Morabito *Boundary Value Problems* 2014, 2014:130
http://www.boundaryvalueproblems.com/content/2014/1/130

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

Higher genus capillary surfaces in the unit ball of $\mathbb{R}^3$

Filippo Morabito*

*Correspondence:
morabito@kaist.ac.kr
Department of Mathematical Sciences, Korea Advanced Institute of Science and Technology, 291 Daehak-ro, Yuseong-gu, Daejeon, South Korea
School of Mathematics, Korea Institute for Advanced Study, 87 Hoegi-ro, Cheongryangry 2-dong, Seoul, South Korea

Abstract

We construct the first examples of capillary surfaces of positive genus, embedded in the unit ball of $\mathbb{R}^3$ with vanishing mean curvature and locally constant contact angles along their three boundary curves. These surfaces come in families depending on one parameter and they converge to the triple equatorial disk. Such surfaces are obtained by deforming the Costa-Hoffman-Meeks minimal surfaces.

MSC: 53A10; 35R35; 53C21

Keywords: minimal surface; perturbation method; nonlinear pde's

1 Introduction

The study of capillarity started in the beginning of the 19th century by the work of PS de Laplace and T Young. They considered a liquid contained in a vertical tube of small radius dipped in a reservoir and studied the shape of the free surface interface between the liquid and the air. Such a surface is called capillary surface. More generally a capillary surface is the surface interface between a liquid situated adjacent to another immiscible liquid or gas.

PS de Laplace proved that the height u of a capillary surface over a domain $\Omega \subset \mathbb{R}^2$ satisfies the differential equation

$$2H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = ku + \lambda, \tag{1}$$

where H is the mean curvature, λ is a constant to be determined by physical condition (volume of the fluid and boundary conditions) and k is positive (resp. negative) when denser fluid lies below (resp. above) the interface.

T Young, who considered the case $\lambda = 0$, understood that the capillary surface meets the tube (or more generally the container) making an angle, called contact angle, which depends on the liquid and on the material which composes the container and not on the gravity. For liquids in tubes (*i.e.* cylindrical containers) we see that the following additional boundary condition (Young condition) is satisfied:

$$\nu \cdot \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} = \cos\alpha.$$

Here ν is the unit normal vector to the tube along the boundary of the surface. It says that the capillary surface meets the tube in a constant contact angle (equal to α). See Finn [1], for a survey on more recent discoveries about capillarity.

Existence and uniqueness for the solution of capillarity problem for graphs over domains of $\mathbb{R}^n$ $n \geq 2$ (also in the more general form where $H = f$, for an assigned function f), has been extensively studied in the past, see *e.g.* Gerhardt [2], Lieberman [3], Simon and Spruck [4], Spruck [5], Uraltseva [6].

A more recent series of works (see *e.g.* [7–9]) deals with the existence and regularity of capillary graphs with constant mean curvature in vertical cylinders containing corners or cusps. Huff and McCuan [10] showed the existence of Scherk-type capillary minimal graphs.

Very recently, Calle and Shahriyari in [11] have solved the prescribed mean curvature equation with a boundary contact angle condition. They show the existence of graphs over domains in $\mathbb{M}^n \times \mathbb{R}$, where $\mathbb{M}^n$ is a n-dimensional Riemannian submanifold of $\mathbb{R}^{n+1}$. In [12] Lira and Wanderley show the existence of Killing graphs with prescribed mean curvature and prescribed contact angle along their boundary in a wide class of Riemannian manifolds endowed with a Killing vector field.

Fall and Mercuri in [13] constructed by a perturbation method disk-type minimal surfaces embedded in an infinite cylinder in $\mathbb{R}^3$ and which intersect its boundary orthogonally. In [14] they extended this result to Riemannian manifolds.

In [15] Fall showed that, given a bounded domain of $\mathbb{R}^3$ there exist embedded constant mean curvature (cmc) surfaces contained in Ω and whose boundary intersects $\partial\Omega$ orthogonally. Also he showed that, given a stable stationary point p for the mean curvature of $\partial\Omega$, there exists near p a family of embedded surfaces with cmc equal to ε^{-1}, which, after scaling and translation, converges to a hemisphere of radius 1 as $\varepsilon \to 0$.

In [16] Fall and Mahmoudi showed that if Ω is a domain of $\mathbb{R}^{m+1}$ and K a k-dimensional non-degenerate minimal submanifold, then there exists a family of embedded constant mean curvature hypersurfaces which, as their mean curvature tends to infinity, concentrate along K and intersect $\partial\Omega$ orthogonally.

In this work we show the existence of higher genus minimal capillary surfaces by a perturbation method. Let B^3 be the unit ball centered at the origin of $\mathbb{R}^3$. For each $k \in [1, \dots, +\infty)$ and $\tau \in (0, \tau_0)$, with τ_0 small enough, there exists a surface S_τ of genus k, embedded in B^3 with non-empty boundary which consists in three simple closed curves λ_t, λ_m, λ_b which lie in ∂B^3 and such that

$$\begin{cases} H(p) = 0, & p \in S_\tau, \\ N_i(p) \cdot \nu_i(p) = \psi_i(\tau), & p \in \lambda_i, i = t, m, b, \end{cases} \tag{2}$$

where $H(p)$ denotes the mean curvature at the point p; $N_i(p)$ and $\nu_i(p)$ denote, respectively, the unit normal vector to the surface S_τ and to ∂B^3 at $p \in \lambda_i$. The functions $(\psi_t(\tau), \psi_m(\tau), \psi_b(\tau)) = (\psi(\tau), 0, \psi(\tau))$ are decreasing smooth and non-zero for $\tau \in (0, \tau_0)$. We will describe them below.

The solution of the previous system is based on the deformation of a compact piece of a scaled Costa-Hoffman-Meeks minimal surface contained in the unit ball. More precisely we consider the image by a homothety of ratio τ. Such a surface is denoted by $M_{k,\tau}$. As we will explain in Section 2.1, $M_{k,\tau}$ is asymptotic to a top half catenoid, to a bottom half catenoid and to a horizontal plane. The functions $(\psi_t(\tau), \psi_m(\tau), \psi_b(\tau))$ are defined to be the values of the scalar product $N_i(p) \cdot \nu_i(p)$ we obtain if we replace S_τ by the two halves catenoid and the plane. In particular $\psi_m = 0$.

We provide the first examples of capillary type surfaces with non-trivial topology, having vanishing mean curvature and locally constant contact angles with the sphere. They are equal to the contact angles made by the asymptotic catenoids and the plane described above with the sphere. Such surfaces are obtained by deformation of minimal surfaces by a function in the space described by Definition 2.1.

Here is the statement of the result we get. The cartesian coordinates in $\mathbb{R}^3$ are denoted by (x_1, x_2, x_3).

Theorem 1.1 *For each $k \in [1, \dots, +\infty)$, there exists $\tau_0 \in \mathbb{R}$ positive and small enough, such that for each $\tau \in (0, \tau_0)$ there exists a surface S_τ embedded in B^3, of genus k, whose boundary $\partial S_\tau \subset \partial B^3$ is composed by three simple Jordan curves λ_t, λ_m, λ_b and satisfying*

$$\begin{cases} H(p) = 0, & p \in S_\tau, \\ N_i(p) \cdot \nu_i(p) = \psi_i(\tau), & p \in \lambda_i. \end{cases} \tag{3}$$

Such surfaces are invariant under the action of the rotation of angle $\frac{2\pi}{k+1}$ about the x_3-axis, under the action of the reflection in the $x_2 = 0$ plane and under the action of the composition of a rotation of angle $\frac{\pi}{k+1}$ about the x_3-axis and the reflection in the $x_3 = 0$ plane.

We observe that for values of τ in the range of validity of our theorem $\psi_t(\tau), \psi_b(\tau) \neq 0$. In other terms the surface cannot make a constant angle equal to $\pi/2$ with ∂B^3 along λ_t, λ_b. We point out that $\lim_{\tau \to 0} \psi_i(\tau) = 0$. As τ is the homothety ratio, this says that, as τ tends to 0 the limit of S_τ consists in the triple equatorial disk.

The proof can easily be modified in order to handle the case of capillary surfaces with boundary on a vertical cylinder.

Among the works dealing with capillary surfaces in a ball we cite [17] by Ros and Souam. They showed that a stable minimal capillary surface (that is, stationary surfaces with non-negative second variation of the area) in a ball of $\mathbb{R}^3$ is a totally geodesic disk or a surfaces of genus 1 with boundary having at most 3 connected components. Consequently, at least for $k > 1$, the surfaces described by Theorem 1.1 are unstable.

The interest in capillary surfaces in the unit ball has been rekindled by the recent works of Fraser and Schoen [18, 19]. They considered free boundary minimal surfaces embedded in the unit ball of $\mathbb{R}^n$, *i.e.* surfaces which meet orthogonally the boundary of the ball.

Free boundary minimal submanifolds are critical for the problem of extremizing the volume among deformations which preserve the ball. Such solutions arise from variational min/max constructions, and examples include equatorial disks, the (critical) catenoid, as well as the cone over any minimal submanifold of the sphere. If Σ is a compact Riemannian surface with $\partial\Sigma \neq \emptyset$ then the Dirichlet-to-Neumann operator maps a function u on $\partial\Sigma$ to the normal derivative of the harmonic extension of u to the interior. A submanifold properly immersed in the unit ball is a free boundary submanifold if and only if its coordinate functions are Steklov eigenfunctions with eigenvalue 1. Using this characterization they prove the existence of free boundary minimal surfaces in the unit ball of $\mathbb{R}^3$ of genus 0 with boundary having k connected components, for any finite $k \geq 1$. The authors conjecture the existence of higher genus examples of free boundary embedded minimal surfaces which have three boundary components and converge to the union of the critical vertical catenoid and the equatorial disk.

The minimal surfaces described in Theorem 1.1 come in 1-parameter families, they have finite genus ≥ 1, they meet orthogonally the boundary of the ball only along the middle boundary curve. Furthermore, for any value of the genus, the limit for values of the parameter close to zero consists in the triple equatorial disk.

2 Preliminaries

The proof of the existence of solutions of the capillarity type problem is based on the deformation of a compact piece of the minimal surfaces $M_{k,\tau}$. We describe this family of surfaces in Section 2.1.

We will show that it is possible to deform a surface Σ in this family in order to get a surface satisfying (3). More precisely we will prove the existence of a function u defined on Σ and of small norm such that its normal graph S_u over Σ has vanishing mean curvature and the scalar product of the unit normal vectors, $(N_{S_u})_i \cdot \nu_i$, equals ψ_i at each point of the ith component of ∂S_u, with $i \in \{1,2,3\}$.

We will adapt to our setting some arguments used in [20, 21].

2.1 The scaled Costa-Hoffman-Meeks surface

The Costa-Hoffman-Meeks surface of genus $k \in [1,\dots,+\infty)$ embedded in $\mathbb{R}^3$ (see [22]) is denoted by M_k.

After suitable rotation and translation, M_k enjoys the following properties.

1. It has one planar end E_m asymptotic to the horizontal plane $x_3 = 0$, one top end E_t and one bottom end E_b that are, respectively, asymptotic to the upper end and to the lower end of a catenoid having the x_3-axis as axis of rotation. The planar end E_m is located between the two catenoidal ends.
2. It is invariant under the action of the rotation of angle $\frac{2\pi}{k+1}$ about the x_3-axis, under the action of the reflection in the $x_2 = 0$ plane and under the action of the composition of a rotation of angle $\frac{\pi}{k+1}$ about the x_3-axis and the reflection in the $x_3 = 0$ plane.
3. It intersects the $x_3 = 0$ plane in $k+1$ straight lines, which intersect themselves at the origin with angles equal to $\frac{\pi}{k+1}$. The intersection of M_k with the plane $x_3 = \text{const}$ ($\neq 0$) is a single Jordan curve. The intersection of M_k with the upper half space $x_3 > 0$ (resp. with the lower half space $x_3 < 0$) is topologically an open annulus.

The parameterization of the end E_i is denoted by X_i, with $i = t, b, m$, and the parameterization of the corresponding end $E_{i,\tau}$ of $M_{k,\tau}$ is denoted by $X_{i,\tau}$. We recall that $M_{k,\tau}$ is the image of M_k by the homothety of ratio τ.

Now we provide a local description of the surface $M_{k,\tau}$ near its ends and we introduce the coordinates that we will use.

2.2 The planar end

The planar end $E_{m,\tau}$ of the surface $M_{k,\tau}$ can be parametrized by

$$X_{m,\tau}(x) := \left(\frac{\tau x}{|x|^2}, \tau u_m(x)\right) \in \mathbb{R}^3, \tag{4}$$

where $x \in \bar{B}_{\rho_0}(0) - \{0\} \subset \mathbb{R}^2$. Here $\rho_0 > 0$ is fixed small enough. In the sequel, where necessary, we will consider on $B_{\rho_0}(0)$ also the polar coordinates $(\rho, \theta) \in [0, \rho_0] \times S^1$. The function u_m satisfies the minimal surface equation, which has the following form:

$$2H_u = \frac{|x|^4}{\tau} \operatorname{div}\left(\frac{\nabla u}{(1 + |x|^4 |\nabla u|^2)^{1/2}} \right) = 0. \tag{5}$$

It can be shown (see [20]) that the function u_m can be extended at the origin continuously by using Weierstrass representation. In particular we can prove that $u_m \in \mathcal{C}^{2,\alpha}(\bar{B}_{\rho_0})$ and $u_m = \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}(|x|^{k+1})$, where the $\mathcal{O}_{\mathcal{C}_b^{n,\alpha}}(g)$ denotes a function that, together with its partial derivatives of order less than or equal to $n + \alpha$ is bounded by a constant times g. Furthermore, by taking into account the symmetries of the surface, it is possible to show the function u_m, in polar coordinates, has to be collinear to $\cos(j(k+1)\theta)$, with $j \geq 1$ and odd.

2.3 The catenoidal ends

The parametrization of the standard catenoid C, whose axis of revolution is the x_3-axis, is denoted by X_c. We have

$$X_c(s, \theta) := (\cosh s \cos \theta, \cosh s \sin \theta, s) \in \mathbb{R}^3,$$

where $(s, \theta) \in \mathbb{R} \times S^1$. The unit normal vector field to C is given by

$$n_c(s, \theta) := \frac{1}{\cosh s} (\cos \theta, \sin \theta, -\sinh s). \tag{6}$$

The catenoid C may be divided in two pieces, denoted by $C_\pm$, which are defined as the image by X_c of $(\mathbb{R}^\pm \times S^1)$. For any $\tau > 0$, we define the catenoid C_τ as the image of C by a homothety of ratio τ. Its parametrization is denoted by $X_{c,\tau} := \tau X_c$. Of course, by this transformation, the two surfaces correspond to $C_\pm$. They are denoted by $C_{\tau,\pm}$.

Up to some dilation, we can assume that the two ends $E_{t,\tau}$ and $E_{b,\tau}$ of $M_{k,\tau}$ are asymptotic to some translated copy of the two halves of the catenoid parametrized by $X_{c,\tau}$ in the vertical direction. Therefore, $E_{t,\tau}$ and $E_{b,\tau}$ can be parametrized, respectively, by

$$X_{t,\tau} := X_{c,\tau} + w_t n_c + \sigma_{t,\tau} e_3 \tag{7}$$

for $(s, \theta) \in (s_0, \infty) \times S^1$,

$$X_{b,\tau} := X_{c,\tau} - w_b n_c - \sigma_{b,\tau} e_3 \tag{8}$$

for $(s, \theta) \in (-\infty, -s_0) \times S^1$, where $\sigma_{t,\tau}, \sigma_{b,\tau} \in \mathbb{R}$, functions w_t, w_b tend exponentially fast to 0 as s goes to $\pm\infty$ reflecting the fact that the ends are asymptotic to a catenoidal end. More precisely it is known that $w_t = \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}(\tau e^{-(k+1)s})$. Furthermore, taking into account the symmetries of the surface, it is easy to show the functions w_t, w_b, in terms of the (s, θ) coordinates, have to be collinear to $\cos(j(k+1)\theta)$, with $j \in \mathbb{N}$ and must satisfy $w_b(s, \theta) = -w_t(-s, \theta - \frac{\pi}{k+1})$. Furthermore we have $\sigma_{t,\tau} = \sigma_{b,\tau}$. In the sequel we will omit the indices t, b and we will use the notation σ_τ. We assume that $\sigma_\tau \leq \kappa \tau^2$, κ being a constant.

For all $\rho < \rho_0$ and $s > s_0$, we define

$$M_{k,\tau}(s,\rho) := M_{k,\tau} - \left[X_{t,\tau}\left((s,\infty) \times S^1\right) \cup X_{b,\tau}\left((-\infty,-s) \times S^1\right) \cup X_{m,\tau}\left(B_\rho(0)\right)\right]. \tag{9}$$

The parametrizations of the three ends of $M_{k,\tau}$ induce a decomposition of $M_{k,\tau}$ into slightly overlapping components: a compact piece $M_{k,\tau}(s_0+1,\rho_0/2)$ and three noncompact pieces $X_{t,\tau}((s_0,\infty) \times S^1)$, $X_{b,\tau}((-\infty,-s_0) \times S^1)$ and $X_{m,\tau}(\bar{B}_{\rho_0}(0))$.

We define a weighted space of functions on $M_{k,\tau}$.

Definition 2.1 Given $\ell \in \mathbb{N}$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$, the space $\mathcal{C}^{\ell,\alpha}_\delta(M_{k,\tau})$ is defined to be the space of functions in $\mathcal{C}^{\ell,\alpha}_{\mathrm{loc}}(M_{k,\tau})$ for which the following norm is finite:

$$\begin{aligned}\|w\|_{\mathcal{C}^{\ell,\alpha}_\delta(M_{k,\tau})} &:= \|w\|_{\mathcal{C}^{\ell,\alpha}(M_{k,\tau}(s_0+1,\rho_0/2))} + \|w \circ X_{m,\tau}\|_{\mathcal{C}^{\ell,\alpha}(\bar{B}_{\rho_0}(0))} \\ &\quad + \|w \circ X_{t,\tau}\|_{\mathcal{C}^{\ell,\alpha}_\delta([s_0,+\infty)\times S^1)} + \|w \circ X_{b,\tau}\|_{\mathcal{C}^{\ell,\alpha}_\delta((-\infty,-s_0]\times S^1)},\end{aligned}$$

where

$$\|f\|_{\mathcal{C}^{\ell,\alpha}_\delta([s_0,+\infty)\times S^1)} = \sup_{s\geq s_0}\left(e^{-\delta s}\|f\|_{\mathcal{C}^{\ell,\alpha}([s,s+1]\times S^1)}\right),$$

$$\|f\|_{\mathcal{C}^{\ell,\alpha}_\delta((-\infty,-s_0]\times S^1)} = \sup_{s\leq -s_0}\left(e^{\delta s}\|f\|_{\mathcal{C}^{\ell,\alpha}([s-1,s]\times S^1)}\right)$$

and which are invariant with respect to the reflection in the $x_2 = 0$ plane, that is, $w(p) = w(\bar{p})$ for all $p \in M_{k,\tau}$, where $\bar{p} := (x_1,-x_2,x_3)$ if $p = (x_1,x_2,x_3)$, invariant with respect to a rotation of angle $\frac{2\pi}{k+1}$ about the x_3 axis and to the composition of a rotation of angle $\frac{\pi}{k+1}$ about the x_3 axis and the reflection in the $x_3 = 0$ plane.

We remark that there is no weight on the middle end. In fact we compactify this end and we consider a weighted space of functions defined on a two ended surface.

The proof of Theorem 1.1 consists of two steps. Firstly we will show that for each choice of the genus k there exists, for τ sufficiently small, a family of functions $u \in \mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})$ such that their normal graph over $M_{k,\tau}$ satisfies the first equation in (3). To do that we need to find the expression of the mean curvature operator for normal graphs of functions defined on $M_{k,\tau}$. This is the aim of following section. Secondly we prove that in the family of solutions described above there is a function satisfying also the capillarity condition in (3).

3 The mean curvature of a graph over $M_{k,\tau}$

It is well known that the mean curvature H_u of the normal graph of a function u over a minimal surface Σ can be decomposed as $2H_u = \mathbb{L}_\Sigma u + Q(u)$, where $\mathbb{L}_\Sigma$ denotes a linear second order elliptic operator and Q is a nonlinear differential operator of higher order. The operator $\mathbb{L}_\Sigma$ is known under the name of Jacobi operator and it is defined as the linearized of the mean curvature operator. For a minimal surface Σ in $\mathbb{R}^3$ its expression is

$$\mathbb{L}_\Sigma := \Delta_\Sigma + |A_\Sigma|^2,$$

where Δ_Σ denotes the Laplace-Beltrami operator and $|A_\Sigma|$ is the norm of the second fundamental form on the surface.

As for the majority of minimal surfaces, unfortunately the explicit expression of the mean curvature operator of the Costa-Hoffman-Meeks surfaces is not known. The knowledge of the geometric behavior of such surfaces (we recall that their ends are asymptotic to the two halves of a catenoid and to a plane) allows us to get information about the operator $\mathbb{L}_{M_{k,\tau}}$ and more generally of the mean curvature operator at the ends of the surfaces.

3.1 Mean curvature operator at the catenoidal ends

The surface parametrized by $X_{c,\tau} + wn_c$ is minimal if and only if the function w satisfies the minimal surface equation

$$2H_w = \frac{1}{\tau^2}\mathbb{L}_C w + Q_\tau(w) = 0, \tag{10}$$

$\mathbb{L}_C$ being the Jacobi operator of the catenoid, *i.e.*

$$\mathbb{L}_C w = \frac{1}{\cosh^2 s}\left(\partial^2_{ss}w + \partial^2_{\theta\theta}w + \frac{2w}{\cosh^2 s}\right),$$

and

$$Q_\tau(w) = \frac{1}{\tau\cosh^2 s}Q_{2,\tau}\left(\frac{w}{\tau\cosh s}\right) + \frac{1}{\tau\cosh s}Q_{3,\tau}\left(\frac{w}{\tau\cosh s}\right). \tag{11}$$

Here Q_2, Q_3 are nonlinear second order differential operators which are bounded in $\mathcal{C}^l(\mathbb{R}\times S^1)$, for every l, and satisfy $Q_2(0) = Q_3(0) = 0$, $\nabla Q_2(0) = \nabla Q_3(0) = 0$, $\nabla^2 Q_3(0) = 0$ together with

$$\left\|Q_j(v_2) - Q_j(v_1)\right\|_{\mathcal{C}^{0,\alpha}([s,s+1]\times S^1)} \le c\left(\sup_{i=1,2}\|v_i\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)}\right)^{j-1}\|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)} \tag{12}$$

for all $s\in\mathbb{R}$ and all v_1, v_2 such that $\|v_i\|_{\mathcal{C}^{2,\alpha}([s,s+1]\times S^1)} \le 1$. The positive constant c does not depend on s.

Finally we observe that the operator $(\cosh s)^2\frac{1}{\tau^2}\mathbb{L}_C$ maps the functional space

$$(\cosh s)^\delta\mathcal{C}^{2,\alpha}\big((s_0,+\infty)\times S^1\big) \text{ into } (\cosh s)^\delta\mathcal{C}^{0,\alpha}\big((s_0,+\infty)\times S^1\big).$$

3.2 Mean curvature operator at the planar end

If we linearize the nonlinear equation (5) we obtain

$$L_u v = \frac{|x|^4}{\tau}\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|x|^4|\nabla u|^2}} - |x|^4\nabla u\frac{\nabla u\cdot\nabla v}{\sqrt{(1+|x|^4|\nabla u|^2)^3}}\right). \tag{13}$$

If we consider $u = 0$ we get an operator which equals, up to a multiplication by τ, the Jacobi operator of the plane, that is, $\mathcal{L}_{\mathbb{R}^2} = |x|^4\Delta_0$. The graph surface of the function u is denoted by Σ_u and its mean curvature by H_u. Then H_{u+v}, the mean curvature of the graph of the function $u+v$, in terms of H_u, is

$$2H_{u+v} = 2H_u + L_u v + \frac{|x|^4}{\tau}Q_u\big(|x|^2\nabla v, |x|^2\nabla^2 v\big), \tag{14}$$

where Q_u satisfies

$$Q_u(0,0) = 0, \qquad \nabla Q_u(0,0) = 0.$$

Since we assume that Σ_u is a minimal surface, we have $H_u = 0$. So we get the following equation:

$$2H_{u+v} = \frac{|x|^4}{\tau}\big(\Delta_0 v + \sqrt{1+|x|^4|\nabla u|^2}\big(\bar{L}_u v + Q_u\big(|x|^2\nabla v, |x|^2\nabla^2 v\big)\big)\big), \tag{15}$$

where $\bar{L}_u v$ is a second order linear operator with operator with coefficients in $\mathcal{O}_{C_b^{2,\alpha}}(|x|^{k+1})$.

We recall that if the function v satisfies the equation $H_{u+v} = 0$ with $u = u_m$ then the graph of the function $\tau(u_m + v)$ is minimal. Now we are interested in finding the equation which a function w must satisfy in such a way the surface parametrized by $X_{m,\tau} + we_3$, that is the graph of w over the middle end $E_{m,\tau}$, is minimal. That is equivalent to require that the graph of $\tau u_m + w$ is minimal. Then we can obtain the wanted equation by replacing v by w/τ in (15). So we get

$$\frac{|x|^4}{\tau}\left(\frac{1}{\tau}\Delta_0 w + \sqrt{1+|x|^4|\nabla u|^2}\left(\frac{1}{\tau}\bar{L}_u w + Q_u\left(\frac{|x|^2}{\tau}\nabla w, \frac{|x|^2}{\tau}\nabla^2 w\right)\right)\right) = 0. \tag{16}$$

If we set $Q_{\tau,u}(\cdot) := \frac{|x|^4}{\tau}\sqrt{1+|x|^4|\nabla u|^2}Q_u(\frac{|x|^2}{\tau}\nabla\cdot, \frac{|x|^2}{\tau}\nabla^2\cdot)$ to simplify the notation, we can write this equation in the following way:

$$\frac{|x|^4}{\tau^2}\Delta_0 w + \frac{|x|^4}{\tau^2}\sqrt{1+|x|^4|\nabla u|^2}\bar{L}_u w + Q_{\tau,u}\tau(w) = 0. \tag{17}$$

We observe that the operator $\frac{1}{|x|^4}\mathbb{L}_{\mathbb{R}^2} = \Delta_0$ clearly maps the space $\mathcal{C}^{2,\alpha}(\bar{B}_{\rho_0})$ into the space $\mathcal{C}^{0,\alpha}(\bar{B}_{\rho_0})$.

3.3 Properties of the Jacobi operator of $M_{k,\tau}$

The Jacobi operator of $M_{k,\tau}$, up to a multiplicative factor, is asymptotic, respectively, to the operators $|x|^4\Delta_0$ and $\mathbb{L}_C$ at the planar end and the catenoidal end.

In this subsection we will describe the mapping properties of an elliptic operator related to $\mathbb{L}_{M_{k,\tau}}$. It will be used to solve the first equation of (3).

The volume form on $M_{k,\tau}$ is denoted by $dvol_{M_{k,\tau}}$. In the parameterization of the ends introduced above, such form can be written as $\gamma_t\, ds\, d\theta$ and $\gamma_b\, ds\, d\theta$ near the catenoidal type ends and as $\gamma_m\, dx_1\, dx_2$ near the middle end. Now we can define globally on $M_{k,\tau}$ a smooth function

$$\gamma : M_{k,\tau} \longrightarrow [0,\infty) \tag{18}$$

that is identically equal to τ^2 on $M_{k,\tau}(s_0 - 1, 2\rho_0)$ and equal to γ_t (resp. γ_b, γ_m) on the end $E_{t,\tau}$ (resp. $E_{b,\tau}$, E_m). They are defined in such a way that for $(s,\theta) \in (s_0,\infty) \times S^1$, $(s,\theta) \in (-\infty, -s_0) \times S^1$ we have, respectively,

$$\gamma \circ X_{t,\tau}(s,\theta) \sim \tau^2\cosh^2 s \quad \text{and} \quad \gamma \circ X_{b,\tau}(s,\theta) \sim \tau^2\cosh^2 s.$$

Finally on B_{ρ_0} we have

$$\gamma \circ X_m(x) \sim \frac{\tau^2}{|x|^4}.$$

It is possible to check that

$$\mathcal{L}_{\tau,\delta} : \mathcal{C}^{2,\alpha}_{\delta}(M_{k,\tau}) \longrightarrow \mathcal{C}^{0,\alpha}_{\delta}(M_{k,\tau}),$$
$$w \longmapsto \gamma \mathbb{L}_{M_{k,\tau}}(w)$$

is a bounded linear operator.

As in [21] (see also [20] for the same result in a less symmetric setting), using the non-degeneracy of the Costa-Hoffman-Meeks surfaces shown in [23, 24], it is possible to show the following result.

Proposition 3.1 *If $\delta \in (1,2)$, then the operator $\mathcal{L}_{\tau,\delta}$ is surjective and has a kernel of dimension one. Moreover, there exists a right inverse $G_{\tau,\delta}$ for $\mathcal{L}_{\tau,\delta}$ whose norm is bounded.*

4 Construction of a family of solutions to $H_{S_u} = 0$

In this section we will prove the existence of a family of embedded minimal surfaces and which are close to the piece of surface $M_{k,\tau}$ contained in the unit ball B^3.

We set

$$\rho_\tau := \tau$$

and we define s_τ to be the value of s such that

$$(\tau \cosh s)^2 + (\sigma_\tau + \tau s)^2 = 1. \tag{19}$$

We get

$$s_\tau = -\ln \tau + \ln 2 + O(\tau).$$

We define r_τ so that

$$s_\tau = \ln\left(\frac{2r_\tau}{\tau}\right).$$

The value of ρ_τ has been chosen so that the image of $x \in B_{\tilde{\rho}}(0)$, with $|x| = \rho_\tau$, by the map $X_{0,\tau}(x) = (\tau x/|x|^2, 0) \in \mathbb{R}^3$ (compare (4)) is the circumference Γ_m of radius 1 in the horizontal plane $x_3 = 0$. Moreover, s_τ is the value of s for which $\pm(\sigma_\tau + \tau s)$ is the height of the curves Γ_t, Γ_b which are the intersection of the unit sphere with the top and bottom halves of the catenoid parametrized by C_τ and translated vertically by $\pm\sigma_\tau$, respectively.

We define $M^T_{k,\tau}$ to be equal to $M_{k,\tau}$ from which we have removed the image of $(s_\tau, +\infty) \times S^1$ by $X_{t,\tau}$, the image of $(-\infty, -s_\tau) \times S^1$ by $X_{b,\tau}$ and the image of $B_{\rho_\tau}(0)$ by $X_{m,\tau}$. The boundary curves of $M^T_{k,\tau}$ do not lie in the unit sphere but they are in a tubular neighborhood of the curves Γ_t, Γ_b, Γ_m. In the sequel we will use also the cylindrical coordinates (r, θ, z) (of course $z = x_3$). The circumferences Γ_t, Γ_b are contained, respectively, in the horizontal planes $z = \pm(\sigma_\tau + \tau s_\tau)$ and their vertical projection on the $z = 0$ plane is the circumference of radius $\tau \cosh s_\tau = 1 - O(\tau^2 \ln^2 1/\tau)$. The middle boundary curve of $M^T_{k,\tau}$ is located in a small neighborhood of Γ_m. Points in the middle boundary curve have a height which can be estimated by $O(\tau^{k+2})$.

By using (4), (7), and (8) we get easily the following lemma. It describes the region of the surface $M_{k,\tau}$ which is a graph over the annular domain $A = \{(r,\theta) \mid |r-1| \leq \tau\}$ of the $x_3 = 0$ plane.

Lemma 4.1 *There exists $\tau_0 > 0$ such that, for all $\tau \in (0,\tau_0)$ an annular part of the ends $E_{t,\tau}$, $E_{b,\tau}$ and $E_{m,\tau}$ of $M_{k,\tau}$ can be written as vertical graphs over the annulus A of the functions*

$$Z_t(r,\theta) = \sigma_\tau + \tau \ln\left(\frac{2r}{\tau}\right) + \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}\left(\tau^3\right), \tag{20}$$

$$Z_b(r,\theta) = -Z_t\left(r, \theta - \frac{\pi}{k+1}\right), \tag{21}$$

$$Z_m(r,\theta) = \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}\left(\tau\left(\frac{r}{\tau}\right)^{-(k+1)}\right). \tag{22}$$

Here (r,θ) are the polar coordinates in the $x_3 = 0$ plane. The functions $\mathcal{O}_{\mathcal{C}_b^{2,\alpha}}(f)$ are defined in the annulus A and are bounded in $\mathcal{C}_b^{2,\alpha}$ topology by a constant (independent by f) multiplied by f, where the partial derivatives are computed with respect to the vector fields $r\partial_r$ and ∂_θ.

We will make a slight modification to the parametrization of the ends $E_{t,\tau}$, $E_{b,\tau}$ and $E_{m,\tau}$, for s and ρ in a small neighborhood of $\pm s_\tau$ and ρ_τ, respectively.

The unit normal vector field to $M_{k,\tau}$ is denoted by n_τ. Firstly we modify the vector field n_τ into a transverse unit vector field $\tilde{n}_\tau$. $\tilde{n}_\tau$ is a smooth interpolation of the following vector fields defined on different pieces of the surface:

- at the top (resp. bottom) catenoidal end, the unit normal vector $n_c(s_\tau,\cdot)$ (resp. $n_c(-s_\tau,\cdot)$) for s in a small neighborhood of $s = s_\tau$ (resp. $s = -s_\tau$); we recall that $n_c(\pm s_\tau,\cdot)$ are the unit normal vectors to the translated copy of the halves catenoid parametrized by $X_{c,\tau} \pm \sigma_\tau e_3$ along the curves Γ_t, Γ_b;
- at the middle planar end, the vertical vector field e_3 for ρ in a small neighborhood of $\rho = \rho_\tau$;
- the normal vector field n_τ on the remaining part of the surface.

We observe that at the top end $E_{t,\tau}$, we can give the following estimate:

$$\left|\tau^2 \cosh^2 s\left(\mathbb{L}_{M_{k,\tau}} v - \left(\tau^2 \cosh^2 s\right)^{-1}(\partial_{ss} v + \partial_{\theta\theta} v)\right)\right| \leq c\left|\left(\cosh^2 s\right)^{-1} v\right|. \tag{23}$$

This follows easily from (10) together with the fact that w_t decays at least like $(\cosh^2 s)^{-1}$ on $E_{t,\tau}$. Similar considerations hold at the bottom end $E_{b,\tau}$. Near the middle planar end $E_{m,\tau}$, we observe that the following estimate holds:

$$\left|\tau^2 |x|^{-4}\left(\mathbb{L}_{M_{k,\tau}} v - |x|^4 \tau^{-2} \Delta_0 v\right)\right| \leq c\left||x|^{2k+3} \nabla v\right|. \tag{24}$$

This follows easily from (13) together with the fact that u_m decays at least like $|x|^{k+1}$ on $E_{m,\tau}$.

The mean curvature of the graph Σ_u of a function u in the direction of the vector field $\tilde{n}_\tau$ is the image of u by a second order nonlinear elliptic operator:

$$2H(\Sigma_u) = \mathbb{L}_{M_{k,\tau}^T} u + \tilde{L}_\tau u + Q_\tau(u),$$

where $\mathbb{L}_{M_{k,\tau}^T}$ is the Jacobi operator of $M_{k,\tau}^T$, Q_τ is a nonlinear second order differential operator and $\tilde{L}_\tau$ is a linear operator which takes into account the change of the normal vector field n_τ into $\tilde{n}_\tau$.

The operator $\tilde{L}_\tau$ is supported in a neighborhood of $\{\pm s_\tau\} \times S^1$ and of $\{\rho_\tau\} \times S^1$. It is possible to show that the coefficients of $\tilde{L}_\tau$ are uniformly bounded by a constant times τ^2. First we observe that $\langle \tilde{n}_\tau, n_\tau \rangle = 1 + \mathcal{O}_{\mathcal{C}_b^{2,\alpha}}(\tau^2)$ in a neighborhood of $\{\pm s_\tau\} \times S^1$ and of $\{\rho_\tau\} \times S^1$ and the result of [20] Appendix B show that the change of vector field induces a linear operator whose coefficients are bounded by a constant times τ^2.

As we will see in the sequel, the function $u \in \mathcal{C}_\delta^{2,\alpha}(M_{k,\tau})$ which solves $H(\Sigma_u) = 0$, depends nonlinearly by a triple of functions defined on the boundary curves of $M_{k,\tau}^T$. Here is the definition of the functional space we will consider.

Definition 4.2 Given $k \geq 1$, $\alpha \in (0,1)$, the space $[\mathcal{C}^{n,\alpha}(S^1)]^3_{\text{sym}}$ is defined to be the space of triples of functions $\Phi = (\varphi_t, \varphi_m, \varphi_b)$ such that $\varphi_j \in \mathcal{C}^{n,\alpha}(S^1)$ and even, φ_t is collinear to $\cos(j(k+1)\theta)$, with $j \geq 1$; φ_m is collinear to $\cos(l(k+1)\theta)$, with $l \geq 1$ and odd, $\varphi_b = -\varphi_t(\theta - \frac{\pi}{k+1})$, and whose norm, defined below, is finite.

$$\|\Phi\|_{[\mathcal{C}^{n,\alpha}(S^1)]^3_{\text{sym}}} := \|\varphi_t\|_{\mathcal{C}^{n,\alpha}(S^1)} + \|\varphi_m\|_{\mathcal{C}^{n,\alpha}(S^1)} + \|\varphi_b\|_{\mathcal{C}^{n,\alpha}(S^1)}. \tag{25}$$

Now we consider the triple of functions $\Phi = (\varphi_t, \varphi_m, \varphi_b) \in [\mathcal{C}^{2,\alpha}(S^1)]^3_{\text{sym}}$,

$$\|\Phi\|_{[\mathcal{C}^{2,\alpha}(S^1)]^3_{\text{sym}}} \leq \kappa\tau^2. \tag{26}$$

We define w_Φ to be the function equal to

1. $\chi_+ H_{\varphi_t}(s_\tau - s, \cdot)$ on the image of $X_{t,\tau}$, where χ_+ is a cut-off function equal to 0 for $s \leq s_0 + 1$ and identically equal to 1 for $s \in [s_0 + 2, s_\tau]$;
2. $\chi_- H_{\varphi_b}(s + s_\tau, \cdot)$ on the image of $X_{b,\tau}$, where χ_- is a cut-off function equal to 0 for $s \geq -s_0 - 1$ and identically equal to 1 for $s \in [-s_\tau, -s_0 - 2]$;
3. $\chi_m \tilde{H}_{\rho_\tau,\varphi_m}(\cdot,\cdot)$ on the image of $X_{m,\tau}$, where χ_m is a cut-off function equal to 0 for $\rho \geq \rho_0$ and identically equal to 1 for $\rho \in [\rho_\tau, \rho_0/2]$;
4. zero on the remaining part of the surface $M_{k,\tau}^T$.

The cut-off functions just introduced must enjoy the same symmetry properties as the functions in $\mathcal{C}_\delta^{2,\alpha}(M_{k,\tau})$. $\tilde{H}$ and H are harmonic extension operators introduced, respectively, in Propositions A.1 and A.2.

We will prove that, under appropriates hypotheses, the graph Σ_u over $M_{k,\tau}^T$ of the function $u = w_\Phi + v$, is a surface whose mean curvature vanishes.

The equation to solve is

$$H(\Sigma_u) = 0.$$

Since we are looking for solutions having the form $u = w_\Phi + v$, we can write it as

$$\mathbb{L}_{M_{k,\tau}^T}(w_\Phi + v) + \tilde{L}_\tau(w_\Phi + v) + Q_\tau(w_\Phi + v) = 0.$$

The resolution of the previous equation is obtained by the one of the following fixed point problem:

$$v = T(\Phi, v) \tag{27}$$

with

$$T(\Phi, v) = G_{\tau,\delta} \circ \mathcal{E}_\tau \left(\gamma \left(-\tilde{L}_\tau(w_\Phi + v) - \mathbb{L}_{M^T_{k,\tau}} w_\Phi - Q_\tau(w_\Phi + v)\right)\right),$$

where $\delta \in (1,2)$, the operator $G_{\tau,\delta}$ is defined in Proposition 3.1 and $\mathcal{E}_\tau$ is a linear extension operator such that

$$\mathcal{E}_\tau : \mathcal{C}^{0,\alpha}_\delta\left(M^T_{k,\tau}\right) \longrightarrow \mathcal{C}^{0,\alpha}_\delta(M_{k,\tau}),$$

where $\mathcal{C}^{0,\alpha}_\delta(M^T_{k,\tau})$ denotes the space of functions of $\mathcal{C}^{0,\alpha}_\delta(M_{k,\tau})$ restricted to $M^T_{k,\tau}$. It is defined by $\mathcal{E}_\tau v = v$ in $M^T_{k,\tau}$, $\mathcal{E}_\tau v = 0$ in the image of $[s_\tau + 1, +\infty) \times S^1$ by $X_{t,\tau}$, in the image of $(-\infty, -s_\tau - 1] \times S^1$ by $X_{b,\tau}$ and in the image of $B_{\rho_\tau/2}$ by $X_{m,\tau}$. Finally $\mathcal{E}_\tau v$ is an interpolation of these values in the remaining part of $M_{k,\tau}$ such that

$$\begin{aligned}
&(\mathcal{E}_\tau v) \circ X_{t,\tau}(s,\theta) = (1 + s_\tau - s)\left(v \circ X_{t,\tau}(s_\tau,\theta)\right), \quad \text{for } (s,\theta) \in [s_\tau, s_\tau + 1] \times S^1, \\
&(\mathcal{E}_\tau v) \circ X_{b,\tau}(s,\theta) = (1 + s_\tau + s)\left(v \circ X_{b,\tau}(s_\tau,\theta)\right), \quad \text{for } (s,\theta) \in [-s_\tau - 1, -s_\tau] \times S^1, \\
&(\mathcal{E}_\tau v) \circ X_{m,\tau}(\rho,\theta) = \left(\frac{2}{\rho_\tau}\rho - 1\right)\left(v \circ X_{m,\tau}(\rho_\tau,\theta)\right) \quad \text{for } (\rho,\theta) \in [\rho_\tau/2, \rho_\tau] \times S^1.
\end{aligned}$$

Remark 4.3 From the definition of $\mathcal{E}_\tau$, if $\operatorname{supp} v \cap (B_{\rho_\tau} - B_{\rho_\tau/2}) \neq \emptyset$, then

$$\left\|(\mathcal{E}_\tau v) \circ X_{m,\tau}\right\|_{\mathcal{C}^{0,\alpha}(\bar{B}_{\rho_0})} \leq c\rho_\tau^{-\alpha} \|v \circ X_{m,\tau}\|_{\mathcal{C}^{0,\alpha}(B_{\rho_0} - B_{\rho_\tau})}.$$

This phenomenon of explosion of the norm does not occur near the catenoidal type ends:

$$\left\|(\mathcal{E}_\tau v) \circ X_{t,\tau}\right\|_{\mathcal{C}^{0,\alpha}([s_0,+\infty)\times S^1)} \leq c\|v \circ X_{t,\tau}\|_{\mathcal{C}^{0,\alpha}([s_0,s_\tau]\times S^1)}.$$

A similar equation holds for the bottom end. In the following we will assume $\alpha > 0$ and close to zero.

The existence of a solution $v \in \mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})$ for (27) is a consequence of the following result, which proves that $T(\Phi, \cdot)$ is a contraction mapping.

Proposition 4.4 *Let $\delta \in (1,2)$, $\alpha \in (0,1/4)$ and $\Phi = (\varphi_t, \varphi_m, \varphi_b) \in [\mathcal{C}^{2,\alpha}(S^1)]^3_{\mathrm{sym}}$ satisfying* (26) *and enjoying the properties given above. There exist constants $c_\kappa > 0$ and $\tau_\kappa > 0$, such that*

$$\left\|T(\Phi,0)\right\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})} \leq c_\kappa \tau^{5/2} \tag{28}$$

and, for all $\tau \in (0, \tau_\kappa)$,

$$\begin{aligned}
&\left\|T(\Phi, v_2) - T(\Phi, v_1)\right\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})} \leq c\tau^{3/2} \|v_2 - v_1\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})}, \\
&\left\|T(\Phi_2, v) - T(\Phi_1, v)\right\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})} \leq c\tau^{3/2} \|\Phi_2 - \Phi_1\|_{[\mathcal{C}^{2,\alpha}(S^1)]^3_{\mathrm{sym}}},
\end{aligned}$$

where c is a positive constant, for all $v, v_1, v_2 \in \mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})$ and satisfying $\|v\|_{\mathcal{C}^{2,\alpha}_\delta} \leq 2c_\kappa \tau^{5/2}$ and for all boundary data $\Phi_i = (\varphi_{t,i}, \varphi_{m,i}, \varphi_{b,i}) \in [\mathcal{C}^{2,\alpha}(S^1)]^3_{\mathrm{sym}}$, $i = 1,2$, enjoying the same properties as Φ.

Proof We recall that the Jacobi operator associated to $M_{k,\tau}$, is asymptotic (up to a multiplication by $1/\tau^2$) to the Jacobi operator of the catenoid (respectively, of the plane) plane at the catenoidal ends (respectively, at the planar end). The function w_Φ is identically zero far from the ends where the explicit expression of $\mathbb{L}_{M_{k,\tau}}$ is not known: this is the reason for our particular choice in the definition of w_Φ. Then from the definition of w_Φ and thanks to Proposition 3.1 we obtain the estimate

$$\begin{aligned}
&\bigl\|\mathcal{E}_\tau(\gamma\mathbb{L}_{M_{k,\tau}}w_\Phi)\bigr\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\tau})}\\
&\quad= \bigl\|\bigl(\gamma\mathbb{L}_{M^T_{k,\tau}}-\bigl(\partial_s^2+\partial_\theta^2\bigr)\bigr)(w_\Phi\circ X_{t,\tau})\bigr\|_{\mathcal{C}^{0,\alpha}_\delta([s_0+1,s_\tau]\times S^1)}\\
&\qquad+\bigl\|\bigl(\gamma\mathbb{L}_{M^T_{k,\tau}}-\bigl(\partial_s^2+\partial_\theta^2\bigr)\bigr)(w_\Phi\circ X_{b,\tau})\bigr\|_{\mathcal{C}^{0,\alpha}_\delta([-s_\tau,-s_0-1]\times S^1)}\\
&\qquad+\rho_\tau^{-\alpha}\bigl\|(\gamma\mathbb{L}_{M^T_{k,\tau}}-\Delta_0)(w_\Phi\circ X_{m,\tau})\bigr\|_{\mathcal{C}^{0,\alpha}([\rho_\tau,\rho_0]\times S^1)}\\
&\quad\le c\bigl\|\cosh^{-2}s(w_\Phi\circ X_{t,\tau})\bigr\|_{\mathcal{C}^{0,\alpha}_\delta([s_0+1,s_\tau]\times S^1)}+c\bigl\|\cosh^{-2}s(w_\Phi\circ X_{b,\tau})\bigr\|_{\mathcal{C}^{0,\alpha}_\delta([-s_\tau,-s_0-1]\times S^1)}\\
&\qquad+c\tau^{-\alpha}\bigl\|\rho^{2k+3}\nabla(w_\Phi\circ X_{m,\tau})\bigr\|_{\mathcal{C}^{0,\alpha}([\rho_\tau,\rho_0]\times S^1)}\\
&\quad\le c_\kappa\tau^4+c_\kappa\tau^{5/2}\le c_\kappa\tau^{5/2}.
\end{aligned}$$

To obtain this estimate we used the following ones:

$$\begin{aligned}
&\sup_{[s_0+1,s_\tau]\times S^1} e^{-\delta s}\bigl\|\cosh^{-2}s(w_\Phi\circ X_{t,\tau})\bigr\|_{\mathcal{C}^{0,\alpha}([s,s+1]\times S^1)}\\
&\quad\le c\sup_{[s_0+1,s_\tau]\times S^1} e^{-\delta s}e^{-2(s_\tau-s)}e^{-2s}\|\phi_t\|_{\mathcal{C}^{2,\alpha}(S^1)}\\
&\quad\le ce^{-2s_\tau}\|\phi_t\|_{\mathcal{C}^{2,\alpha}(S^1)}\le c_\kappa\tau^4
\end{aligned}$$

(a similar estimate holds for the bottom end) and

$$\begin{aligned}
&\rho_\tau^{-\alpha}\bigl\|\rho^{2k+3}\nabla(w_\Phi\circ X_{m,\tau})\bigr\|_{\mathcal{C}^{0,\alpha}([\rho_\tau,\rho_0]\times S^1)}\\
&\quad\le c\tau^{-\alpha}\rho_\tau\|\varphi_m\|_{\mathcal{C}^{2,\alpha}(S^1)}\le c_\kappa\tau^{5/2}
\end{aligned}$$

together with the fact that $s_\tau=-\ln\tau+\ln 2+O(\tau)$ and $\rho_\tau=\tau$, from which $e^{-2s_\tau}\le c\tau^2$.

Using the estimates of the coefficients of $\tilde{L}_\tau$ and the definition of γ (see (18)), we obtain

$$\begin{aligned}
\bigl\|\mathcal{E}_\tau(\gamma\tilde{L}_\tau w_\Phi)\bigr\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\tau})}&\le c\tau^2\|w_\Phi\circ X_{t,\tau}\|_{\mathcal{C}^{0,\alpha}_\delta([s_0+1,s_\tau]\times S^1)}\\
&\quad+c\tau^2\|w_\Phi\circ X_{b,\tau}\|_{\mathcal{C}^{0,\alpha}_\delta([-s_\tau,-s_0-1]\times S^1)}\\
&\quad+c\tau^{2-\alpha}\|w_\Phi\circ X_{m,\tau}\|_{\mathcal{C}^{0,\alpha}([\rho_\tau,\rho_0]\times S^1)}\le c_\kappa\tau^{4-\alpha}.
\end{aligned}$$

As for the last term, we recall that the expression of the operator Q_τ depends on the type of end we are considering (see (17) and (11)). We have

$$\bigl\|\mathcal{E}_\tau\bigl(\gamma Q_\tau(w_\Phi)\bigr)\bigr\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\tau})}\le c_\kappa\tau^{5/2}.$$

In fact

$$\begin{aligned}
&\bigl\| \mathcal{E}_\tau\bigl(\gamma Q_\tau(w_\Phi)\bigr)\bigr\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\tau})} \\
&\quad \le c\tau \left\| \frac{w_\Phi}{\tau\cosh s}\circ X_{t,\tau}\right\|^2_{\mathcal{C}^{2,\alpha}_{\delta/2}([s_0+1,s_\tau]\times S^1)} + c\tau \left\| \frac{w_\Phi}{\tau\cosh s}\circ X_{b,\tau}\right\|^2_{\mathcal{C}^{2,\alpha}_{\delta/2}([-s_\tau,-s_0-1]\times S^1)} \\
&\qquad + c\tau^{(1-2\alpha)} \left\| \frac{|x|^2}{\tau} w_\Phi \circ X_{m,\tau}\right\|^2_{\mathcal{C}^{2,\alpha}([\rho_\tau,\rho_0]\times S^1)} \le c_\kappa \tau^{5/2}.
\end{aligned}$$

As for the second estimate, we recall that

$$T(\Phi,\nu) := G_{\tau,\delta}\circ \mathcal{E}_\tau\bigl(\gamma\bigl(-\tilde{L}_\tau(w_\Phi+\nu) - \mathbb{L}_{M_{k,\tau}} w_\Phi - Q_\tau(w_\Phi+\nu)\bigr)\bigr).$$

Then

$$\begin{aligned}
&\bigl\| T(\Phi,\nu_2) - T(\Phi,\nu_1)\bigr\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})} \\
&\quad \le \bigl\| \mathcal{E}_\tau\bigl(\gamma \tilde{L}_\tau(\nu_2-\nu_1)\bigr)\bigr\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\tau})} + \bigl\| \mathcal{E}_\tau\bigl(\gamma\bigl(Q_\tau(w_\Phi+\nu_1) - Q_\tau(w_\Phi+\nu_2)\bigr)\bigr)\bigr\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\tau})}.
\end{aligned}$$

We observe that from the considerations above it follows that

$$\bigl\| \mathcal{E}_\varepsilon\bigl(\gamma \tilde{L}_\tau(\nu_2-\nu_1)\bigr)\bigr\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\tau})} \le c\tau^2 \|\nu_2-\nu_1\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})}$$

and

$$\begin{aligned}
&\bigl\| \mathcal{E}_\tau\bigl(\gamma\bigl(Q_\tau(w_\Phi+\nu_1) - Q_\tau(w_\Phi+\nu_2)\bigr)\bigr)\bigr\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\tau})} \\
&\quad \le c\|\nu_2-\nu_1\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})}\left(\tau \left\| \frac{w_\Phi}{\tau\cosh s}\circ X_{t,\tau}\right\|_{\mathcal{C}^{0,\alpha}([s_0+1,s_\tau]\times S^1)}\right. \\
&\qquad \left. + \tau \left\| \frac{w_\Phi}{\tau\cosh s}\circ X_{b,\tau}\right\|_{\mathcal{C}^{0,\alpha}([-s_\tau,-s_0-1]\times S^1)} + \tau^{1-2\alpha}\left\| \frac{|x|^2}{\tau} w_\Phi\circ X_{m,\tau}\right\|_{\mathcal{C}^{0,\alpha}([\rho_\tau,\rho_0]\times S^1)}\right) \\
&\quad \le c_\kappa \tau^{3/2}\|\nu_2-\nu_1\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})}.
\end{aligned}$$

Then

$$\bigl\| T(\Phi,\nu_2) - T(\Phi,\nu_1)\bigr\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})} \le c\tau^{3/2}\|\nu_2-\nu_1\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})}.$$

To get the last estimate it suffices to observe that

$$\begin{aligned}
&\bigl\| T(\Phi_2,\nu) - T(\Phi_1,\nu)\bigr\|_{\mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})} \\
&\quad \le \bigl\| \mathcal{E}_\tau\bigl(\gamma \tilde{L}_\tau(w_{\Phi_2} - w_{\Phi_1})\bigr)\bigr\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\tau})} + \bigl\| \mathcal{E}_\varepsilon\bigl(\gamma\bigl(Q_\varepsilon(w_{\Phi_2}+\nu) - Q_\tau(w_{\Phi_1}+\nu)\bigr)\bigr)\bigr\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\tau})} \\
&\quad \le c\tau^{3/2}\|\Phi_2-\Phi_1\|_{[\mathcal{C}^{2,\alpha}(S^1)]^3_{\mathrm{sym}}} + c\|\nu\|_{\mathcal{C}^{0,\alpha}_\delta(M_{k,\tau})}\|\Phi_2-\Phi_1\|_{[\mathcal{C}^{2,\alpha}(S^1)]^3_{\mathrm{sym}}} \\
&\quad \le c\tau^{3/2}\|\Phi_2-\Phi_1\|_{[\mathcal{C}^{2,\alpha}(S^1)]^3_{\mathrm{sym}}}.
\end{aligned}$$

□

Theorem 4.5 *Let $\delta \in (1,2)$, $\alpha \in (0,1/4)$ and $B := \{w \in \mathcal{C}^{2,\alpha}_\delta(M_{k,\tau}) \mid \|w\|_{\mathcal{C}^{2,\alpha}_\delta} \le 2c_\kappa \tau^{5/2}\}$. Then the nonlinear mapping $T(\Phi,\cdot)$ defined above has a unique fixed point ν in B.*

Proof The previous lemma shows that, if τ is chosen small enough, the nonlinear mapping $T(\Phi,\cdot)$ is a contraction mapping from the ball B of radius $2c_\kappa\tau^{5/2}$ in $\mathcal{C}^{2,\alpha}_\delta(M_{k,\tau})$ into itself. This value follows from the estimate of the norm of $T(\Phi,0)$. Consequently thanks to Schäuder fixed point theorem, $T(\Phi,\cdot)$ has a unique fixed point w in this ball. □

This argument provides a new surface $M^T_{k,\tau}(\Phi)$ whose mean curvature equals zero, which is close to $M^T_{k,\tau}$ and has three boundary curves.

The surface $M^T_{k,\tau}(\Phi)$ is, close to its upper and lower boundary curve, the graph over the catenoidal ends in the direction given by the vector $\tilde{n}_\tau$ of the functions

$$U_t(r,\theta) = H_{\varphi_t}\left(s_\tau - \ln\frac{2r}{\tau},\theta\right) + V_t(r,\theta),$$
$$U_b(r,\theta) = -U_t\left(r,\theta - \frac{\pi}{k+1}\right),$$

where $s_\tau = -\ln\tau + \ln 2 + O(\tau)$. Nearby the middle boundary the surface is the vertical graph of

$$U_m(r,\theta) = \tilde{H}_{\rho_\tau,\varphi_m}\left(\frac{\tau}{r},\theta\right) + V_m(r,\theta),$$

with $\rho_\tau = \tau$. All the functions V_i, $i = t,b,m$, depend nonlinearly on τ,Φ.

Lemma 4.6 *The function* $V_i(\tau,\varphi_i)$, *for* $i = t,b$, *satisfies* $\|V_i(\tau,\varphi_i)(\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{3/4})} \leq c\tau^{4-\delta}$ *and*

$$\left\|V_i(\tau,\varphi_{i,2})(\cdot,\cdot) - V_i(\tau,\varphi_{i,1})(\cdot,\cdot)\right\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{3/4})} \leq c\tau^{3/2-\delta}\|\varphi_{i,2} - \varphi_{i,1}\|_{\mathcal{C}^{2,\alpha}(S^1)}. \tag{29}$$

The function $V_m(\tau,\varphi_m)$ *satisfies* $\|V_m(\tau,\varphi_m)(\cdot,\cdot)\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{3/4})} \leq c\tau^{5/2}$ *and*

$$\left\|V_m(\tau,\varphi_{m,2})(\cdot,\cdot) - V_m(\tau,\varphi_{m,1})(\cdot,\cdot)\right\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{3/4})} \leq c\tau^{3/2}\|\varphi_{m,2} - \varphi_{m,1}\|_{\mathcal{C}^{2,\alpha}(S^1)}. \tag{30}$$

Proof We recall that the functions V_t, V_b, V_m are the restrictions to $E_{t,\tau}$, $E_{b,\tau}$, $E_{m,\tau}$ of a fixed point v for the operator $T(\Phi,\cdot)$. The estimates of their norm are a consequence of Proposition 4.4. Observe that to derive the estimate of the norm of V_t and V_b we use the better estimate for the norm of the fixed point v which holds at the catenoidal type ends. Precisely stated: $\|v \circ X_i\|_{\mathcal{C}^{2,\alpha}_\delta} \leq c_\kappa\tau^4$ with $i = t,b$. Then (29) follows from

$$\begin{aligned}&\left\|V_i(\tau,\varphi_{i,2})(\cdot,\cdot) - V_i(\tau,\varphi_{i,1})(\cdot,\cdot)\right\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{3/4})}\\ &\quad\leq ce^{\delta s_\tau}\left\|\bigl(T(\Phi_2,V_i) - T(\Phi_1,V_i)\bigr)\circ X_{i,\tau}\right\|_{\mathcal{C}^{2,\alpha}_\delta(\Omega_i\times S^1)},\end{aligned}$$

for $i = t,b$, with $\Omega_t = [s_0,s_\tau]$ and $\Omega_b = [-s_\tau,-s_0]$. To get the estimate (30) we observe that

$$\begin{aligned}&\left\|V_m(\tau,\varphi_{m,2})(\cdot,\cdot) - V_m(\tau,\varphi_{m,1})(\cdot,\cdot)\right\|_{\mathcal{C}^{2,\alpha}(\bar{B}_1 - B_{3/4})}\\ &\quad\leq c\left\|\bigl(T(\Phi_2,V_m) - T(\Phi_1,V_m)\bigr)\circ X_{m,\tau}\right\|_{\mathcal{C}^{2,\alpha}([\rho_\tau,\rho_0]\times S^1)}.\end{aligned}$$

□

Remark 4.7 In next section we will use previous result to prove Theorem 1.1 under the additional assumption $\delta \in (1, 5/4)$. Consequently in (29) it appears a positive power of τ. The previous result can be reformulated as follows: all of the mappings $V_i(\tau, \cdot)$ are contracting. Furthermore the norm $\|V_i\|$ is $O(\tau^{\frac{5}{2}})$.

5 Proof of Theorem 1.1

The surface $M_{k,\tau}^T(\Phi)$ we constructed in previous section, has three boundary curves. Such curves do not lie in the sphere ∂B^3. So we introduce a new surface $\tilde{M}_{k,\tau}^T(\Phi) := M_{k,\tau}^T(\Phi) \cap B^3$.

To prove the main theorem we need to show that there exists Φ such that also the second equation of (3) is satisfied.

We recall that we modified the immersion of $M_{k,\tau}^T$ in $\mathbb{R}^3$ in order to have the normal vector $\tilde{n}_\tau$ to $M_{k,\tau}^T$ equal to the normal vectors $n_c(\pm s_\tau, \cdot)$ in a neighborhood of its top and bottom boundary curves and equal to e_3 in a neighborhood of the middle boundary curve. Precisely, at the catenoidal type ends, from (6), $\tilde{n}_\tau$ in a neighborhood of the boundary curves equals the vector fields (here we use the basis $(\vec{r}, \vec{\theta}, \vec{z})$)

$$n_t = (n_{t,r}, n_{t,\theta}, n_{t,z}) = \left(\frac{1}{\cosh(s_\tau)}, 0, -\frac{\sinh(s_\tau)}{\cosh(s_\tau)} \right),$$
$$n_b = (n_{b,r}, n_{b,\theta}, n_{b,z}) = \left(\frac{1}{\cosh(s_\tau)}, 0, +\frac{\sinh(s_\tau)}{\cosh(s_\tau)} \right).$$

Near the boundary curves, the surface $\tilde{M}_{k,\tau}^T(\Phi)$ is the graph in the direction of the vectors n_i, $i = t, m, b$, over the ends of $M_{k,\tau}^T$ of the functions $U_i(r, \theta)$ for $i = t, m, b$.

As a consequence the top and bottom ends of $M_{k,\tau}^T(\Phi)$, near the boundary curves, can be parametrized as follows:

$$\tilde{U}_t(r, \theta) = \left(r, \theta, \sigma_\tau + \tau \ln\left(\frac{2r}{\tau}\right) + O(\tau^3) \right) + U_t(r, \theta) n_t,$$

where

$$U_t(r, \theta) = H_{\varphi_t}\left(s_\tau - \ln\left(\frac{2r}{\tau}\right), \theta \right) + O(\tau^{\frac{5}{2}}),$$
$$\tilde{U}_b(r, \theta) = \left(r, \theta, -\sigma_\tau - \tau \ln\left(\frac{2r}{\tau}\right) + O(\tau^3) \right) - U_b(r, \theta) n_b,$$

where

$$U_b(r, \theta) = -U_t\left(r, \theta - \frac{\pi}{k+1} \right), \qquad \tilde{U}_m(r, \theta) := \left(r, \theta, U_m(r, \theta) \right).$$

Let $\tilde{r}_i$ be the function of θ defined as

$$\left| \tilde{U}_i(\tilde{r}_i, \theta) \right|^2 = 1. \tag{31}$$

In other terms $\tilde{r}_i(\theta)$ is the value of the r-variable for which $\tilde{U}_i(\tilde{r}_i(\theta), \theta)$ is the parametrization of a curve on the sphere ∂B^3. More precisely it is one of the boundary curves of $\tilde{M}_{k,\tau}^T(\Phi)$.

Using the expression of $\tilde{U}_i$ we get the following estimate:

$$\left\|\tilde{r}_i(\theta) - r_i\right\|_{C^{2,\alpha}(S^1)} \leq c\tau^2, \tag{32}$$

where r_i denotes r_τ for $i = t, b$ and $r_i = 1$ when $i = m$. They are the values taken by r for $s = s_\tau$ and $\rho = \rho_\tau$.

In order to compute a unit normal vector to ∂B^3 along the boundary curves of $\tilde{M}^T_{k,\tau}(\Phi)$ we will consider cylindrical coordinates (r, θ, z). It is clear that the vector $\tilde{\nu} = (r, 0, z)$ is orthogonal to ∂B^3 at the point (r, θ, z). So three unit normal vectors to ∂B^3 along the top (resp. bottom, middle) boundary curve of $\tilde{M}^T_{k,\tau}(\Phi)$ are obtained replacing r by $\tilde{r}_i(\theta)$ and z by $\tilde{U}_i(\theta, \tilde{r}_i)$ in the formula giving $\tilde{\nu}$. We get

$$\tilde{\nu}_t = \left(\tilde{r}_t + U_t(\tilde{r}_t, \theta)n_{t,r}, 0, \sigma_\tau + \tau \ln\left(\frac{2\tilde{r}_t}{\tau}\right) + U_t(\tilde{r}_t, \theta)n_{t,z} + O(\tau^3)\right),$$

$$\tilde{\nu}_b = \left(\tilde{r}_b - U_b(\tilde{r}_b, \theta)n_{b,r}, 0, -\sigma_\tau - \tau \ln\left(\frac{2\tilde{r}_b}{\tau}\right) - U_b(\tilde{r}_b, \theta)n_{b,z} + O(\tau^3)\right),$$

and $\tilde{\nu}_m = (\tilde{r}_m, 0, U_m(\tilde{r}_m, \theta) + O(\tau^3))$.

A non-unit normal vector to $\tilde{M}^T_{k,\tau}(\Phi)$ along its boundary curves is given, in the frame $(\vec{r}, \vec{\theta}, \vec{z})$, by

$$\tilde{N}_m = \left(-\tilde{N}_{m,r}(\theta), -\tilde{N}_{m,\theta}(\theta), 1\right),$$

where

$$\tilde{N}_{m,r}(\theta) = \partial_r U_m(r, \theta)_{|_{r=\tilde{r}_m(\theta)}}, \qquad \tilde{N}_{m,\theta}(\theta) = \partial_\theta U_m(r, \theta)_{|_{r=\tilde{r}_m(\theta)}};$$

$$\tilde{N}_i = (\partial_r \tilde{U}_i \wedge \partial_\theta \tilde{U}_i)_{|_{r=\tilde{r}_i(\theta)}}$$

for $i = t, b$ with

$$\partial_r \tilde{U}_t = \left(1 + \partial_r U_t n_{t,r}, 0, \frac{\tau}{r} + \partial_r U_t n_{t,z} + O(\tau^3)\right),$$

$$\partial_\theta \tilde{U}_t = \left(\partial_\theta U_t n_{t,r}, 1, \partial_\theta U_t n_{t,z} + O(\tau^3)\right),$$

$$\partial_r \tilde{U}_b = \left(1 - \partial_r U_b n_{b,r}, 0, -\frac{\tau}{r} - \partial_r U_b n_{b,z} + O(\tau^3)\right),$$

$$\partial_\theta \tilde{U}_b = \left(-\partial_\theta U_b n_{b,r}, 1, -\partial_\theta U_b n_{b,z} + O(\tau^3)\right).$$

We get

$$\tilde{N}_i = (\tilde{N}_{i,r}, \tilde{N}_{i,\theta}, \tilde{N}_{i,z}),$$

where

$$\tilde{N}_{t,r} = -\frac{\tau}{r} - \partial_r U_t n_{t,z} + O(\tau^3), \qquad \tilde{N}_{t,z} = 1 + \partial_r U_t n_{t,r},$$

$$\tilde{N}_{t,\theta} = -\partial_\theta U_t \left(n_{t,z} - \frac{\tau}{r} n_{t,r}\right) + O(\tau^3)(1 + \partial_r U_t n_{t,r} - \partial_\theta U_t n_{t,r}),$$

$$\tilde{N}_{b,r} = \frac{\tau}{r} + \partial_r U_b n_{b,z} + O(\tau^3), \qquad \tilde{N}_{b,z} = 1 - \partial_r U_b n_{b,r},$$
$$\tilde{N}_{b,\theta} = \partial_\theta U_b \left(n_{b,z} + \frac{\tau}{r} n_{b,r} \right) + O(\tau^3)(1 - \partial_r U_b n_{b,r} + \partial_\theta U_b n_{b,r}).$$

In Section 4 we proved the existence of a family of solutions to the first equation of (3). It remains to show the existence of one solution in such a family which satisfies the other equations in (3). It is clear that the last equations in (3) are equivalent to

$$\begin{cases} A_t := \frac{\tilde{N}_t \cdot \tilde{\nu}_t}{|\tilde{N}_t|} - \frac{N_t \cdot \nu_t}{|N_t|} = 0, \\ A_m := \tilde{N}_m \cdot \tilde{\nu}_m = 0, \\ A_b := \frac{\tilde{N}_b \cdot \tilde{\nu}_b}{|\tilde{N}_b|} - \frac{N_b \cdot \nu_b}{|N_b|} = 0. \end{cases} \tag{33}$$

$|\nu|$ denotes the length of the vector ν. If Γ_t, Γ_b are the intersection curves of the asymptotic halves catenoid and ∂B^3, then N_t, N_b and ν_t, ν_b are, respectively, the normal vectors to the asymptotic halves catenoid and to ∂B^3 along Γ_t, Γ_b. We observe that ν_t, ν_b have unit length. Such normal vectors can be computed as done for $\tilde{N}_t$, $\tilde{N}_b$, and $\tilde{\nu}_t$, $\tilde{\nu}_b$.

The computation of the scalar product yields

$$\tilde{N}_t \cdot \tilde{\nu}_{t|_{r=\tilde{r}_t(\theta)}} = \left[-\tau + \sigma_\tau + \tau \ln\left(\frac{2r}{\tau}\right) + O(\tau^3) + (U_t - r\partial_r U_t) n_{t,z} - \frac{\tau}{r} U_t n_{t,r} + \left(\sigma_\tau + \tau \ln\left(\frac{2r}{\tau}\right) \right) \partial_r U_t n_{t,r} \right]_{|_{r=\tilde{r}_t(\theta)}},$$
$$\tilde{N}_b \cdot \tilde{\nu}_{b|_{r=\tilde{r}_b(\theta)}} = \left[\tau - \sigma_\tau - \tau \ln\left(\frac{2r}{\tau}\right) + O(\tau^3) - (U_b - r\partial_r U_b) n_{b,z} - \frac{\tau}{r} U_b n_{b,r} + \left(\sigma_\tau + \tau \ln\left(\frac{2r}{\tau}\right) \right) \partial_r U_b n_{b,r} \right]_{|_{r=\tilde{r}_b(\theta)}}.$$

We can compute $N_i \cdot \nu_i$ by using previous formula: indeed it suffices to assume $U_i = 0$ and to replace $\tilde{r}_i(\theta)$ by r_i. We get

$$N_t \cdot \nu_{t|_{r=r_t}} = -\tau + \sigma_\tau + \tau \ln\left(\frac{2r_t}{\tau}\right) + O(\tau^3), \qquad N_b \cdot \nu_{b|_{r=r_b}} = \tau - \sigma_\tau - \tau \ln\left(\frac{2r_b}{\tau}\right) + O(\tau^3).$$

The square of the length of the normal vectors $\tilde{N}_t$, $\tilde{N}_b$ are

$$|\tilde{N}_t|^2 = \left(1 + \frac{\tau^2}{r^2} + 2\partial_r U_t \left(n_{t,r} + \frac{\tau}{r} n_{t,z} \right) + (\partial_r U_t)^2 + (\partial_\theta U_t)^2 \left(n_{t,z} - \frac{\tau}{r} n_{t,r} \right)^2 + O(\tau^3) \right)_{r=\tilde{r}_t(\theta)},$$
$$|\tilde{N}_b|^2 = \left(1 + \frac{\tau^2}{r^2} + 2\partial_r U_b \left(-n_{b,r} + \frac{\tau}{r} n_{b,z} \right) + (\partial_r U_b)^2 + (\partial_\theta U_b)^2 \left(n_{b,z} + \frac{\tau}{r} n_{b,r} \right)^2 + O(\tau^3) \right)_{r=\tilde{r}_b(\theta)}.$$

By construction of U_t, U_b and the fact that $n_{t,r}, n_{b,r} = O(\tau^2)$, it follows that $|\tilde{N}_t|^2$, $|\tilde{N}_b|^2$ can be estimated as $1 + \frac{\tau^2}{\tilde{r}_t^2} + O(\tau^3)$ and $1 + \frac{\tau^2}{\tilde{r}_b^2} + O(\tau^3)$, respectively. If we replace $\tilde{r}_t(\theta)$, $\tilde{r}_b(\theta)$

by $r_t = r_b$, and we set $U_t = U_b = 0$, we get the values of $|N_t|^2$, $|N_b|^2$. In conclusion $|\tilde{N}_t|^2$, $|\tilde{N}_b|^2$ are small perturbations of $|N_t|^2$, $|N_b|^2$.

The equation $A_t = \frac{\tilde{N}_t \cdot \tilde{\nu}_t}{|\tilde{N}_t|} - \frac{N_t \cdot \nu_t}{|N_t|} = 0$ is equivalent to $(\tilde{N}_t \cdot \tilde{\nu}_t)|N_t| = (N_t \cdot \nu_t)|\tilde{N}_t|$. In view of previous observations this last equation can be seen as a small perturbation of the simpler equation $|N_t|(\tilde{N}_t \cdot \tilde{\nu}_t - N_t \cdot \nu_t) = 0$.

The advantage of solving $\tilde{N}_t \cdot \tilde{\nu}_t = N_t \cdot \nu_t$ is that it reduces to

$$T_t(\varphi_t) = \biggl[(U_t - r\partial_r U_t)n_{t,z} + O(\tau^3) - \frac{\tau}{r}U_t n_{t,r} + \biggl(\sigma_\tau + \tau\ln\biggl(\frac{2r}{\tau}\biggr)\biggr)\partial_r U_t n_{t,r} + \tau\ln\biggl(\frac{r}{r_t}\biggr)\biggr]_{|r=\tilde{r}_t(\theta)} = 0.$$

Similarly, instead of solving $A_b = 0$, we consider the simpler equation $\tilde{N}_b \cdot \tilde{\nu}_b = N_b \cdot \nu_b$, which reduces to

$$T_b(\varphi_b) = \biggl[-(U_b - r\partial_r U_b)n_{b,z} + O(\tau^3) - \frac{\tau}{r}U_b n_{b,r} + \biggl(\sigma_\tau + \tau\ln\biggl(\frac{2r}{\tau}\biggr)\biggr)\partial_r U_b n_{b,r} + \tau\ln\biggl(\frac{r_b}{r}\biggr)\biggr]_{|r=\tilde{r}_b(\theta)} = 0.$$

The equation $\tilde{N}_m \cdot \tilde{\nu}_m = 0$ is equivalent to

$$T_m(\varphi_m) = \bigl[U_m - r\partial_r U_m + O(\tau^3)\bigr]_{|r=\tilde{r}_m(\theta)} = 0.$$

To establish the proof we need to find a more explicit expression of $\partial_r U_i$. We get easily

$$\partial_r U_t(r,\theta) = \partial_r H_{\varphi_t}\biggl(s_\tau - \ln\frac{2r}{\tau},\theta\biggr) + O_{\mathcal{C}_b^{1,\alpha}}\bigl(\tau^{\frac{5}{2}}\bigr) = -\frac{1}{r}\partial_s H_{\varphi_t}(s,\theta)_{|s=s_\tau-\ln\frac{2r}{\tau}} + O_{\mathcal{C}_b^{1,\alpha}}\bigl(\tau^{\frac{5}{2}}\bigr),$$

$$\partial_r U_b(r,\theta) = -\partial_r U_t\biggl(r,\theta - \frac{\pi}{k+1}\biggr),$$

$$\partial_r U_m(r,\theta) = -\frac{\tau}{r^2}\partial_\rho \tilde{H}_{\rho_\tau,\varphi_m}(\rho,\theta)_{|\rho=\frac{\tau}{r}} + O_{\mathcal{C}_b^{1,\alpha}}\bigl(\tau^{\frac{5}{2}}\bigr).$$

Observe that if we evaluate first two functions at $r = r_\tau (= r_t = r_b)$ (the value taken by r if $s = s_\tau$) and third one at $r = 1 (= r_m)$ (the value taken by r if $\rho = \rho_\tau$) then we get

$$\partial_r U_t(r,\theta)_{|r=r_\tau} = -\frac{1}{r_\tau}\partial^* \varphi_t(\theta) + O_{\mathcal{C}_b^{1,\alpha}}\bigl(\tau^{\frac{5}{2}}\bigr),$$

$$\partial_r U_m(r,\theta)_{|r=1} = -\partial^* \varphi_m(\theta) + O_{\mathcal{C}_b^{1,\alpha}}\bigl(\tau^{\frac{5}{2}}\bigr),$$

$$\partial_r U_b(r,\theta)_{|r=r_\tau} = -\frac{1}{r_\tau}\partial^* \varphi_b(\theta) + O_{\mathcal{C}_b^{1,\alpha}}\bigl(\tau^{\frac{5}{2}}\bigr),$$

where ∂^* is the operator defined as follows. If $\phi = \sum_{j\geq 1}\phi_j\cos(j\theta)$, then

$$\partial^*\phi = -\sum_{j\geq 1} j\phi_j\cos(j\theta).$$

Let us consider the operator

$$A_\tau : \left[\mathcal{C}^{2,\alpha}(S^1)\right]^3_{\mathrm{sym}} \to \left[\mathcal{C}^{1,\alpha}(S^1)\right]^3_{\mathrm{sym}},$$
$$\Phi = (\varphi_t, \varphi_m, \varphi_b) \to \big(A_t(\varphi_t), A_m(\varphi_m), A_b(\varphi_b)\big),$$

see (33).

The definition of the space $[\mathcal{C}^{1,\alpha}(S^1)]^3_{\mathrm{sym}}$ is similar to definition 4.2, with the unique difference of the lower regularity.

We want to show the existence of a solution of $A_\tau(\Phi) = 0$.

We define the operator $T_\tau : [\mathcal{C}^{2,\alpha}(S^1)]^3_{\mathrm{sym}} \to [\mathcal{C}^{1,\alpha}(S^1)]^3_{\mathrm{sym}}$ by

$$\Phi \to \big(T_t(\varphi_t), T_m(\varphi_m), T_b(\varphi_b)\big).$$

Proposition 5.1 *There exists* $\kappa_0 > 0$ *such that if* $\kappa > \kappa_0$ *then there exists* $\tau_0 > 0$ *for which, for each* $\tau \in (0, \tau_0)$, *then* $A_\tau(\Phi) = 0$ *has a solution in* $\mathcal{B}_\kappa$, *the ball centered at* $(0,0,0)$ *and of radius* $\kappa\tau^2$ *in* $[\mathcal{C}^{2,\alpha}(S^1)]^3_{\mathrm{sym}}$.

Proof Let us consider the operator

$$I_\tau : \left[\mathcal{C}^{2,\alpha}(S^1)\right]^3_{\mathrm{sym}} \to \left[\mathcal{C}^{1,\alpha}(S^1)\right]^3_{\mathrm{sym}},$$
$$\Phi \to \left[(\varphi_t + \partial^*\varphi_t)n_{t,z} + O(\tau^3), \varphi_m + \partial^*\varphi_m + O(\tau^3), -(\varphi_b + \partial^*\varphi_b)n_{b,z} + O(\tau^3)\right].$$

The operator I_τ can be seen as an approximation of T_τ: indeed we get I_τ from T_τ omitting some nonlinear terms and evaluating the remaining ones at r_i instead of $\tilde{r}_i(\theta)$.

Equation $I_\tau(\Phi) = 0$ has a unique solution, because the operator $\mathrm{Id} + \partial^* : H^1(S^1) \to L^2(S^1)$ is easily seen to be invertible. By elliptic regularity theory this result extends to the operator

$$\mathrm{Id} + \partial^* : \mathcal{C}^{2,\alpha}(S^1) \to \mathcal{C}^{1,\alpha}(S^1).$$

From (32) and Lemma 4.6 we obtain $\|(A_\tau - I_\tau)(\Psi)\|_{[\mathcal{C}^{1,\alpha}(S^1)]^3_{\mathrm{sym}}} \le c\tau^{5/2}$, for any $\Psi \in \mathcal{B}_\kappa \subset [\mathcal{C}^{1,\alpha}(S^1)]^3_{\mathrm{sym}}$. We would like to show existence of a solution to $A_\tau(\Phi) = 0$ by the Leray-Schauder degree theory but the nonlinear operator $I_\tau - A_\tau$ is not compact. We apply the same technique as in Proposition 15 of [25].

Let us introduce a family of smoothing operators S^q, for $q \in (0,1)$, defined by

$$S^q(\psi_1, \psi_2, \psi_3) := \big(D^q\psi_1, D^q\psi_2, D^q\psi_3\big)$$

with

$$D^q : \sum_{i\ge 1} a_i \cos(i\theta) \to \sum_{i\ge 1} i^{-q} a_i \cos(i\theta)$$

for $(\psi_1, \psi_2, \psi_3) \in [\mathcal{C}^{1,\alpha}(S^1)]^3$. The operator S^q satisfies for fixed $0 < \alpha' < \alpha < 1$

$$\begin{aligned} &\|S^q\Psi\|_{[\mathcal{C}^{1,\alpha}(S^1)]^3} \le c\|\Psi\|_{[\mathcal{C}^{1,\alpha}(S^1)]^3}, \\ &\|\Psi - S^q\Psi\|_{[\mathcal{C}^{1,\alpha'}(S^1)]^3} \le cq^{\alpha'-\alpha}\|\Psi\|_{[\mathcal{C}^{1,\alpha}(S^1)]^3}, \end{aligned} \tag{34}$$

where c does not depend on q.

We approximate A_τ by the family of compact operators A_τ^q defined as follows:

$$A_\tau^q := I_\tau + S^q \circ (A_\tau - I_\tau).$$

Now we can apply Leray-Schauder degree theory to prove the existence of a solution Φ_q to $A_\tau^q(\Phi) = 0$ in $\mathcal{B}_\kappa$ for $\tau \in (0, \tau_0)$, with τ_0 small enough and $\kappa > \kappa_0$ with κ_0 chosen large enough.

Since the norm of Φ_q is bounded uniformly in q, we can extract a sequence $\{q_j\}$ converging to 0 such that $\{\Phi_{q_j}\}$ converges in $[\mathcal{C}^{2,\alpha'}(S^1)]^3_{\mathrm{sym}}$ for any fixed $\alpha' < \alpha$. Thanks to the continuity of A_τ^q and to (34), the limit of this sequence converges to a solution of $A_\tau(\Phi) = 0$ for all $\tau \in (0, \tau_0)$. □

The zero of A_τ provides the boundary data Φ for which the surface $\tilde{M}_{k,\tau}^T(\Phi)$ meets ∂B^3 in order to make (33) satisfied. That finishes the proof of Theorem 1.1.

Appendix

Results in this section are about the existence of some harmonic extension operators.

The following result gives a harmonic extension of a function on $\mathbb{R}^2 \setminus D_{\bar{\rho}}$.

Proposition A.1 *There exists an operator*

$$\tilde{H}_{\bar{\rho}} : C^{2,\alpha}\big(S^1\big) \longrightarrow C^{2,\alpha}\big(S^1 \times [\bar{\rho}, +\infty)\big),$$

such that for each even function $\varphi(\theta) \in C^{2,\alpha}(S^1)$, which is L^2-orthogonal to the constant function then $w = \tilde{H}_{\bar{\rho},\varphi}$ solves

$$\begin{cases} \Delta w = 0 & \text{on } S^1 \times [\bar{\rho}, +\infty), \\ w = \varphi & \text{on } S^1 \times \{\bar{\rho}\}. \end{cases}$$

Moreover,

$$\|\tilde{H}_{\bar{\rho},\varphi}\|_{C^{2,\alpha}(S^1 \times [\bar{\rho},+\infty))} \leq c\|\varphi\|_{C^{2,\alpha}(S^1)}, \tag{35}$$

for some constant $c > 0$.

Proof We consider the decomposition of the function φ with respect to the basis $\{\cos(i\theta)\}$, that is,

$$\varphi = \sum_{i=1}^{\infty} \varphi_i \cos(i\theta).$$

Then the solution w_φ is given by

$$w(\rho, \theta) = \sum_{i=1}^{\infty} \left(\frac{\bar{\rho}}{\rho}\right)^i \varphi_i \cos(i\theta).$$

Since $\frac{\bar{\rho}}{\rho} \leq 1$, then $(\frac{\bar{\rho}}{\rho})^i \leq (\frac{\bar{\rho}}{\rho})$, we can conclude that $|w(\theta,\rho)| \leq c|\varphi(\theta)|$ and then $\|w\|_{C^{2,\alpha}} \leq c\|\varphi\|_{C^{2,\alpha}}$. □

Proposition A.2 *There exists an operator*

$$H : \mathcal{C}^{2,\alpha}(S^1) \longrightarrow \mathcal{C}^{2,\alpha}_{-2}(S^1 \times [0,+\infty)),$$

such that for all $\varphi \in \mathcal{C}^{2,\alpha}(S^1)$, *even function and orthogonal to* e_i, $i = 0, 1$ *in the* L^2*-sense, the function* $w = H_\varphi$ *solves*

$$\begin{cases} (\partial_s^2 + \partial_\theta^2)w = 0 & \text{in } S^1 \times [0,+\infty), \\ w = \varphi & \text{on } S^1 \times \{0\}. \end{cases}$$

Moreover

$$\|H_\varphi\|_{\mathcal{C}^{2,\alpha}_{-2}(S^1\times[0,+\infty))} \leq c\|\varphi\|_{\mathcal{C}^{2,\alpha}(S^1)},$$

for some constant $c > 0$.

The proof is immediate once we observe that, if $\varphi = \sum_{j\geq 2} \varphi_j \cos(j\theta)$, then the solution is $H_\varphi = \sum_{j\geq 2} e^{-js}\varphi_j \cos(j\theta)$.

Competing interests

The author declares that he has no competing interests.

Acknowledgements

This research was supported by Basic Science Research Program through the National Research Foundation of South Korea (NRF) funded by the Ministry of Education, Grant NRF-2013R1A1A1013521.

Received: 16 December 2013 Accepted: 30 April 2014 Published: 22 May 2014

References

1. Finn, R: Capillary surface interfaces. Not. Am. Math. Soc. **46**(7), 770-781 (1999)
2. Gerhardt, C: Global regularity of the solutions to the capillarity problem. Ann. Sc. Norm. Super. Pisa, Cl. Sci. **3**(4), 157-175 (1976)
3. Lieberman, G: Gradient estimates for capillary-type problems via the maximum principle. Commun. Partial Differ. Equ. **13**(1), 33-59 (1988)
4. Simon, L, Spruck, J: Existence and regularity of a capillary surface with prescribed contact angle. Arch. Ration. Mech. Anal. **61**(1), 19-34 (1976)
5. Spruck, J: On the existence of a capillary surface with prescribed contact angle. Commun. Pure Appl. Math. **28**, 189-200 (1975)
6. Uraltseva, N: Solvability of the capillary problem. Vestnik Leningrad. Univ. n. 19 (1973), 54-64, n. 1 (1975)
7. Aori, Y, Siegel, D: Bounded and unbounded capillary surfaces in a cusp domain. Pac. J. Math. **257**(1), 143-165 (2012)
8. Athanassenas, M, Lancaster, D: Cmc capillary surfaces at reentrant corners. Pac. J. Math. **234**(2), 201-228 (2008)
9. Chen, JT, Finn, R, Miesermann, E: Capillary surfaces in wedge domains. Pac. J. Math. **236**(2), 283-305 (2008)
10. Huff, R, McCuan, J: Scherk-type capillary graphs. J. Math. Fluid Mech. **8**, 99-119 (2006)
11. Calle, M, Shahriyari, L: Existence of a capillary surface with prescribed contact angle in $\mathbb{M} \times \mathbb{R}$. Preprint. arXiv:1012.5490v1
12. Lira, JH, Wanderley, GA: Existence of nonparametric solutions for a capillary problem in warped products. Trans. Am. Math. Soc. (to appear)
13. Fall, MM, Mercuri, C: Minimal disc-type surfaces embedded in a perturbed cylinder. Differ. Integral Equ. **22**(11-12), 1115-1124 (2009)
14. Fall, MM, Mercuri, C: Foliations of small tubes in Riemannian manifolds by capillary minimal discs. Nonlinear Anal. **70**(12), 4422-4440 (2009)
15. Fall, MM: Embedded disc-type surfaces with large constant mean curvature and free boundaries. Commun. Contemp. Math. **14**(6), 1250037 (2012)
16. Fall, MM, Mahmoudi, F: Hypersurfaces with free boundary and large constant mean curvature: concentration along submanifolds. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **7**(3), 407-446 (2008)
17. Ros, A, Souam, R: On stability of capillary surfaces in a ball. Pac. J. Math. **178**(2), 345-361 (1997)

18. Fraser, A, Schoen, R: The first Steklov eigenvalue, conformal geometry, and minimal surfaces. Adv. Math. **226**, 4011-4030 (2011)
19. Fraser, A, Schoen, R: Minimal surfaces and eigenvalue problems. arXiv:1304.0851v1
20. Hauswirth, L, Pacard, F: Higher genus Riemann minimal surfaces. Invent. Math. **169**(3), 569-620 (2007)
21. Morabito, F: A Costa-Hoffman-Meeks type surface in $\mathbb{H}^2 \times \mathbb{R}$. Trans. Am. Math. Soc. **363**(1), 1-36 (2011)
22. Hoffman, D, Meeks, WH III: Embedded minimal surfaces of finite topology. Ann. Math. **131**, 1-34 (1990)
23. Morabito, F: Index and nullity of the Gauss map of the Costa-Hoffman-Meeks surfaces. Indiana Univ. Math. J. **58**(2), 677-707 (2009)
24. Nayatani, S: Morse Index of Complete Minimal Surfaces. In: Rassias, TM (ed.) The Problem of Plateau, pp. 181-189 (1992)
25. Mazzeo, R, Pacard, F, Pollack, D: Connected sums of constant mean curvature surfaces in Euclidean 3 space. J. Reine Angew. Math. **536**, 115-165 (2001)

10.1186/1687-2770-2014-130
Cite this article as: Morabito: **Higher genus capillary surfaces in the unit ball of $\mathbb{R}^3$**. *Boundary Value Problems* 2014, 2014:130