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The Cauchy problem for the seventh-order dispersive equation in Sobolev space

Hongjun Wang[1*] and Yan Zheng[2]

[*]Correspondence:
hsdwhj@163.com
[1]College of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan 453007, P.R. China
Full list of author information is available at the end of the article

Abstract
This paper is devoted to the Cauchy problem for the higher-order dispersive equation $u_t + \partial_x^7 u = \partial_x^2(u^2)$, $x, t \in \mathbf{R}$. The local well-posedness of the associated Cauchy problem is established in Sobolev space $H^s(\mathbf{R})$ with $s > -\frac{7}{4}$ with the aid of the Fourier restriction norm method.

MSC: 35K30

Keywords: Cauchy problem; well-posedness; Sobolev spaces

1 Introduction

In this paper, we are concerned with the Cauchy problem for the following seventh-order dispersive equation:

$$u_t + \partial_x^7 u = \partial_x^2(u^2), \quad x, t \in \mathbf{R}, \tag{1.1}$$

$$u(x,0) = u_0(x). \tag{1.2}$$

Kenig *et al.* [1] established that

$$u_t + \partial_x^{2j+1} u + P(u, \partial_x u, \dots, \partial_x^{2j} u) = 0, \quad j \in \mathbf{N}, x, t \in \mathbf{R}, \tag{1.3}$$

$$u(x,0) = u_0(x), \tag{1.4}$$

is locally well-posed in some weighted Sobolev spaces for small initial data and for arbitrary initial data. Recently, Pilod [2] studied the following higher-order nonlinear dispersive equation:

$$u_t + \partial_x^{2j+1} u = \sum_{0 \le j_1 + j_2 \le 2j} a_{j_1, j_2} \partial_x^{j_1} u \partial_x^{j_2} u, \tag{1.5}$$

where $x, t \in \mathbf{R}$ and u is a real- (or complex-) valued function and proved it is locally well-posed in weighted Besov and Sobolev spaces for small initial data and proved ill-posedness results when $a_{0,k} \neq 0$ for some $k > j$ in the sense that (1.5) cannot have its flow map C^2 at the origin in $H^s(R)$. Very recently, Guo *et al.* [3] studied the Cauchy problem for

$$u_t + \partial_x^5 + c_1 \partial_x u \partial_x^2 u + c_2 u \partial_x^3 u = 0, \tag{1.6}$$

and he proved that it is locally well-posed in $H^s(\mathbf{R})$ with $s \geq \frac{5}{4}$ with the aid of a short time Bourgain space.

In this paper, inspired by [1–5], by using the Fourier restriction norm method, we establish that (1.1)-(1.2) is locally well-posed in Sobolev space H^s with $s > -\frac{7}{4}$.

Now we give some notations and definitions. Throughout this paper, we always assume that ψ is a smooth function, $\psi_\delta(t) = \psi(\frac{t}{\delta})$, satisfying $0 \leq \psi \leq 1$, $\psi = 1$ when $t \in [0,1]$, $\operatorname{supp}\psi \subset [-1,2]$ and $\sigma = \tau - \xi^7$, $\sigma_k = \tau_k - \xi_k^7$ $(k = 1,2)$,

$$W(t)u_0 = \int_{\mathbf{R}} e^{i(x\xi - t\xi^7)} \mathscr{F}_x u_0(\xi)\, d\xi,$$

$$\|f\|_{L_t^q L_x^p} = \left(\int_{\mathbf{R}}\left(\int_{\mathbf{R}}\big|f(x,t)\big|^p\, dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}}, \qquad \|f\|_{L_t^p L_x^p} = \|f\|_{L_{xt}^p}.$$

$\langle\xi\rangle^s = (1+\xi^2)^{\frac{s}{2}}$ for any $\xi \in \mathbf{R}$, and $\mathscr{F}u$ denotes the Fourier transformation of u with respect to its all variables. $\mathscr{F}^{-1}u$ denotes the Fourier inverse transformation of u with respect to its all variables. $\mathscr{F}_x u$ denotes the Fourier transformation of u with respect to its space variable. $\mathscr{F}_x^{-1}u$ denotes the Fourier inverse transformation of u with respect to its space variable. $\mathscr{S}(\mathbf{R}^n)$ is the Schwarz space and $\mathscr{S}'(\mathbf{R}^n)$ is its dual space. $H^s(\mathbf{R})$ is the Sobolev space with norm $\|f\|_{H^s(\mathbf{R})} \stackrel{\triangle}{=} \|\langle\xi\rangle^s \mathscr{F}_x f\|_{L_\xi^2(\mathbf{R})}$. For any $s, b \in \mathbf{R}$, $X_{s,b}(\mathbf{R}^2)$ is the Bourgain space with phase function $\phi(\xi) = -\xi^7$. That is, a function $u(x,t)$ in $\mathscr{S}'(\mathbf{R}^2)$ belongs to $X_{s,b}(\mathbf{R}^2)$ iff

$$\|u\|_{X_{s,b}(\mathbf{R}^2)} \stackrel{\triangle}{=} \big\|\langle\xi\rangle^s\langle\sigma\rangle^b \mathscr{F}u(\xi,\tau)\big\|_{L_\tau^2(\mathbf{R})L_\xi^2(\mathbf{R})} < \infty.$$

For any given interval L, $X_{s,b}(\mathbf{R}\times L)$ is the space of the restriction of all functions in $X_{s,b}(\mathbf{R}^2)$ on $\mathbf{R}\times L$, and for $u \in X_{s,b}(\mathbf{R}\times L)$ its norm is

$$\|u\|_{X_{s,b}(\mathbf{R}\times L)} = \inf\big\{\|U\|_{X_{s,b}(\mathbf{R}^2)}; U|_{\mathbf{R}\times L} = u\big\}.$$

When $L = [0,T]$, $X_{s,b}(\mathbf{R}\times L)$ is abbreviated as $X_{s,b}^T$.

The main result of this paper is as follows.

Theorem 1.1 *Assume that $u_0(x) \in H^s(\mathbf{R})$ with $s > -\frac{7}{4}$. Then the Cauchy problem for* (1.1) *is locally well-posed.*

The remainder of paper is arranged as follows. In Section 2, we make some preliminaries. In Section 3, we give an important bilinear estimate. In Section 4, we establish Theorem 1.1.

2 Preliminaries

Lemma 2.1 *Let $b > \frac{1}{2}$. Then*

$$\|u\|_{L_{xt}^4} \leq C\|u\|_{X_{0,\frac{4b}{7}}}, \tag{2.1}$$

$$\big\|D_x^{\frac{5}{4}}u\big\|_{L_t^4 L_x^\infty} \leq C\|u\|_{X_{0,b}}, \tag{2.2}$$

$$\|u\|_{L_t^4 L_x^2} \leq C\|u\|_{X_{0,\frac{1}{2}b}}, \tag{2.3}$$

$$\|u\|_{X_{0,-\frac{1}{2}b}} \leq C\|u\|_{L_t^{\frac{4}{3}} L_x^2}, \tag{2.4}$$

$$\left\|D_x^{\frac{5}{8}} u\right\|_{L_{xt}^4} \leq C\|u\|_{X_{0,\frac{3}{4}b}}. \tag{2.5}$$

Proof For the proof of (2.1)-(2.5), we refer the readers to Lemma 2.1 of [5].

We have completed the proof of Lemma 2.1. □

Lemma 2.2 *Assume that* $b = \frac{1}{2} + \epsilon$. *Then*

$$\left\|I^{\frac{1}{2}}(u_1, u_2)\right\|_{L_{xt}^2} \leq C \prod_{k=1}^{2} \|u_k\|_{X_{0,b}}, \tag{2.6}$$

where

$$\mathscr{F} I^{\frac{1}{2}}(u_1, u_2)(\xi, \tau) = \int_{\substack{\xi=\xi_1+\xi_2 \\ \tau=\tau_1+\tau_2}} \left|\xi_1^6 - \xi_2^6\right|^{\frac{1}{2}} \mathscr{F} u_1(\xi_1, \tau_1) \mathscr{F} u_2(\xi_2, \tau_2)\, d\xi_1\, d\tau_1.$$

Lemma 2.2 is the case of $n = 3$ of Lemma 3.1 of [5].

Lemma 2.3 *For any* $0 < \delta < \frac{1}{2}$, *and* $s \in \mathbf{R}$, *for* $b > 0$, *we have*

$$\left\|\psi_\delta(t) W(t) u_0\right\|_{X_{s,b}} \leq C \delta^{\frac{1}{2}-b} \|u_0\|_{H^s}. \tag{2.7}$$

For $-\frac{1}{2} < b' \leq 0 \leq b' + 1$, *we have*

$$\left\|\psi_\delta(t) \int_0^t W(t-\tau) u\, d\tau\right\|_{X_{s,b}} \leq C \delta^{1+b'-b} \|u\|_{X_{s,b'}}. \tag{2.8}$$

Lemma 2.3 can be found as Lemma 2.4 of [6].

3 Bilinear estimates

In this section, we will give an important bilinear estimate.

We give an important relation before proving the bilinear estimate.

$$|\sigma - \sigma_1 - \sigma_2| = \left|\xi^7 - \xi_1^7 - \xi_2^7\right| \sim \xi_{\min} \xi_{\max}^6,$$

where

$$\begin{aligned} \xi_{\min} &= \min\left\{|\xi|, |\xi_1|, |\xi_2|\right\}, \\ \xi_{\max} &= \max\left\{|\xi|, |\xi_1|, |\xi_2|\right\}. \end{aligned} \tag{3.1}$$

Lemma 3.1 *Let* $s \geq -\frac{7}{4} + 21\epsilon$, $b = \frac{1}{2} + \epsilon$, *where* $0 \ll \epsilon \leq 1$, $b' = -\frac{1}{2} + 2\epsilon$. *Then*

$$\left\|\partial_x^2 \prod_{k=1}^{2} (u_k)\right\|_{X_{s,b'}} \leq C \prod_{k=1}^{2} \|u_k\|_{X_{s,b}}. \tag{3.2}$$

Proof Let

$$F_k(\xi_k,\tau_k)=\langle\xi_k\rangle^s\langle\sigma_k\rangle^b\mathcal{F}u_k(\xi_k,\tau_k),$$
$$F(\xi,\tau)=\langle\xi\rangle^{-s}\langle\sigma\rangle^{b'}\mathcal{F}u(\xi,\tau),$$
$$\sigma=\tau-\xi^7,\qquad \sigma_k=\tau_k-\xi_k^7,\quad k=1,2.$$

To establish (3.2), it is sufficient to derive the following inequality:

$$\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}}K_1(\xi_1,\tau_1,\xi,\tau)|F|\prod_{k=1}^{2}|F_k|\,d\xi_1\,d\tau_1\,d\xi\,d\tau\le C\|F\|_{L^2_{\xi\tau}}\|F_1\|_{L^2_{\xi\tau}}\|F_2\|_{L^2_{\xi\tau}},\tag{3.3}$$

where

$$K_1(\xi_1,\tau_1,\xi,\tau)=\frac{|\xi|^2\langle\sigma\rangle^{b'}\langle\xi\rangle^s}{\prod_{k=1}^2\langle\xi_k\rangle^s\langle\sigma_k\rangle^b}.\tag{3.4}$$

Without loss of generality, we assume that $F\ge 0$, $F_k\ge 0$ ($k=1,2$). To derive (3.3), it suffices to prove that

$$\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}}K_1(\xi_1,\tau_1,\xi,\tau)F\prod_{k=1}^{2}F_k\,d\xi_1\,d\tau_1\,d\xi\,d\tau\le C\|F\|_{L^2_{\xi\tau}}\|F_1\|_{L^2_{\xi\tau}}\|F_2\|_{L^2_{\xi\tau}}.\tag{3.5}$$

By using the symmetry between $|\xi_1|$ and ξ_2, without loss of generality, we can assume that $|\xi_1|\le|\xi_2|$. Obviously,

$$\{\xi=\xi_1+\xi_2,\tau=\tau_1+\tau_2,|\xi_2|\ge|\xi_1|\}\subset\bigcup_{k=1}^{6}\Omega_k,$$

where

$$\Omega_1=\{(\xi_1,\tau_1,\xi,\tau)\in\mathbf{R}^4,\xi=\xi_1+\xi_2,\tau=\tau_1+\tau_2,|\xi_1|\le|\xi_2|\le 18\},$$
$$\Omega_2=\{(\xi_1,\tau_1,\xi,\tau)\in\mathbf{R}^4,\xi=\xi_1+\xi_2,\tau=\tau_1+\tau_2,|\xi_2|\ge 18,|\xi_2|\ge 4|\xi_1|,|\xi_1|\le 1\},$$
$$\Omega_3=\{(\xi_1,\tau_1,\xi,\tau)\in\mathbf{R}^4,\xi=\xi_1+\xi_2,\tau=\tau_1+\tau_2,|\xi_2|\ge 18,|\xi_2|\ge 4|\xi_1|,|\xi_1|\ge 1\},$$
$$\Omega_4=\Big\{(\xi_1,\tau_1,\xi,\tau)\in\mathbf{R}^4,\xi=\sum_{k=1}^2\xi_k,\tau=\sum_{k=1}^2\tau_k,|\xi_2|\ge 18,$$
$$|\xi_1|\le|\xi_2|\le 4|\xi_1|,|\xi|\le\frac{1}{2}|\xi_1|,\xi_1\xi_2\le 0\Big\},$$
$$\Omega_5=\Big\{(\xi_1,\tau_1,\xi,\tau)\in\mathbf{R}^4,\xi=\xi_1+\xi_2,\tau=\tau_1+\tau_2,|\xi_2|\ge 18,|\xi_1|\le|\xi_2|\le 4|\xi_1|,|\xi_2|\ge\frac{|\xi|}{2}\Big\},$$
$$\Omega_6=\{(\xi_1,\tau_1,\xi,\tau)\in\mathbf{R}^4,\xi=\xi_1+\xi_2,\tau=\tau_1+\tau_2,|\xi_2|\ge 18,|\xi_1|\le|\xi_2|\le 4|\xi_1|,\xi_1\xi_2\ge 0\}.$$

We will denote the integrals in (3.5) corresponding to Ω_k ($1\le k\le 6$) by J_k ($1\le k\le 6$), respectively. Let $f=\mathscr{F}^{-1}\frac{F}{\langle\sigma\rangle^{-b'}}$, $f_k=\mathscr{F}^{-1}\frac{F_k}{\langle\sigma_k\rangle^b}$, $k=1,2$.

(1) *Subregion* Ω_1. Since $|\xi_1| \le |\xi_2| \le 18$, we have $|\xi| \le 36$, which yields

$$K_1(\xi_1,\tau_1,\xi,\tau) \le \frac{C}{\langle\sigma\rangle^{-b'}\prod_{k=1}^{2}\langle\sigma_k\rangle^{b}}.$$

Then, by the Plancherel identity, the Hölder inequality, and $\frac{4}{7}b < b$, we derive

$$\begin{aligned}
J_1 &\le C\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} K_1(\xi_1,\tau_1,\xi,\tau)F\prod_{k=1}^{2}F_k\,d\xi_1\,d\tau_1\,d\xi\,d\tau \\
&\le C\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{F\prod_{k=1}^{2}F_k}{\langle\sigma\rangle^{-b'}\prod_{k=1}^{2}\langle\sigma_k\rangle^{b}}\,d\xi_1\,d\tau_1\,d\xi\,d\tau \le C\int_{\mathbf{R}^2} f_1f_2f\,dx\,dt \\
&\le C\|f\|_{L^2_{xt}}\prod_{k=1}^{2}\|f_k\|_{L^4_{xt}} \le C\|F\|_{L^2_{\xi\tau}}\prod_{k=1}^{2}\|f_k\|_{X_{0,\frac{4}{7}b}} \le C\|F\|_{L^2_{\xi\tau}}\prod_{k=1}^{2}\|F_k\|_{L^2_{\xi\tau}}.
\end{aligned}$$

(2) *Subregion* Ω_2. In this subregion, obviously, $|\xi_2| \sim |\xi|$.
It is easily checked that

$$K_1(\xi_1,\tau_1,\xi,\tau) \le C\frac{|\xi|^2\langle\sigma\rangle^{b'}}{\prod_{k=1}^{2}\langle\sigma_k\rangle^{b}} \le C\frac{|\xi_1^6-\xi_2^6|^{1/2}}{\prod_{k=1}^{2}\langle\sigma_k\rangle^{b}}.$$

Consequently, by the Cauchy-Schwarz inequality and Lemma 2.2, we have

$$\begin{aligned}
J_2 &\le C\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} K_1(\xi_1,\tau_1,\xi,\tau)F\prod_{k=1}^{2}F_k\,d\xi_1\,d\tau_1\,d\xi\,d\tau \\
&\le C\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{F\prod_{k=1}^{2}F_k|\xi_1^6-\xi_2^6|^{\frac{1}{2}}}{\prod_{k=1}^{2}\langle\sigma_k\rangle^{b}}\,d\xi_1\,d\tau_1\,d\xi\,d\tau \\
&\le C\left\|\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{|\xi_1^6-\xi_2^6|^{\frac{1}{2}}\prod_{k=1}^{2}F_k}{\prod_{k=1}^{2}\langle\sigma_k\rangle^{b}}\,d\xi_1\,d\tau_1\right\|_{L^2_{\xi\tau}}\|F\|_{L^2_{\xi\tau}} \\
&\le C\|F\|_{L^2_{\xi\tau}}\prod_{k=1}^{2}\|F_k\|_{L^2_{\xi\tau}}.
\end{aligned}$$

(3) *Subregion* Ω_3. In this subregion, we derive $|\xi| \sim |\xi_2|$. Thus,

$$K_1(\xi_1,\tau_1,\xi,\tau) \le C\frac{|\xi|^2|\xi_1|^{-s}\langle\sigma\rangle^{b'}}{\prod_{k=1}^{2}\langle\sigma_k\rangle^{b}}.$$

(i) Case $|\sigma| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}$. By (3.1), we derive

$$K_1(\xi_1,\tau_1,\xi,\tau) \le C\frac{|\xi|^{2+6b'}|\xi_1|^{-s+b'}}{\prod_{k=1}^{2}\langle\sigma_k\rangle^{b}} \le C\frac{|\xi_1^6-\xi_2^6|^{1/2}}{\prod_{k=1}^{2}\langle\sigma_k\rangle^{b}}.$$

If $-s+b' \le 0$, then

$$K_1(\xi_1,\tau_1,\xi,\tau) \le C\frac{|\xi|^{2+6b'}|\xi_1|^{-s+b'}}{\prod_{k=1}^{2}\langle\sigma_k\rangle^{b}} \le C\frac{|\xi_1^6-\xi_2^6|^{1/2}}{\prod_{k=1}^{2}\langle\sigma_k\rangle^{b}}.$$

If $-s+b' \geq 0$, then

$$K_1(\xi_1,\tau_1,\xi,\tau) \leq C\frac{|\xi|^{2-s+7b'}}{\prod_{k=1}^2 \langle\sigma_k\rangle^b} \leq C\frac{|\xi_1^6-\xi_2^6|^{1/2}}{\prod_{k=1}^2 \langle\sigma_k\rangle^b}.$$

This case can be proved similarly to Ω_2.

(ii) Case $|\sigma_1| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}$. Since $\langle\sigma\rangle^{b+b'} \leq \langle\sigma_1\rangle^{b+b'}$, we have

$$K_1(\xi_1,\tau_1,\xi,\tau) \leq C\frac{|\xi|^2|\xi_1|^{-s}\langle\sigma_1\rangle^{b'}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b} \leq C\frac{|\xi|^{2+6b'}|\xi_1|^{-s+b'}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b}.$$

If $-s+b' \leq 0$, we have

$$K_1(\xi_1,\tau_1,\xi,\tau) \leq C\frac{|\xi_2|^{5/4}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b},$$

consequently, by using the Cauchy-Schwarz inequality and (2.5) and (2.4), we have

$$\begin{aligned}
&\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{|\xi_2|^{\frac{2n-1}{4}} F\prod_{j=1}^2 F_j}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b}\,d\xi_1\,d\tau_1\,d\xi\,d\tau \\
&\quad\leq \left\|\langle\sigma\rangle^{-b}\int_{\mathbf{R}^2}|\xi_2|^{\frac{5}{4}}\langle\sigma\rangle^{-b}\prod_{j=1}^2 F_j\,d\xi_1\,d\tau_1\right\|_{L^2_{\xi\tau}}\|F\|_{L^2_{\xi\tau}} \\
&\quad\leq C\left\|\mathscr{F}^{-1}(F_1)\left(D_x^{\frac{5}{4}}f_2\right)\right\|_{L_t^{\frac{4}{3}}L_x^2}\|F\|_{L^2_{\xi\tau}} \\
&\quad\leq C\left\|\mathscr{F}^{-1}(F_1)\right\|_{L^2_{xt}}\left\|D_x^{5/4}f_2\right\|_{L_t^4L_x^\infty}\|F\|_{L^2_{\xi\tau}} \\
&\quad\leq C\|F_1\|_{L^2_{\xi\tau}}\|F\|_{L^2_{\xi\tau}}\|f_2\|_{X_{0,b}} \\
&\quad\leq C\|F\|_{L^2_{\xi\tau}}\prod_{j=1}^2\|F_j\|_{L^2_{\xi\tau}}.
\end{aligned}$$

If $-s+b' \geq 0$, since $s \geq -\frac{7}{4}+21\epsilon$, we have

$$K_1(\xi_1,\tau_1,\xi,\tau) \leq C\frac{|\xi|^{3-s+7b'}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b} \leq C\frac{|\xi_2|^{5/4}}{\langle\sigma_2\rangle^b\langle\sigma\rangle^b}.$$

This case can be proved similarly to the above case.

(iii) Case $|\sigma_2| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}$. This case is similar to (ii) case $|\sigma_1| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}$.

(4) *Subregion* Ω_4. In this subregion, $|\xi_1| \sim |\xi_2|$, and it is easy to obtain

$$\left|\xi_1^6-\xi_2^6\right| \geq C|\xi||\xi_1|^5, \qquad \left|\xi^6-\xi_1^6\right| \geq C|\xi_1|^6, \qquad \left|\xi^6-\xi_2^6\right| \geq C|\xi_2|^6.$$

(i) Case $|\sigma| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}$. By using, $|\xi_1| \sim |\xi_2|$, when $s \geq 0$, we have

$$K_1(\xi_1,\tau_1,\xi,\tau) \leq C\frac{|\xi|^3\langle\sigma\rangle^{b'}}{\prod_{k=1}^2 \langle\sigma_k\rangle^b} \leq C\frac{|\xi_1^6-\xi_2^6|^{1/2}\langle\sigma\rangle^{b'}}{\prod_{k=1}^2 \langle\sigma_k\rangle^b}.$$

This case can be proved similarly to *Subregion* Ω_2. When $s \leq 0$, we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi|^3 |\xi_1|^{-2s}}{\langle\sigma\rangle^{-b'} \prod_{j=1}^{2} \langle\sigma_j\rangle^b}.$$

If $|\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$, since $-\frac{7}{4} + 21\epsilon \leq s \leq 0$, then

$$\begin{aligned} K_1(\xi_1, \tau_1, \xi, \tau) &\leq C \frac{|\xi|^{3+b'} |\xi_1|^{-2s+6b'}}{\prod_{k=1}^{2} \langle\sigma_k\rangle^b} \\ &\leq C \frac{|\xi|^{1/2} |\xi_1|^{5/2} |\xi|^{\frac{5}{2}+b'} |\xi_1|^{-2s+6b'-5/2}}{\prod_{k=1}^{2} \langle\sigma_k\rangle^b} \\ &\leq C \frac{|\xi_1^6 - \xi_2^6|^{1/2}}{\prod_{k=1}^{2} \langle\sigma_k\rangle^b}. \end{aligned}$$

By using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} J_4 &\leq C \int_{\mathbf{R}^2} \int_{\substack{\xi=\xi_1+\xi_2 \\ \tau=\tau_1+\tau_2}} K_1(\xi_1, \tau_1, \xi, \tau) F \prod_{k=1}^{2} F_k \, d\xi_1 \, d\tau_1 \, d\xi \, d\tau \\ &\leq C \int_{\mathbf{R}^2} \int_{\substack{\xi=\xi_1+\xi_2 \\ \tau=\tau_1+\tau_2}} \frac{|\xi_1^6 - \xi_2^6|^{1/2}}{\prod_{k=1}^{2} \langle\sigma_k\rangle^b} F \prod_{k=1}^{2} F_k \, d\xi_1 \, d\tau_1 \, d\xi \, d\tau \\ &\leq C \left\| \int_{\substack{\xi=\xi_1+\xi_2 \\ \tau=\tau_1+\tau_2}} \frac{|\xi_1^6 - \xi_2^6|^{1/2}}{\prod_{k=1}^{2} \langle\sigma_k\rangle^b} \prod_{k=1}^{2} F_k \, d\xi_1 \, d\tau_1 \right\|_{L^2_{\xi\tau}} \|F\|_{L^2_{\xi\tau}} \\ &\leq C \|F\|_{L^2_{\xi\tau}} \prod_{j=1}^{2} \|F_j\|_{L^2}. \end{aligned}$$

(ii) Case $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} = |\sigma_1|$. Since $\langle\sigma\rangle^{b+b'} \leq \langle\sigma_1\rangle^{b+b'}$, by using $-\frac{7}{4} + 21\epsilon \leq s \leq 0$, we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi|^3 |\xi_1|^{-2s}}{\langle\sigma\rangle^b \langle\sigma_2\rangle^b \langle\sigma_1\rangle^{-b'}} \leq C \frac{|\xi|^{3+b'} |\xi_1|^{-2s+6b'}}{\langle\sigma\rangle^b \langle\sigma_2\rangle^b} \leq C \frac{|\xi^6 - \xi_2^6|^{1/2}}{\langle\sigma\rangle^b \langle\sigma_2\rangle^b}.$$

This case can be proved similarly to $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} = |\sigma|$.

(iii) Case $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} = |\sigma_2|$.

This case can be proved similarly to $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} = |\sigma_1|$.

(5) *Subregion* Ω_5. In this region $|\xi| \sim |\xi_1| \sim |\xi_2|$, thus, we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi|^{3-s} \langle\sigma\rangle^{b'}}{\prod_{k=1}^{2} \langle\sigma_k\rangle^b}.$$

(i) If $|\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\}$, by using (3.1) and $s \geq -\frac{9}{4} + \frac{21}{2}\epsilon$, we have

$$K_1(\xi_1, \tau_1, \xi, \tau) \leq C \frac{|\xi|^{3-s+7b'}}{\prod_{k=1}^{2} \langle\sigma_k\rangle^b} \leq C \frac{\prod_{k=1}^{2} |\xi_k|^{\frac{5}{8}}}{\prod_{k=1}^{2} \langle\sigma_k\rangle^b}.$$

By using the Plancherel identity, the Hölder inequality, and $\frac{3}{4}b < b$ as well as (2.5), we have

$$\begin{aligned}
J_5 &\le C\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} K_1(\xi_1,\tau_1,\xi,\tau)F\prod_{k=1}^{2}F_k\,d\xi_1\,d\tau_1\,d\xi\,d\tau\\
&\le C\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{\prod_{k=1}^{2}|\xi_k|^{\frac{5}{8}}}{\prod_{k=1}^{2}\langle\sigma_k\rangle^{b}}\,d\xi_1\,d\tau_1\,d\xi\,d\tau\\
&\le C\int_{\mathbf{R}^2}\mathscr{F}^{-1}F\prod_{k=1}^{2}D_x^{\frac{5}{8}}f_k\,dx\,dt\\
&\le C\big\|\mathscr{F}^{-1}F\big\|_{L^2_{xt}}\prod_{k=1}^{2}\big\|D_x^{\frac{5}{8}}f_k\big\|_{L^4_{xt}}\\
&\le C\|F\|_{L^2_{\xi\tau}}\prod_{k=1}^{2}\|f_k\|_{X_{0,\frac{3}{4}b}} \le C\|F\|_{L^2_{\xi\tau}}\prod_{k=1}^{2}\|F_k\|_{L^2_{\xi\tau}}.
\end{aligned}$$

(ii) If $|\sigma_1| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}$, then $\langle\sigma\rangle^{b'}\langle\sigma_1\rangle^{-b} \le \langle\sigma_1\rangle^{b'}\langle\sigma\rangle^{-b}$. By using (3.1), we have

$$K_2(\xi_1,\tau_1,\xi,\tau) \le C\frac{|\xi|^{2-s}\langle\sigma_1\rangle^{b'}}{\langle\sigma_2\rangle^{b}\langle\sigma\rangle^{b}} \le C\frac{|\xi|^{2-s+7b'}}{\langle\sigma_2\rangle^{b}\langle\sigma\rangle^{b}} \le C\frac{|\xi|^{\frac{5}{8}}|\xi_2|^{\frac{5}{8}}}{\langle\sigma\rangle^{b}\langle\sigma_2\rangle^{b}}.$$

By using the Plancherel identity, the Hölder inequality, (2.5) and $\frac{3}{4}b < b$, we have

$$\begin{aligned}
J_5 &\le C\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} K_1(\xi_1,\tau_1,\xi,\tau)F\prod_{k=1}^{2}F_k\,d\xi_1\,d\tau_1\,d\xi\,d\tau\\
&\le C\int_{\mathbf{R}^2}\int_{\substack{\xi=\xi_1+\xi_2\\ \tau=\tau_1+\tau_2}} \frac{|\xi|^{\frac{5}{8}}|\xi_2|^{\frac{5}{8}}}{\langle\sigma_2\rangle^{b}\langle\sigma_2\rangle^{b}}F\prod_{k=1}^{2}F_k\,d\xi_1\,d\tau_1\,d\xi\,d\tau\\
&\le C\int_{\mathbf{R}^2}\big(\mathscr{F}^{-1}F_1\big)\left(D_x^{\frac{5}{8}}\mathscr{F}^{-1}\left(\frac{F}{\langle\sigma\rangle^{b}}\right)\right)D_x^{\frac{5}{8}}f_2\,dx\,dt\\
&\le C\big\|\mathscr{F}^{-1}F_1\big\|_{L^2_{xt}}\big\|D_x^{\frac{5}{8}}f_2\big\|_{L^4_{xt}}\left\|D_x^{\frac{5}{8}}\mathscr{F}^{-1}\left(\frac{F}{\langle\sigma\rangle^{b}}\right)\right\|_{L^4_{xt}}\\
&\le C\|F_1\|_{L^2_{\xi\tau}}\|f_2\|_{X_{0,\frac{3}{4}b}}\left\|\frac{F}{\langle\sigma\rangle^{b}}\right\|_{X_{0,\frac{3}{4}b}} \le C\|F\|_{L^2_{\xi\tau}}\prod_{k=1}^{2}\|F_k\|_{L^2_{\xi\tau}}.
\end{aligned}$$

(iii) If $|\sigma_2| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}$.

This case can be proved similarly to the case $|\sigma_1| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}$.

(6) *Subregion* Ω_6. In this region, we have $|\xi| \sim |\xi_1| \sim |\xi_2|$.

This case can be proved similarly to the *Subregion* Ω_5.

We have completed the proof of Lemma 3.1. □

4 Proof of Theorem 1.1

The system (1.1)-(1.2) is equivalent to the following integral equation:

$$u(t) = W(t)u_0 + \int_0^t W(t-\tau)\partial_x^2\big(u^2\big)\,d\tau. \tag{4.1}$$

We define

$$\Phi(u) = \Psi(t)W(t)u_0 + \Psi_\delta(t)\int_0^t W(t-\tau)\partial_x^2\bigl(u^2\bigr)\,d\tau. \tag{4.2}$$

Combining Lemmas 2.3 and 3.1 with the fixed point theorem, we easily obtain Theorem 1.1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]College of Mathematics and Information Science, Henan Normal University, Xinxiang, Henan 453007, P.R. China. [2]Henan Vocational College of Agriculture, Zhengzhou, Henan 451450, P.R. China.

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