Ashyralyev and Agirseven *Boundary Value Problems* 2014, 2014:126
http://www.boundaryvalueproblems.com/content/2014/1/126

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Well-posedness of delay parabolic equations with unbounded operators acting on delay terms

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Abstract
In the present paper, the well-posedness of the initial value problem for the delay differential equation $\frac{dv(t)}{dt} + Av(t) = B(t)v(t-\omega) + f(t)$, $t \geq 0$; $v(t) = g(t)$ $(-\omega \leq t \leq 0)$ in an arbitrary Banach space E with the unbounded linear operators A and $B(t)$ in E with dense domains $D(A) \subseteq D(B(t))$ is studied. Two main theorems on well-posedness of this problem in fractional spaces E_α are established. In practice, the coercive stability estimates in Hölder norms for the solutions of the mixed problems for delay parabolic equations are obtained.
MSC: 35G15

Keywords: delay parabolic equations; well-posedness; fractional spaces; coercive stability estimates

1 Introduction

The stability of delay ordinary differential and difference equations and delay partial differential and difference equations with bounded operators acting on delay terms has been studied extensively in a large cycle of works (see [1–13] and the references therein) and insight has developed over the last three decades. The theory of stability and coercive stability of delay partial differential and difference equations with unbounded operators acting on delay terms has received less attention than delay ordinary differential and difference equations (see [14–19]). It is well known that various initial-boundary value problems for linear evolutionary delay partial differential equations can be reduced to an initial value problem of the form

$$\begin{cases} \frac{dv(t)}{dt} + Av(t) = B(t)v(t-\omega) + f(t), \quad t \geq 0, \\ v(t) = g(t) \quad (-\omega \leq t \leq 0) \end{cases} \tag{1}$$

in an arbitrary Banach space E with the unbounded linear operators A and $B(t)$ in E with dense domains $D(A) \subseteq D(B(t))$. Let A be a strongly positive operator, *i.e.* $-A$ is the generator of the analytic semigroup $\exp\{-tA\}$ $(t \geq 0)$ of the linear bounded operators with exponentially decreasing norm when $t \to \infty$. That means the following estimates hold:

$$\left\|\exp\{-tA\}\right\|_{E\mapsto E} \leq Me^{-\delta t}, \qquad \left\|tA\exp\{-tA\}\right\|_{E\mapsto E} \leq M, \quad t > 0 \tag{2}$$

for some $M > 1$, $\delta > 0$. Let $B(t)$ be closed operators.

A function $v(t)$ is called a solution of the problem (1) if the following conditions are satisfied:

(i) $v(t)$ is continuously differentiable on the interval $[-\omega,\infty)$. The derivative at the endpoint $t=-\omega$ is understood as the appropriate unilateral derivative.

(ii) The element $v(t)$ belongs to $D(A)$ for all $t\in[-\omega,\infty)$, and the function $Av(t)$ is continuous on the interval $[-\omega,\infty)$.

(iii) $v(t)$ satisfies the equation and the initial condition (1).

A solution $v(t)$ of the initial value problem (1) is said to be coercive stable (well-posed) if

$$\|Av(t)\|_E \le \max_{-\omega\le t\le 0}\|Ag(t)\|_E + \sup_{0\le s\le t}\|f(t)\|_E \tag{3}$$

for every t, $-\omega\le t<\infty$. We are interested in studying the coercive stability of solutions of the initial value problem under the assumption that

$$\|B(t)A^{-1}\|_{E\mapsto E}\le 1 \tag{4}$$

holds for every $t\ge 0$. We have not been able to obtain the estimate (3) in the arbitrary Banach space E. Nevertheless, we can establish the analog of estimates (3) where the space E is replaced by the fractional spaces E_α $(0<\alpha<1)$ under an assumption stronger than (4). The coercive stability estimates in Hölder norms for the solutions of the mixed problem of the delay differential equations of the parabolic type are obtained.

The present paper is organized as follows. Section 1 is introduction. In Section 2, two main theorems on well-posedness of the initial value problem (1) are established. In Section 3, the coercive stability estimates in Hölder norms for the solutions of the initial-boundary value problem for delay parabolic equations are obtained. Finally, Section 4 is our conclusion.

2 Theorems on well-posedness

The strongly positive operator A defines the fractional spaces $E_\alpha=E_\alpha(E,A)$ $(0<\alpha<1)$ consisting of all $u\in E$ for which the following norms are finite:

$$\|u\|_{E_\alpha}=\sup_{\lambda>0}\|\lambda^{1-\alpha}A\exp\{-\lambda A\}u\|_E.$$

We consider the initial value problem (1) for delay differential equations of parabolic type in the space $C(E_\alpha)$ of all continuous functions $v(t)$ defined on the segment $[0,\infty)$ with values in a Banach space E_α. First, we consider the problem (1) when A^{-1} and $B(t)$ commute, *i.e.*

$$A^{-1}B(t)u=B(t)A^{-1}u,\quad u\in D(A). \tag{5}$$

Theorem 2.1 *Assume that the condition*

$$\|B(t)A^{-1}\|_{E\mapsto E}\le\frac{(1-\alpha)}{M2^{2-\alpha}} \tag{6}$$

holds for every $t \geq 0$, *where* M *is the constant from* (2). *Then for every* t, $(n-1)\omega \leq t \leq n\omega$, $n = 1, \ldots,$ *we have the following coercive stability estimate*:

$$\begin{aligned} &\left\|v'(t)\right\|_{E_\alpha} + \left\|Av(t)\right\|_{E_\alpha} \\ &\quad \leq M(\alpha)\left[\max_{-\omega \leq t \leq 0}\left\|Ag(t)\right\|_{E_\alpha} + \sum_{k=1}^{n-1}\max_{-(k-1)\omega \leq s \leq k\omega}\left\|f(s)\right\|_{E_\alpha} + \max_{(n-1)\omega \leq s \leq t}\left\|f(s)\right\|_{E_\alpha}\right], \end{aligned} \tag{7}$$

where $M(\alpha)$ *does not depend on* $g(t)$ *and* $f(t)$. *Here, we put* $\sum_{k=1}^{m} a_k = 0$ *when* $m < 1$.

Proof It is clear that

$$v(t) = u(t) + w(t), \tag{8}$$

where $u(t)$ is the solution of the problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = B(t)u(t-\omega), \quad t \geq 0, \\ u(t) = g(t) \quad (-\omega \leq t \leq 0), \end{cases} \tag{9}$$

and $w(t)$ is the solution of the problem

$$\begin{cases} \frac{dw(t)}{dt} + Aw(t) = B(t)w(t-\omega) + f(t), \quad t \geq 0, \\ w(t) = 0 \quad (-\omega \leq t \leq 0). \end{cases} \tag{10}$$

First, we consider the problem (9). Using the formula

$$u(t) = \exp\{-tA\}g(0) + \int_0^t \exp\bigl\{-(t-s)A\bigr\}B(s)g(s-\omega)\,ds, \tag{11}$$

the semigroup property, condition (5), and the estimates (2), (6), we obtain

$$\begin{aligned} &\lambda^{1-\alpha}\left\|A\exp\{-\lambda A\}Av(t)\right\|_E \\ &\quad \leq \lambda^{1-\alpha}\left\|A\exp\bigl\{-(\lambda+t)A\bigr\}Ag(0)\right\|_E \\ &\qquad + \lambda^{1-\alpha}\int_0^t \left\|A\exp\left\{-\frac{\lambda+t-s}{2}A\right\}\right\|_{E\to E}\left\|B(s)A^{-1}\right\|_{E\to E} \\ &\qquad \times \left\|A\exp\left\{-\frac{\lambda+t-s}{2}A\right\}Ag(s-\omega)\right\|_E\,ds \\ &\quad \leq \frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}}\left\|Ag(0)\right\|_{E_\alpha} + \frac{1-\alpha}{M2^{2-\alpha}}\int_0^t \frac{M\lambda^{1-\alpha}2^{2-\alpha}}{(\lambda+t-s)^{2-\alpha}}\,ds \max_{0 \leq s \leq \omega}\left\|Ag(s-\omega)\right\|_{E_\alpha} \\ &\quad \leq \max_{-\omega \leq t \leq 0}\left\|Ag(t)\right\|_{E_\alpha} \end{aligned}$$

for every t, $0 \leq t \leq \omega$ and λ, $\lambda > 0$. This shows that

$$\left\|Au(t)\right\|_{E_\alpha} \leq \max_{-\omega \leq t \leq 0}\left\|Ag(t)\right\|_{E_\alpha} \tag{12}$$

for every t, $0 \leq t \leq \omega$. Applying mathematical induction, one can easily show that it is true for every t. Namely, assume that the inequality

$$\|Au(t)\|_{E_\alpha} \leq \max_{-\omega\leq s\leq 0}\|Ag(s)\|_{E_\alpha}$$

is true for t, $(n-1)\omega \leq t \leq n\omega$, $n = 1, 2, 3, \ldots$ for some n. Letting $t = s + n\omega$, we have

$$\frac{du(s+n\omega)}{ds} + Au(s+n\omega) = B(s+n\omega)u\bigl(s+(n-1)\omega\bigr), \quad 0 \leq s \leq \omega.$$

Using the estimate (12), we obtain

$$\max_{n\omega\leq s\leq (n+1)\omega}\|Au(s-\omega)\|_{E_\alpha} \leq \max_{-(n-1)\omega\leq t\leq n\omega}\|Au(t)\|_{E_\alpha}$$

for every t, $n\omega \leq t \leq (n+1)\omega$, $n = 1, 2, 3, \ldots$ and λ, $\lambda > 0$. This shows that

$$\|Au(t)\|_{E_\alpha} \leq \max_{-\omega\leq t\leq 0}\|Ag(t)\|_{E_\alpha}$$

for every t, $n\omega \leq t \leq (n+1)\omega$, $n = 1, 2, 3, \ldots$. Therefore

$$\|Au(t)\|_{E_\alpha} \leq \max_{-\omega\leq t\leq 0}\|Ag(t)\|_{E_\alpha} \tag{13}$$

is true for every $t \geq 0$. Applying (9), the triangle inequality, condition (5), and the estimates (6) and (13), we get

$$\begin{aligned}\|u'(t)\|_{E_\alpha} &\leq \|Au(t)\|_{E_\alpha} + \|B(t)A^{-1}\|_{E\mapsto E}\|Au(t-\omega)\|_{E_\alpha}\\ &\leq \left(1 + \frac{1-\alpha}{M2^{2-\alpha}}\right)\max_{-\omega\leq t\leq 0}\|Ag(t)\|_{E_\alpha}\end{aligned} \tag{14}$$

for every $t \geq 0$. Second, we consider the problem (10). To prove the theorem it suffices to establish the following stability inequality:

$$\|Aw(t)\|_{E_\alpha} \leq \frac{M2^{2-\alpha}}{1-\alpha}\left[\sum_{k=1}^{n-1}\max_{-(k-1)\omega\leq s\leq k\omega}\|f(s)\|_{E_\alpha} + \max_{(n-1)\omega\leq s\leq t}\|f(s)\|_{E_\alpha}\right] \tag{15}$$

for the solution of the problem (10) for every t, $(n-1)\omega \leq t \leq n\omega$, $n = 1, \ldots$. Using the formula

$$w(t) = \int_0^t \exp\bigl\{-(t-s)A\bigr\}f(s)\,ds, \tag{16}$$

the semigroup property, and the definition of the spaces E_a, we obtain

$$\begin{aligned}&\lambda^{1-\alpha}\|A\exp\{-\lambda A\}Aw(t)\|_E\\ &\quad\leq \lambda^{1-\alpha}\int_0^t \|A^2\exp\bigl\{-(\lambda+t-s)A\bigr\}f(s)\|_E\end{aligned}$$

$$\begin{aligned}
&\leq M2^{2-\alpha}\int_0^t \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{2-\alpha}}\left\|f(s)\right\|_{E_\alpha} ds \\
&\leq M2^{2-\alpha}\left(\int_0^t \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{2-\alpha}} ds\right)\max_{0\leq s\leq t}\left\|f(s)\right\|_{E_\alpha} \\
&\leq \frac{M2^{2-\alpha}}{1-\alpha}\max_{0\leq s\leq t}\left\|f(s)\right\|_{E_\alpha}
\end{aligned}$$

for every t, $0 \leq t \leq \omega$ and λ, $\lambda > 0$. This shows that

$$\left\|Aw(t)\right\|_{E_\alpha} \leq \frac{M2^{2-\alpha}}{1-\alpha}\max_{0\leq s\leq t}\left\|f(s)\right\|_{E_\alpha} \tag{17}$$

for every t, $0 \leq t \leq \omega$. Applying mathematical induction, one can easily show that it is true for every t. Namely, assume that the inequality (15) is true for t, $(n-1)\omega \leq t \leq n\omega$, $n = 1, \dots,$ for some n. Using the formula

$$\begin{aligned}
w(t) = {} & \exp\{-(t-n\omega)A\}w(n\omega) + \int_{n\omega}^t \exp\{-(t-s)A\}B(s)w(s-\omega)\,ds \\
&+ \int_{n\omega}^t \exp\{-(t-s)A\}f(s)\,ds,
\end{aligned} \tag{18}$$

the semigroup property, the definition of the spaces E_a, the estimate (2), and condition (6), we obtain

$$\begin{aligned}
&\lambda^{1-\alpha}\left\|A\exp\{-\lambda A\}Aw(t)\right\|_E \\
&\quad\leq \lambda^{1-\alpha}\left\|A\exp\{-(\lambda+t-n\omega)A\}Aw(n\omega)\right\|_E \\
&\qquad + \lambda^{1-\alpha}\int_{n\omega}^t \left\|A\exp\left\{-\frac{\lambda+t-s}{2}A\right\}\right\|_{E\to E}\left\|B(s)A^{-1}\right\|_{E\to E} \\
&\qquad \times \left\|A\exp\left\{-\frac{\lambda+t-s}{2}A\right\}Aw(s-\omega)\right\|_E ds \\
&\qquad + \lambda^{1-\alpha}\int_{n\omega}^t \left\|A^2\exp\{-(\lambda+t-s)A\}f(s)\right\|_E ds \\
&\quad\leq \frac{\lambda^{1-\alpha}}{(\lambda+t-n\omega)^{1-\alpha}}\left\|Aw(n\omega)\right\|_{E_\alpha} + \lambda^{1-\alpha}(1-\alpha)\int_{n\omega}^t \frac{1}{(\lambda+t-s)^{2-\alpha}}\left\|Aw(s-\omega)\right\|_{E_\alpha} ds \\
&\qquad + M2^{2-\alpha}\int_{n\omega}^t \frac{\lambda^{1-\alpha}}{(\lambda+t-s)^{2-\alpha}}\left\|f(s)\right\|_{E_\alpha} ds \\
&\quad\leq \left(\frac{\lambda^{1-\alpha}}{(\lambda+t-n\omega)^{1-\alpha}} + \lambda^{1-\alpha}(1-\alpha)\int_{n\omega}^t \frac{1}{(\lambda+t-s)^{2-\alpha}} ds\right)\max_{(n-1)\omega\leq t\leq n\omega}\left\|Aw(t)\right\|_{E_\alpha} \\
&\quad\leq \frac{M2^{2-\alpha}}{1-\alpha}\max_{(n-1)\omega\leq t\leq n\omega}\left[\sum_{k=1}^{n-1}\max_{-(k-1)\omega\leq s\leq k\omega}\left\|f(s)\right\|_{E_\alpha} + \max_{(n-1)\omega\leq s\leq t}\left\|f(s)\right\|_{E_\alpha}\right] \\
&\qquad + \frac{M2^{2-\alpha}}{1-\alpha}\sup_{n\omega\leq s\leq t}\left\|f(s)\right\|_{E_\alpha} = \frac{M2^{2-\alpha}}{1-\alpha}\left[\sum_{k=1}^{n}\max_{-(k-1)\omega\leq s\leq k\omega}\left\|f(s)\right\|_{E_\alpha} + \max_{n\omega\leq s\leq t}\left\|f(s)\right\|_{E_\alpha}\right]
\end{aligned}$$

for every t, $n\omega \leq t \leq (n+1)\omega$, $n = 1, 2, 3, \ldots$ and λ, $\lambda > 0$. This shows that

$$\left\|Aw(t)\right\|_{E_\alpha} \leq \frac{M2^{2-\alpha}}{1-\alpha}\left[\sum_{k=1}^{n} \max_{-(k-1)\omega\leq s\leq k\omega}\left\|f(s)\right\|_{E_\alpha} + \max_{n\omega\leq s\leq t}\left\|f(s)\right\|_{E_\alpha}\right] \tag{19}$$

for every t, $n\omega \leq t \leq (n+1)\omega$, $n = 1, 2, 3, \ldots$. Applying equation (9), the triangle inequality, and condition (5) and estimates (6) and (19), we get

$$\begin{aligned}\left\|w'(t)\right\|_{E_\alpha} &\leq \left\|Aw(t)\right\|_{E_\alpha} + \left\|B(t)A^{-1}\right\|_{E\mapsto E}\left\|Aw(t-\omega)\right\|_{E_\alpha} + \left\|f(t)\right\|_{E_\alpha} \\ &\leq \left(2 + \frac{M2^{2-\alpha}}{1-\alpha}\right)\left[\sum_{k=1}^{n} \max_{-(k-1)\omega\leq s\leq k\omega}\left\|f(s)\right\|_{E_\alpha} + \max_{n\omega\leq s\leq t}\left\|f(s)\right\|_{E_\alpha}\right]\end{aligned}$$

for every $t \geq 0$. This result completes the proof of Theorem 2.1. □

Now, we consider the problem (1) when

$$A^{-1}B(t)x \neq B(t)A^{-1}x, \quad x \in D(A)$$

for some $t \geq 0$. Note that A is a strongly positive operator in a Banach spaces E iff its spectrum $\sigma(A)$ lies in the interior of the sector of angle φ, $0 < 2\varphi < \pi$, symmetric with respect to the real axis, and if on the edges of this sector, $S_1 = [z = \rho\exp(i\varphi) : 0 \leq \rho < \infty]$ and $S_2 = [z = \rho\exp(-i\varphi) : 0 \leq \rho < \infty]$ and outside it the resolvent $(z-A)^{-1}$ is the subject to the bound

$$\left\|(z-A)^{-1}\right\|_{E\to E} \leq \frac{M_1}{1+|z|} \tag{20}$$

for some $M_1 > 0$. First of all let us give lemmas from the paper [18] that will be needed in the sequel.

Lemma 2.1 *For any z on the edges of the sector,*

$$S_1 = \left[z = \rho\exp(i\varphi) : 0 \leq \rho < \infty\right]$$

and

$$S_2 = \left[z = \rho\exp(-i\varphi) : 0 \leq \rho < \infty\right]$$

and outside it the estimate

$$\left\|A(z-A)^{-1}x\right\|_E \leq \frac{M_1^\alpha M^\alpha(1+M_1)^{1-\alpha}2^{(2-\alpha)\alpha}}{\alpha(1-\alpha)(1+|z|)^\alpha}\|x\|_{E_\alpha}$$

holds for any $x \in E_\alpha$. Here and in the future M and M_1 are the same constants of the estimates (2) *and* (20).

Lemma 2.2 *Let for all $s \geq 0$ the operator $B(s)A^{-1} - A^{-1}B(s)$ with domain which coincide with $D(A)$ admit a closure $Q = \overline{B(s)A^{-1} - A^{-1}B(s)}$ bounded in E. Then for all $\tau > 0$ the following estimate holds*:

$$\begin{aligned}
&\bigl\|A^{-1}\bigl[A\exp\{-\tau A\}B(s) - B(s)A\exp\{-\tau A\}\bigr]x\bigr\|_E \\
&\quad \leq \frac{e(\alpha+1)M^\alpha M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha}2^{(2-\alpha)\alpha}\|Q\|_{E\mapsto E}\|x\|_{E_\alpha}}{\tau^{1-\alpha}\pi\alpha^2(1-\alpha)}.
\end{aligned}$$

Here $Q = \overline{A^{-1}(AB(s) - B(s)A)A^{-1}}$.

Suppose that

$$\begin{aligned}
&\overline{\bigl\|A^{-1}\bigl(AB(t) - B(t)A\bigr)A^{-1}\bigr\|}_{E\mapsto E} \\
&\quad \leq \frac{\pi(1-\alpha)^2\alpha^2\varepsilon}{eM^{1+\alpha}M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha}2^{2+\alpha-\alpha^2}(1+\alpha)}
\end{aligned} \tag{21}$$

holds for every $t \geq 0$. Here and in the future ε is some constant, $0 \leq \varepsilon \leq 1$.

The application of Lemmas 2.1 and 2.2 enables us to establish the following fact.

Theorem 2.2 *Assume that the condition*

$$\overline{\bigl\|A^{-1}B(t)\bigr\|}_{E\mapsto E} \leq \frac{(1-\alpha)(1-\varepsilon)}{M2^{2-\alpha}} \tag{22}$$

holds for every $t \geq 0$. Then for every $t \geq 0$ the coercive stability estimate (7) *holds.*

Proof In a similar manner as in the proof of Theorem 2.1 we establish estimates for the solution of the problems (9) and (10), separately. First, we consider the problem (9). Let $0 \leq t \leq \omega$ and λ, $\lambda > 0$. Then using (11), we have

$$\begin{aligned}
&\lambda^{1-\alpha}A\exp\{-\lambda A\}Au(t) \\
&\quad = \lambda^{1-\alpha}A\exp\bigl\{-(\lambda+t)A\bigr\}Ag(0) \\
&\qquad + \lambda^{1-\alpha}\int_0^t \exp\biggl\{-\frac{\lambda+t-s}{2}A\biggr\}B(s)A\exp\biggl\{-\frac{\lambda+t-s}{2}A\biggr\}Ag(s-\omega)\,ds \\
&\qquad + \lambda^{1-\alpha}\int_0^t \exp\biggl\{-\frac{\lambda+t-s}{2}A\biggr\}\biggl[A\exp\biggl\{-\frac{\lambda+t-s}{2}A\biggr\}B(s) - B(s)A \\
&\qquad \times\exp\biggl\{-\frac{\lambda+t-s}{2}A\biggr\}\biggr]Ag(s-\omega)\,ds \\
&\quad = I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
&I_1 = \lambda^{1-\alpha}A\exp\bigl\{-(\lambda+t)A\bigr\}Ag(0), \\
&I_2 = \lambda^{1-\alpha}\int_0^t \exp\biggl\{-\frac{\lambda+t-s}{2}A\biggr\}B(s)A\exp\biggl\{-\frac{\lambda+t-s}{2}A\biggr\}Ag(s-\omega)\,ds,
\end{aligned}$$

$$I_3 = \lambda^{1-\alpha} \int_0^t \exp\left\{-\frac{\lambda+t-s}{2}A\right\}\left[A\exp\left\{-\frac{\lambda+t-s}{2}A\right\}B(s) - B(s)A \times \exp\left\{-\frac{\lambda+t-s}{2}A\right\}\right]Ag(s-\omega)\,ds.$$

Using the estimates (2), (20), and condition (22), we obtain

$$\begin{aligned}
\|I_1\|_E &= \lambda^{1-\alpha}\left\|A\exp\{-(\lambda+t)A\}Ag(0)\right\|_E \\
&\leq \frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}}\left\|Ag(0)\right\|_{E_\alpha} \leq \frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}}\max_{-\omega\leq t\leq 0}\left\|Ag(t)\right\|_{E_\alpha}, \\
\|I_2\|_E &\leq \lambda^{1-\alpha}\int_0^t \left\|A\exp\left\{-\frac{\lambda+t-s}{2}A\right\}\right\|_{E\mapsto E}\|\overline{A^{-1}B(s)}\|_{E\mapsto E} \\
&\quad\times\left\|A\exp\left\{-\frac{\lambda+t-s}{2}A\right\}Ag(s-\omega)\right\|_E\,ds \\
&\leq \max_{0\leq t\leq\omega}\|\overline{A^{-1}B(t)}\|_{E\mapsto E}\int_0^t \frac{M\lambda^{1-\alpha}2^{2-\alpha}}{(\lambda+t-s)^{2-\alpha}}\,ds\max_{0\leq s\leq\omega}\left\|Ag(s-\omega)\right\|_{E_\alpha} \\
&\leq \max_{-\omega\leq t\leq 0}\left\|Ag(t)\right\|_{E_\alpha}\left(1-\frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}}\right)(1-\varepsilon)
\end{aligned}$$

for every t, $0 \leq t \leq \omega$ and λ, $\lambda > 0$. Now let us estimate I_3. By Lemma 2.1 and using the estimate (21), we obtain

$$\begin{aligned}
\|I_3\|_E &\leq \lambda^{1-\alpha}\int_0^t \left\|A\exp\left\{-\frac{\lambda+t-s}{2}A\right\}\right\|_{E\mapsto E} \\
&\quad\times\left\|A^{-1}\left[A\exp\left\{-\frac{\lambda+t-s}{2}A\right\}B(s) - B(s)A\exp\left\{-\frac{\lambda+t-s}{2}A\right\}\right]Ag(s-\omega)\right\|_E\,ds \\
&\leq \lambda^{1-\alpha}e(1+\alpha)M^{1+\alpha}M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha}2^{(2-\alpha)\alpha} \\
&\quad\times\int_0^t \frac{\|\overline{A^{-1}(AB(s)-B(s)A)A^{-1}}\|_{E\mapsto E}2^{2-\alpha}\|Ag(s-\omega)\|_{E_\alpha}}{(\lambda+t-s)^{2-\alpha}\pi\alpha^2(1-\alpha)}\,ds \\
&\leq \max_{0\leq s\leq\omega}\overline{\left\|A^{-1}\big(AB(s)-B(s)A\big)A^{-1}\right\|}_{E\mapsto E} \\
&\quad\times\int_0^t \frac{\lambda^{1-\alpha}e(1+\alpha)M^{1+\alpha}M_1^{1+\alpha}(1+2M_1)(1+M_1)^{1-\alpha}2^{(2-\alpha)\alpha}2^{2-\alpha}}{(\lambda+t-s)^{2-\alpha}\pi\alpha^2(1-\alpha)}\,ds \\
&\quad\times\max_{-\omega\leq t\leq 0}\left\|Ag(t)\right\|_{E_\alpha} \leq \max_{-\omega\leq t\leq 0}\left\|Ag(t)\right\|_{E_\alpha}\left(1-\frac{\lambda^{1-\alpha}}{(\lambda+t)^{1-\alpha}}\right)\varepsilon
\end{aligned}$$

for every t, $0 \leq t \leq \omega$ and λ, $\lambda > 0$. Using the triangle inequality, we obtain

$$\lambda^{1-\alpha}\left\|A\exp\{-\lambda A\}Au(t)\right\|_E \leq \max_{-\omega\leq t\leq 0}\left\|Ag(t)\right\|_{E_\alpha}$$

for every t, $0 \leq t \leq \omega$ and λ, $\lambda > 0$. This shows that

$$\left\|Au(t)\right\|_{E_\alpha} \leq \max_{-\omega\leq t\leq 0}\left\|Ag(t)\right\|_{E_\alpha}$$

for every t, $0 \leq t \leq \omega$. In a similar manner as with Theorem 2.1 applying mathematical induction, one can easily show that it is true for every t. Therefore, to prove the theorem it suffices to establish the coercive stability inequality (15) for the solution of the problem (10). Now, we consider the problem (10). Exactly in the same manner, using (16), the semigroup property, and the definition of the spaces E_a, we obtain (15) for every t, $0 \leq t \leq \omega$. Applying mathematical induction, one can easily show that it is true for every t. Namely, assume that the inequality (15) is true for t, $(n-1)\omega \leq t \leq n\omega$, $n = 1, \dots$ for some n. Using (18) and the semigroup property, we write

$$
\begin{aligned}
&\lambda^{1-\alpha} A \exp\{-\lambda A\} A w(t) \\
&\quad = \lambda^{1-\alpha} A \exp\bigl\{-(\lambda + t - n\omega)A\bigr\} A w(n\omega) \\
&\qquad + \lambda^{1-\alpha} \int_{n\omega}^{t} \exp\biggl\{-\frac{\lambda + t - s}{2} A\biggr\} B(s) A \exp\biggl\{-\frac{\lambda + t - s}{2} A\biggr\} A w(s-\omega)\, ds \\
&\qquad + \lambda^{1-\alpha} \int_{n\omega}^{t} \exp\biggl\{-\frac{\lambda + t - s}{2} A\biggr\} \biggl[A \exp\biggl\{-\frac{\lambda + t - s}{2} A\biggr\} B(s) - B(s) A \\
&\qquad \times \exp\biggl\{-\frac{\lambda + t - s}{2} A\biggr\}\biggr] A w(s-\omega)\, ds + \lambda^{1-\alpha} \int_{n\omega}^{t} A^2 \exp\bigl\{-(\lambda + t - s)A\bigr\} f(s)\, ds \\
&\quad = I_1 + I_2 + I_3 + I_4,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \lambda^{1-\alpha} A \exp\bigl\{-(\lambda + t - n\omega)A\bigr\} A w(n\omega), \\
I_2 &= \lambda^{1-\alpha} \int_{n\omega}^{t} \exp\biggl\{-\frac{\lambda + t - s}{2} A\biggr\} B(s) A \exp\biggl\{-\frac{\lambda + t - s}{2} A\biggr\} A w(s-\omega)\, ds, \\
I_3 &= \lambda^{1-\alpha} \int_{n\omega}^{t} \exp\biggl\{-\frac{\lambda + t - s}{2} A\biggr\} \biggl[A \exp\biggl\{-\frac{\lambda + t - s}{2} A\biggr\} B(s) - B(s) A \\
&\quad \times \exp\biggl\{-\frac{\lambda + t - s}{2} A\biggr\}\biggr] A w(s-\omega)\, ds, \\
I_4 &= \lambda^{1-\alpha} \int_{n\omega}^{t} A^2 \exp\bigl\{-(\lambda + t - s)A\bigr\} f(s)\, ds.
\end{aligned}
$$

Using the estimate (2) and condition (22), we obtain

$$
\begin{aligned}
\|I_1\|_E &= \lambda^{1-\alpha} \bigl\|A \exp\bigl\{-(\lambda + t - n\omega)A\bigr\} A w(n\omega)\bigr\|_E \leq \frac{\lambda^{1-\alpha}}{(\lambda + t - n\omega)^{1-\alpha}} \bigl\|A w(n\omega)\bigr\|_{E_\alpha}, \\
\|I_2\|_E &\leq \lambda^{1-\alpha} \int_{n\omega}^{t} \biggl\|A \exp\biggl\{-\frac{\lambda + t - s}{2} A\biggr\}\biggr\|_{E \mapsto E} \bigl\|\overline{A^{-1}B(s)}\bigr\|_{E \mapsto E} \\
&\quad \times \biggl\|A \exp\biggl\{-\frac{\lambda + t - s}{2} A\biggr\} A w(s-\omega)\biggr\|_E\, ds \\
&\leq \max_{n\omega \leq t \leq (n+1)\omega} \bigl\|\overline{A^{-1}B(t)}\bigr\|_{E \mapsto E} \int_{n\omega}^{t} \frac{M \lambda^{1-\alpha} 2^{2-\alpha}}{(\lambda + t - s)^{2-\alpha}}\, ds \max_{n\omega \leq s \leq (n+1)\omega} \bigl\|A w(s-\omega)\bigr\|_{E_\alpha} \\
&\leq \biggl(1 - \frac{\lambda^{1-\alpha}}{(\lambda + t - n\omega)^{1-\alpha}}\biggr)(1-\varepsilon) \max_{(n-1)\omega \leq s \leq n\omega} \bigl\|A w(s)\bigr\|_{E_\alpha}, \\
\|I_4\|_E &\leq M 2^{2-\alpha} \int_{n\omega}^{t} \frac{\lambda^{1-\alpha}}{(\lambda + t - s)^{2-\alpha}} \bigl\|f(s)\bigr\|_{E_\alpha}\, ds \leq \frac{M 2^{2-\alpha}}{1-\alpha} \sup_{n\omega \leq s \leq t} \bigl\|f(s)\bigr\|_{E_\alpha}
\end{aligned}
$$

for every t, $n\omega \le t \le (n+1)\omega$, $n = 1,2,3,\ldots$, and λ, $\lambda > 0$. Now let us estimate I_3. By Lemma 2.2 and using the estimate (2) and condition (21), we obtain

$$\begin{aligned}
\|I_3\|_E &\le \lambda^{1-\alpha} \int_{n\omega}^{t} \left\| A \exp\left\{-\frac{\lambda + t - s}{2} A\right\} \right\|_{E\mapsto E} \\
&\quad \times \left\| A^{-1}\left[A \exp\left\{-\frac{\lambda + t - s}{2} A\right\} B(s) - B(s) A \exp\left\{-\frac{\lambda + t - s}{2} A\right\} \right] A w(s-\omega) \right\|_E ds \\
&\le \lambda^{1-\alpha} e(1+\alpha) M^{1+\alpha} M_1^{1+\alpha} (1+2M_1)(1+M_1)^{1-\alpha} 2^{(2-\alpha)\alpha} \\
&\quad \times \int_{n\omega}^{t} \frac{\overline{\|A^{-1}(AB(s) - B(s)A)A^{-1}\|}_{E\mapsto E} 2^{2-\alpha} \|Aw(s-\omega)\|_{E_\alpha}}{(\lambda + t - s)^{2-\alpha} \pi \alpha^2 (1-\alpha)} ds \\
&\le \max_{0\le s\le \omega} \overline{\left\| A^{-1}\left(AB(s) - B(s)A\right)A^{-1} \right\|}_{E\mapsto E} \\
&\quad \times \int_{n\omega}^{t} \frac{\lambda^{1-\alpha} e(1+\alpha) M^{1+\alpha} M_1^{1+\alpha} (1+2M_1)(1+M_1)^{1-\alpha} 2^{(2-\alpha)\alpha} 2^{2-\alpha}}{(\lambda + t - s)^{2-\alpha} \pi \alpha^2 (1-\alpha)} ds \\
&\quad \times \max_{(n-1)\omega\le s\le n\omega} \left\| Aw(s) \right\|_{E_\alpha} \\
&\le \left(1 - \frac{\lambda^{1-\alpha}}{(\lambda + t - n\omega)^{1-\alpha}} \right) \varepsilon \max_{(n-1)\omega\le s\le n\omega} \left\| Aw(s) \right\|_{E_\alpha}
\end{aligned}$$

for every t, $n\omega \le t \le (n+1)\omega$, $n = 1,2,3,\ldots$ and λ, $\lambda > 0$. Using the triangle inequality and estimates for all $\|I_k\|_E$, $k = 1,2,3,4$, we obtain

$$\lambda^{1-\alpha} \left\| A \exp\{-\lambda A\} A w(t) \right\|_E \le \frac{M 2^{2-\alpha}}{1-\alpha} \left[\sum_{k=1}^{n} \max_{-(k-1)\omega\le s\le k\omega} \left\| f(s) \right\|_{E_\alpha} + \max_{n\omega\le s\le t} \left\| f(s) \right\|_{E_\alpha} \right]$$

for every t, $n\omega \le t \le (n+1)\omega$, $n = 1,2,3,\ldots$ and λ, $\lambda > 0$. This shows that

$$\left\| Aw(t) \right\|_{E_\alpha} \le \frac{M 2^{2-\alpha}}{1-\alpha} \left[\sum_{k=1}^{n} \max_{-(k-1)\omega\le s\le k\omega} \left\| f(s) \right\|_{E_\alpha} + \max_{n\omega\le s\le t} \left\| f(s) \right\|_{E_\alpha} \right]$$

for every t, $n\omega \le t \le (n+1)\omega$, $n = 1,2,3,\ldots$. This result completes the proof of Theorem 2.2. □

Note that these abstract results are applicable to the study of stability of various delay parabolic equations with local and nonlocal boundary conditions with respect to the space variables. However, it is important to study the structure of E_α for space operators in Banach spaces. The structure of E_α for some space differential and difference operators in Banach spaces has been investigated (see [20–30]). In Section 3, applications of Theorem 2.1 to the study of the coercive stability of initial-boundary value problem for delay parabolic equations are given.

3 Applications

First, we consider the initial-boundary value problem for one dimensional delay differential equations of parabolic type

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) \\ \quad = b(t)(-a(x)\frac{\partial^2 u(t-\omega,x)}{\partial x^2} + \delta u(t-\omega,x)) + f(t,x), \quad 0 < t < \infty, x \in (0,l), \\ u(t,x) = g(t,x), \quad -\omega \leq t \leq 0, x \in [0,l], \\ u(t,0) = u(t,l) = 0, \quad -\omega \leq t < \infty, \end{cases} \tag{23}$$

where $a(x)$, $b(t)$, $g(t,x)$, $f(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is a sufficiently large number. We will assume that $a(x) \geq a > 0$. The problem (23) has a unique smooth solution. This allows us to reduce the initial-boundary value problem (23) to the initial value problem (1) in Banach space $E = C[0,l]$ with a differential operator A^x defined by the formula

$$A^x u = -a(x)\frac{d^2 u}{dx^2} + \delta u \tag{24}$$

with domain $D(A^x) = \{u \in C^{(2)}[0,1] : u(0) = u(1) = 0\}$. Let us give a number of corollaries of the abstract Theorem 2.1.

Theorem 3.1 *Assume that*

$$\sup_{0 \leq t < \infty} |b(t)| \leq \frac{1-\alpha}{M2^{2-\alpha}}. \tag{25}$$

Then for all $t \geq 0$ the solutions of the initial-boundary value problem (23) *satisfy the following coercive stability estimates*:

$$\begin{aligned} &\|u'(t,\cdot)\|_{C^{2\alpha}[0,l]} + \|u(t,\cdot)\|_{C^{2+2\alpha}[0,l]} \\ &\quad \leq M(\alpha)\Bigg[\max_{-\omega \leq t \leq 0} \|g(t,\cdot)\|_{C^{2+2\alpha}[0,l]} \\ &\qquad + \sum_{k=1}^{n} \max_{-(k-1)\omega \leq s \leq k\omega} \|f(s,\cdot)\|_{C^{2\alpha}[0,l]} + \max_{n\omega \leq s \leq t} \|f(s,\cdot)\|_{C^{2\alpha}[0,l]}\Bigg], \quad 0 < \alpha < \frac{1}{2}, \end{aligned}$$

where $M(\alpha)$ is not dependent on $g(t,x)$ and $f(t,x)$. Here $C^{\beta}[0,l]$ is the space of functions satisfying a Hölder condition with the indicator $\beta \in (0,1)$.

The proof of Theorem 3.1 is based on the estimate

$$\|\exp\{-tA^x\}\|_{C[0,l] \to C[0,l]} \leq M, \quad t \geq 0,$$

and on the abstract Theorem 2.1, on the strong positivity of the operator A^x in $C[0,l]$ (see [31, 32]) and on Theorem 3.2 on the structure of the fractional space $E_\alpha = E_\alpha(C[0,l], A^x)$ for $0 < \alpha < \frac{1}{2}$.

Theorem 3.2 *For $\alpha \in (0, \frac{1}{2})$, the norms of the space $E_\alpha(C[0,l], A^x)$ and the Hölder space $C^{2\alpha}[0,l]$ are equivalent* [21].

Second, we consider the initial nonlocal boundary value problem for one dimensional delay differential equations of parabolic type,

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - a(x)\frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) \\ \quad = b(t)(-a(x)\frac{\partial^2 u(t-\omega,x)}{\partial x^2} + \delta u(t-\omega,x)) + f(t,x), \quad 0 < t < \infty, x \in (0,l), \\ u(t,x) = g(t,x), \quad -\omega \leq t \leq 0, x \in [0,l], \\ u(t,0) = u(t,l), \qquad u_x(t,0) = u_x(t,l), \quad -\omega \leq t < \infty, \end{cases} \tag{26}$$

where $a(x)$, $b(t)$, $g(t,x)$, $f(t,x)$ are given sufficiently smooth functions and $\delta > 0$ is a sufficiently large number. We will assume that $a(x) \geq a > 0$. The problem (26) has a unique smooth solution. This allows us to reduce the initial-boundary value problem (26) to the initial value problem (1) in Banach space $E = C[0,l]$ with a differential operator A^x defined by the formula

$$A^x u = -a(x)\frac{d^2 u}{dx^2} + \delta u \tag{27}$$

with domain $D(A^x) = \{u \in C^{(2)}[0,1] : u(0) = u(1), u'(0) = u'(1)\}$. Let us give a number of corollaries of the abstract Theorem 2.1.

Theorem 3.3 *Assume that condition* (25) *holds. Then for all* $t \geq 0$ *the solutions of the initial-boundary value problem* (26) *satisfy the following coercive stability estimates*:

$$\begin{aligned} &\|u'(t,\cdot)\|_{C^{2\alpha}[0,l]} + \|u(t,\cdot)\|_{C^{2+2\alpha}[0,l]} \\ &\quad \leq M(\alpha)\Bigg[\max_{-\omega\leq t\leq 0}\|g(t,\cdot)\|_{C^{2+2\alpha}[0,l]} \\ &\qquad + \sum_{k=1}^{n}\max_{-(k-1)\omega\leq s\leq k\omega}\|f(s,\cdot)\|_{C^{2\alpha}[0,l]} + \max_{n\omega\leq s\leq t}\|f(s,\cdot)\|_{C^{2\alpha}[0,l]}\Bigg], \quad 0 < \alpha < \frac{1}{2}, \end{aligned}$$

where $M(\alpha)$ *is not dependent on* $g(t,x)$ *and* $f(t,x)$.

The proof of Theorem 3.3 is based on the estimate

$$\|\exp\{-tA^x\}\|_{C[0,l]\to C[0,l]} \leq M, \quad t \geq 0,$$

and on the abstract Theorem 2.1, on the strong positivity of the operator A^x in $C[0,l]$ (see [6]) and on Theorem 3.4 on the structure of the fractional space $E_\alpha = E_\alpha(C[0,l], A^x)$ for $0 < \alpha < \frac{1}{2}$.

Theorem 3.4 *For* $\alpha \in (0, \frac{1}{2})$, *the norms of the space* $E_\alpha(C[0,l], A^x)$ *and the Hölder space* $C^{2\alpha}[0,l]$ *are equivalent* [6].

Third, we consider the initial value problem on the range

$$\{0 \leq t \leq 1, x = (x_1, \dots, x_n) \in \mathbb{R}^n, r = (r_1, \dots, r_n)\}$$

for $2m$th order multidimensional delay differential equations of parabolic type,

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} + \sum_{|r|=2m} a_\tau(x) \frac{\partial^{|r|} u(t,x)}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} + \delta u(t,x) \\ \quad = b(t)(\sum_{|r|=2m} a_\tau(x) \frac{\partial^{|r|} u(t-\omega,x)}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} + \delta u(t-\omega,x)) + f(t,x), \quad 0 < t < \infty, x \in \mathbb{R}^n, \\ u(t,x) = g(t,x), \quad -\omega \le t \le 0, x \in \mathbb{R}^n, |r| = r_1 + \cdots + r_n, \end{cases} \tag{28}$$

where $a_r(x)$, $b(t)$, $g(t,x)$, and $f(t,x)$ are sufficiently smooth functions and $\delta > 0$ is a sufficiently large number. We will assume that the symbol $[\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n]$

$$A_1^x(\xi) = \sum_{|r|=2m} a_r(x)(i\xi_1)^{r_1} \cdots (i\xi_n)^{r_n}$$

of the differential operator of the form

$$A_1^x = \sum_{|r|=2m} a_r(x) \frac{\partial^{|r|}}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}}, \tag{29}$$

acting on functions defined on the space $\mathbb{R}^n$, satisfies the inequalities

$$0 < M_1|\xi|^{2m} \le (-1)^m A_1^x(\xi) \le M_2|\xi|^{2m} < \infty$$

for $\xi \ne 0$, where $|\xi| = \sqrt{|\xi_1|^2 + \cdots + |\xi_n|^2}$. The problem (28) has a unique smooth solution. This allows us to reduce the initial value problem (28) to the initial value problem (1) in Banach space E with a strongly positive operator $A^x = A_1^x + \delta I$ defined by (29). Let us give a number of corollaries of the abstract Theorem 2.1.

Theorem 3.5 *Assume that condition* (25) *holds. Then for all $t \ge 0$ the solutions of the initial-boundary value problem* (28) *satisfy the following coercive stability estimates*:

$$\begin{aligned} &\|u'(t,\cdot)\|_{C^{2m\alpha}(\mathbb{R}^n)} + \sum_{|r|=2m} \left\| \frac{\partial^{|r|} u(t,\cdot)}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} \right\|_{C^{2m\alpha}(\mathbb{R}^n)} \\ &\quad \le M(\alpha) \left[\max_{-\omega \le t \le 0} \sum_{|r|=2m} \left\| \frac{\partial^{|r|} g(t,\cdot)}{\partial x_1^{r_1} \cdots \partial x_n^{r_n}} \right\|_{C^{2m\alpha}(\mathbb{R}^n)} \right. \\ &\qquad \left. + \sum_{k=1}^{n} \max_{-(k-1)\omega \le s \le k\omega} \|f(s,\cdot)\|_{C^{2m\alpha}(\mathbb{R}^n)} + \max_{n\omega \le s \le t} \|f(s,\cdot)\|_{C^{2m\alpha}(\mathbb{R}^n)} \right], \quad 0 < \alpha < \frac{1}{2m}, \end{aligned}$$

where $M_2(\alpha)$ does not depend on $g(t,x)$ and $f(t,x)$. Here $C^\varepsilon(\mathbb{R}^n)$ is the space of functions satisfying a Hölder condition with the indicator $\varepsilon \in (0,1)$.

The proof of Theorem 3.5 is based on the estimate

$$\|\exp\{-tA^x\}\|_{C(\mathbb{R}^n) \to C(\mathbb{R}^n)} \le M, \quad t \ge 0,$$

and on the abstract Theorem 2.1, on the strong positivity of the operator A^x in $C(\mathbb{R}^n)$, and on the equivalence of the norms in the spaces $E_\alpha = E_\alpha(A, C(\mathbb{R}^n))$ and $C^{2m\alpha}(\mathbb{R}^n)$ when $0 < \alpha < \frac{1}{2m}$ [20, 23].

4 Conclusion

In the present paper, two theorems on the well-posedness of the initial value problem for the delay parabolic differential equations with unbounded operators acting on delay terms in fractional spaces E_α are established. In practice, the coercive stability estimates in Hölder norms for the solutions of the mixed problems for delay parabolic equations are obtained.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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Acknowledgements
This work is supported by Trakya University Scientific Research Projects Unit (Project No: 2010-91).

Received: 31 March 2014 Accepted: 8 May 2014 Published: 20 May 2014

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10.1186/1687-2770-2014-126
Cite this article as: Ashyralyev and Agirseven: **Well-posedness of delay parabolic equations with unbounded operators acting on delay terms.** *Boundary Value Problems* 2014, 2014:126