbb{Z})$ follows from the proof of the theorem:

(1) By the given numbers $\{\lambda_n, \alpha_n\}$ $(n \in \mathbb{Z})$ the functions $F_0(x,t)$ and $F(x,t)$ are constructed, respectively, by (14) and (15).
(2) The function $A(x,t)$ is found from (13).
(3) $\Omega(x)$ is calculated by (4).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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