Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

On a certain way of proving the solvability for boundary value problems

Arnold Y Lepin*

*Correspondence: ctr34@inbox.lv
Institute of Mathematics and Computer Science, University of Latvia, Riga, Latvia

Abstract

A certain way of replacing a given boundary value problem by another one, a solution of which solves also the original problem, is considered.

MSC: 34B15

Keywords: boundary value problems; upper and lower functions

Consider the solvability of the boundary value problem (BVP)

$$\left(\varphi\left(t,x,x'\right)\right)' = f\left(t,x,x'\right), \quad t \in I = [a,b], \tag{1}$$

$$H_1 x = h_1, \qquad H_2 x = h_2, \quad \alpha \le x \le \beta, \tag{2}$$

where $\varphi \in C(I \times R^2, R)$ is strictly increasing in x' for fixed t and x, $f : I \times R^2 \to R$ satisfies the Caratheodory conditions, that is, $f(t,\cdot,\cdot)$ is measurable in I for fixed $x, x' \in R$, $f(\cdot,x,x')$ is continuous on R^2 for fixed $t \in I$, and for any compact set $P \subset R^2$ there exists function $g \in L(I,R)$ such, that for any $(t,x,x') \in I \times P$, the estimate $|f(t,x,x')| \le g(t)$ holds, $H_1, H_2 \in C(C^1(I,R),R)$, $h_1, h_2 \in R$, α is the lower function, β the upper function.

This boundary value problem is replaced by another one, which is dependent on the parameter $M \in (M_0, +\infty)$, $M_0 > 0$,

$$\begin{aligned} &\left(\varphi_M\left(t,x,x'\right)\right)' = f_M\left(t,x,x'\right), \quad t \in I = [a,b], \\ &H_1 x = h_1, \qquad H_2 x = h_2, \quad \alpha \le x \le \beta, \end{aligned} \tag{3}$$

where $\varphi_M \in C(I \times R^2, R)$ is strictly increasing in x' for fixed t and x, and $f : I \times R^2 \to R$ satisfies the Caratheodory conditions.

Definition 1 A function $x \in C^1(I,R)$ is a solution of (1), if $\varphi(t,x(t),x'(t))$ is absolutely continuous on I and (1) is satisfied almost everywhere on I.

We provide below definitions of generalized upper and lower functions and the generalized solution along with Theorem 1 from [1–3]. This is needed to prove the main result.

Definition 2 The class $BB^+(I,R)$ consists of functions $\alpha : I \to R$, which possess the property: for any $t \in (a,b]$ there exist the left derivative $\alpha'_l(t)$ and the limit $\lim_{\tau\to t-} \alpha'_l(\tau)$, and

$\alpha_l'(t) \geq \lim_{\tau\to t-} \alpha_l'(\tau)$; for any $t \in [a,b)$ there exist the right derivative $\alpha_r'(t)$ and the limit $\lim_{\tau\to t+} \alpha_r'(\tau)$, and $\alpha_r'(t) \leq \lim_{\tau\to t+} \alpha_r'(\tau)$, and, for any $t \in (a,b)$, $\alpha_l'(t) \leq \alpha_r'(t)$.

The class $BB^-(I,R)$ consists of functions $\beta \colon I \to R$, which possess the following property: for any $t \in (a,b]$ there exist the left derivative $\beta_l'(t)$ and the limit $\lim_{\tau\to t-} \beta_l'(\tau)$, and $\beta_l'(t) \leq \lim_{\tau\to t-} \beta_l'(\tau)$; for any $t \in [a,b)$ there exist the right derivative $\beta_r'(t)$ and the limit $\lim_{\tau\to t+} \beta_r'(\tau)$, and $\beta_r'(t) \geq \lim_{\tau\to t+} \beta_r'(\tau)$, and, for any $t \in (a,b)$, $\beta_l'(t) \geq \beta_r'(t)$.

Definition 3 We call a bounded function $\alpha \in BB^+(I,R)$ a *generalized lower function* and write $\alpha \in AG(I,R)$, if in any interval $[c,d] \in I$, where this function satisfies the Lipschitz condition, for any $t_1 \in (c,d)$ and $t_2 \in (t_1,d)$ where the derivative exists, the inequality

$$\varphi\bigl(t_2,\alpha(t_2),\alpha'(t_2)\bigr) - \varphi\bigl(t_1,\alpha(t_1),\alpha'(t_1)\bigr) \geq \int_{t_1}^{t_2} f\bigl(s,\alpha(s),\alpha'(s)\bigr)\,ds$$

holds. We will call a bounded function $\beta \in BB^-(I,R)$ a *generalized upper function* and write $\beta \in BG(I,R)$, if in any interval $[c,d] \in I$, where this function satisfies the Lipschitz condition, for any $t_1 \in (c,d)$ and $t_2 \in (t_1,d)$ where the derivative exists, the inequality

$$\varphi\bigl(t_2,\beta(t_2),\beta'(t_2)\bigr) - \varphi\bigl(t_1,\beta(t_1),\beta'(t_1)\bigr) \leq \int_{t_1}^{t_2} f\bigl(s,\beta(s),\beta'(s)\bigr)\,ds$$

holds.

A function $x \colon I \to R$ will be called a *generalized solution*, if $x \in AG(I,R) \cap BG(I,R)$.

A generalized solution has a derivative at any point, possibly infinite, either $-\infty$ or $+\infty$, and x' is continuous on $[-\infty,+\infty]$; if in some interval the derivative x' does not attain the values $-\infty$ or $+\infty$, then x is a solution of (1) in this interval.

Theorem 1 *Let $\alpha \in AG(I,R)$, $\beta \in BG(I,R)$ and $\alpha \leq \beta$. Then for any $A \in [\alpha(a),\beta(a)]$ and $B \in [\alpha(b),\beta(b)]$ there exists a generalized solution of the Dirichlet problem*

$$\bigl(\varphi\bigl(t,x,x'\bigr)\bigr)' = f\bigl(t,x,x'\bigr), \qquad x(a) = A, \qquad x(b) = B, \quad \alpha \leq x \leq \beta. \tag{4}$$

In addition to conditions on α and β the compactness conditions are needed for solvability of the boundary value problem (1)-(2). The Nagumo condition [4] for φ-Laplacian and the Schrader condition [5] are sufficient conditions for compactness of a set of solutions. We accept the following compactness conditions.

Definition 4 We say that the compactness condition is fulfilled, if for all $A \in [\alpha(a),\beta(a)]$ and $B \in [\alpha(b),\beta(b)]$ any *generalized solution* of the Dirichlet problem (4) is a solution.

It is clear that this condition is weaker than the Schrader condition.

A set of solutions of the Dirichlet problem (4) will be denoted by S.

Remark 1 If $\alpha \in AG(I,R)$, $\beta \in BG(I,R)$, $\alpha \leq \beta$ and the compactness condition is fulfilled, then the Dirichlet problem (4) has a solution.

Theorem 2 *Let $\alpha \in AG(I,R)$, $\beta \in BG(I,R)$ and the compactness condition be fulfilled. If the boundary value problem* (3) *has a solution u_M for all $M \in (M_0,+\infty)$ and for $t \in I$*

$$\varphi_M(t,x,x') = \varphi(t,x,x'), \qquad f_M(t,x,x') = f(t,x,x'), \quad \alpha \le x \le \beta, |x'| \le M,$$

then there exists $M_1 \in (M_0,+\infty)$ such that u_{M_1} solves the boundary value problem (1)-(2).

Proof Notice that the results in [6] imply that $\sup\{\|x'\|_C : x \in S\} = M_0 < +\infty$. Suppose the contrary. Let the sequence $\{M_i\}$, where $M_i \in (M_0,+\infty)$, $i = 1,2,\ldots$ tend to infinity. Consider the sequence $\{u_i\}$, where $u_i = u_{M_i}$, $i = 1,2,\ldots$. We can assume, without loss of generality, that it converges in any rational points of the interval (a,b) to the function u, located between α and β. Notice that without loss of generality for any interval $(a_1,b_1) \subset (a,b)$ it follows from the boundedness of u and the Mean Value Formula that there exists an interval $[c,d] \subset (a_1,b_1)$ such that

$$\sup\{|u_i'(t)| : i \in \{1,2,\ldots\}, t \in [c,d]\} = L < +\infty.$$

It is clear that u_i, $i \in \{1,2,\ldots\}$, and u satisfy the Lipschitz condition with constant L in $[c,d]$. The u can be extended by continuity to the entire interval $[c,d]$, and thus we obtain a function u that satisfies the Lipschitz condition. It follows from the Lipschitz condition that $\{u_i(t)\}$ converges to $u(t)$ for any $t \in [c,d]$. It is clear that the derivatives $\{u_i'(t)\}$ converge to the derivative $u'(t)$ for any $t \in [c,d]$. Therefore, $u(t)$ is a solution of (1) in the interval $[c,d]$. Continuing the construction of $u(t)$ on both sides, one gets a solution of (1) on the maximal interval (c_1,d_1). If $c_1 > a$, then $\lim_{t\to c_1+} u'(t)$ is either $-\infty$ or $+\infty$. Similarly, if $d_1 < b$, then $\lim_{t\to d_1-} u'(t)$ is either $-\infty$ or $+\infty$. If $c_1 = a$ and $\lim_{t\to a+} u'(t)$ is not $-\infty$ or $+\infty$, then $u(t)$ can be continued to a. Similarly, if $d_1 = b$ and $\lim_{t\to b-} u'(t)$ is not $-\infty$ nor $+\infty$, then $u(t)$ can be continued to b. By repeating this construction, find an open set I_1 in I, where the function $u(t)$ is defined and $u(t)$ is a solution of (1) on intervals from I_1. A set $I_2 = I\backslash I_1$ is closed and nowhere dense. For $t \in I_2$ the limit $\lim_{i\to\infty} u_i'(t)$ is equal to $-\infty$ or $+\infty$. Indeed, assuming the contrary and acting as above, we get $t \in I_1$. Extend $u(t)$ to irrational points of I_2. If $a \in I_2$, then $u(a) = \lim_{t\to a+} u(t)$, and in the remaining cases $u(\tau) = \lim_{t\to\tau-} u(t)$. The above limits exist since $u(t)$ is monotone in neighborhood of any point from I_2. Similarly we get for $t \in I_2$, $u'(t) = \lim_{i\to\infty} u_i'(t)$ and $\lim_{\tau\to t} u'(\tau) = u'(t)$. Therefore $u(t)$ is a generalized solution of (1). It follows from the compactness condition that $u(t)$ is a solution of (1). Let us show that the sequence $\{u_i'(t)\}$ uniformly converges to $u'(t)$. Suppose the contrary is true. We assume, without loss of generality, that there exist $\varepsilon > 0$ and a sequence $\{t_i\}$, where $t_i \in I$, $i = 1,2,\ldots$ such that $|u'(t_i) - u_i'(t_i)| > \varepsilon$, $i = 1,2,\ldots$ and $\lim_{i\to\infty} t_i = t_0$. Consider the case $u_i'(t_i) > u'(t_i) + \varepsilon$, $i = 1,2,\ldots$. We can assume, without loss of generality, that $u_i'(t_0) > u'(t_0) + \varepsilon/2$, $i = 1,2,\ldots$, and this contradicts the equality $\lim_{i\to\infty} u_i'(t_0) = u'(t_0)$. The uniform convergence is proved. We can conclude now that all $u_i(t)$ are the solutions of the boundary value problem (1)-(2). □

Remark 2 Theorem 2 gives the possibility to prove the solvability of boundary value problems if the solvability of more simple boundary value problems is known.

Remark 3 If $\alpha'(a) \ge \beta'(a)$ and the inequalities $\alpha'(a) \ge x'(a) \ge \beta'(a)$ hold for a solution x of the boundary value problem (1)-(2), then the compactness condition (Definition 4) can be weakened.

Definition 5 We will say that the compactness condition holds if for any $A_1 \in [\beta'(a), \alpha'(a)]$ and $B \in [\alpha(b), \beta(b)]$ all generalized solutions of the problem

$$\bigl(\varphi\bigl(t,x,x'\bigr)\bigr)' = f\bigl(t,x,x'\bigr), \qquad x'(a) = A_1, \qquad x(b) = B, \quad \alpha \le x \le \beta,$$

are classical solutions.

Example One way to use Theorem 2 is to verify that for all $t \in I$, $x, x' \in R$ and $M \in (M_0, +\infty)$, $M_0 > 0$, the following conditions are satisfied:

$$\varphi_M\bigl(t,x,x'\bigr) = \varphi\bigl(t,x,x'\bigr),$$
$$f_M\bigl(t,x,x'\bigr) = f\bigl(t,x,\delta\bigl(-M,x',M\bigr)\bigr),$$

where $\delta(u,v,w) = u$ if $v < u$, $\delta(u,v,w) = v$ if $u \le v \le w$, $\delta(u,v,w) = w$ if $w < v$.

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author participated in drafting, revising and commenting on the manuscript. The author read and approved the final manuscript.

Acknowledgements
The author sincerely thanks the reviewers for their valuable suggestions and useful comments. This research was supported by the Institute of Mathematics and Computer Science, University of Latvia.

Received: 13 December 2013 Accepted: 28 April 2014 Published: 13 May 2014

References
1. Lepin, LA: Generalized solutions and solvability of boundary value problems for the second order differential equations. Differ. Equ. **18**, 1323-1330 (1982)
2. Lepin, AY, Lepin, LA: Boundary Value Problems for Ordinary Differential Equations. Zinatne, Riga (1988) (Russian)
3. Lepin, AY, Lepin, LA: Generalized lower and upper functions for φ-Laplacian equations. Differ. Equ. (2014, in press)
4. Nagumo, M: Über die Differentialgleichung $y'' = f(x,y,y')$. Proc. Phys. Math. Soc. Jpn. **19**(3), 861-866 (1937)
5. Schrader, KW: Existence theorems for second order boundary value problems. J. Differ. Equ. **5**(3), 572-584 (1969)
6. Lepin, AY: Compactness of generalized solutions between the generalized lower and upper functions. Proc. LUMII Math. Differ. Equ. **11**, 22-24 (2011)

10.1186/1687-2770-2014-111
Cite this article as: Lepin: On a certain way of proving the solvability for boundary value problems. *Boundary Value Problems* 2014, 2014:111