Jung and Choi *Boundary Value Problems* 2014, 2014:30
http://www.boundaryvalueproblems.com/content/2014/1/30

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Nonlinear biharmonic boundary value problem

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Abstract

We consider the nonlinear biharmonic equation with variable coefficient and polynomial growth nonlinearity and Dirichlet boundary condition. We get two theorems. One theorem says that there exists at least one bounded solution under some condition. The other one says that there exist at least two solutions, one of which is a bounded solution and the other of which has a large norm under some condition. We obtain this result by the variational method, generalized mountain pass geometry and the critical point theory of the associated functional.

MSC: 35J20; 35J25; 35Q72

Keywords: biharmonic boundary value problem; polynomial growth; variational method; generalized mountain pass geometry; critical point theory; (*PS*) condition

1 Introduction

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$ and $L^2(\Omega)$ be a square integrable function space defined on Ω. Let Δ be the elliptic operator and Δ^2 be the biharmonic operator. Let $c \in R$. In this paper we study the following nonlinear biharmonic equation with Dirichlet boundary condition:

$$\begin{aligned} &\Delta^2 u + c\Delta u = a(x)g(u) \quad \text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $a : \bar{\Omega} \to R$ is a continuous function which changes sign in Ω.

We assume that g satisfies the following conditions:

(g1) $g \in C(R, R)$,

(g2) there are constants $a_1, a_2 \ge 0$ such that

$$|g(u)| \le a_1 + a_2|u|^{\mu-1},$$

where $2 < \mu < \frac{2n}{n-2}$ if $n \ge 3$,

(g3) there exists a constant $r_0 \ge 0$ such that

$$0 < \mu G(\xi) = \mu \int_0^\xi g(t)\,dt \le \xi g(\xi) \quad \text{for } |\xi| \ge r_0,$$

(g4) $g(u) = o(|u|)$ as $u \to 0$.

We note that (g3) implies the existence of the positive constants a_3, a_4, a_5 such that

$$\frac{1}{\mu}\left(\xi g(\xi)+a_3\right)\geq G(\xi)+a_4\geq a_5|\xi|^{\mu}\quad \text{for } \xi\in R. \tag{1.2}$$

Remark 1.1 The real number ξ in the definition (g3) is not automatically nonnegative. The reason is as follows.

Since $0 < \mu G(\xi) < \xi g(\xi)$ and $\mu > 2$, $G(\xi) > 0$ and $\xi g(\xi) > 0$. By $\xi g(\xi) > 0$, we have two cases: one case is that $\xi > 0$ and $g(\xi) > 0$. The other case is that $\xi < 0$ and $g(\xi) < 0$. Thus ξ is not nonnegative.

Remark 1.2 We obtain the boundedness of $\frac{1}{2}g(u)u - G(u)$ as follows.

By the condition (g3), $\frac{1}{2}g(u)u \geq \frac{1}{2}\mu G(u)$ for $|u| \geq r_0$. Since $\mu > 2$, $\frac{1}{2} - \frac{1}{\mu} > 0$, and $G(u) + a_4 \geq a_5|u|^{\mu}$ in (1.2),

$$\begin{aligned}\frac{1}{2}g(u)u-G(u)&\geq\frac{1}{2}\mu G(u)-G(u)\\&=\mu\left(\frac{1}{2}-\frac{1}{\mu}\right)G(u)\\&\geq\mu\left(\frac{1}{2}-\frac{1}{\mu}\right)\left(a_5|u|^{\mu}-a_4\right).\end{aligned}$$

Thus we obtain the boundedness of $\frac{1}{2}g(u)u - G(u)$.

Remark 1.3 (i) Assumption (g4) implies that (1.1) has a trivial solution.

(ii) If $n = 1$, (g2) can be dropped. If $n = 2$, it suffices that

$$\left|g(u)\right|\leq a_1\exp p(\xi),$$

where $p(\xi)\xi^{-2} \to 0$ as $|\xi| \to \infty$.

(iii) If $n \geq 3$ and $g(\xi) = \xi^{+1+\epsilon}$, where $\xi^+ = \max\{\xi, 0\}$ and $\epsilon > 0$ is a small number, then (g1)-(g4) are satisfied.

The eigenvalue problem

$$\begin{aligned}&\Delta u+\lambda u=0\quad \text{in } \Omega,\\&u=0\quad \text{on } \partial\Omega\end{aligned}$$

has infinitely many eigenvalues λ_k, $k \geq 1$, and corresponding eigenfunctions ϕ_k, $k \geq 1$, suitably normalized with respect to the $L^2(\Omega)$ inner product, where each eigenvalue λ_k is repeated as often as its multiplicity. The eigenvalue problem

$$\begin{aligned}&\Delta^2 u+c\Delta u=\Lambda u\quad \text{in } \Omega,\\&u=0,\qquad \Delta u=0\quad \text{on } \partial\Omega\end{aligned}$$

has also infinitely many eigenvalues $\lambda_k(\lambda_k - c)$, $k \geq 1$ and corresponding eigenfunctions ϕ_k, $k \geq 1$. We note that $\lambda_1(\lambda_1 - c) \leq \lambda_2(\lambda_2 - c) \leq \cdots \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$.

Khanfir and Lassoued [1] showed the existence of at least one solution for the nonlinear elliptic boundary problem when g is locally Hölder continuous on R_+. Choi and Jung [2] showed that the problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = bu^+ + s \quad \text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.3}$$

has at least two nontrivial solutions when ($c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$) or ($\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$). The authors obtained these results by using the variational reduction method. The authors [3] also proved that when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$ and $s < 0$, (1.3) has at least three nontrivial solutions by using degree theory. Tarantello [4] also studied

$$\begin{aligned} &\Delta^2 u + c\Delta u = b\big((u+1)^+ - 1\big), \\ &u = 0, \qquad \Delta u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

She showed that if $c < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - c)$, then (1.4) has a negative solution. She obtained this result by degree theory. Micheletti and Pistoia [5] also proved that if $c < \lambda_1$ and $b \geq \lambda_2(\lambda_2 - c)$ then (1.4) has at least four solutions by the variational linking theorem and Leray-Schauder degree theory.

In this paper we are trying to find the weak solutions of (1.1), that is,

$$\int_\Omega \big(\Delta^2 u + c\Delta u - a(x)g(u)\big)v\,dx = 0 \quad \text{for any } v \in H,$$

where the space H is introduced in Section 2. Let us set

$$\Omega^+ = \big\{x \in \Omega \mid a(x) > 0\big\}, \qquad \Omega^- = \big\{x \in \Omega \mid a(x) < 0\big\}$$

and let

$$a^+ = a \cdot \chi_{\Omega^+}, \qquad a^- = -a \cdot \chi_{\Omega^-}.$$

Since $a(x)$ changes sign, the open subsets Ω^+ and Ω^- are nonempty. Now we can write $a = a^+ - a^-$. Our main results are as follows.

Theorem 1.1 *Assume that* $\lambda_k < c < \lambda_{k+1}$, g *satisfies* (g1)-(g4) *and* $g(u)u - \mu G(u)$ *is bounded. Then* (1.1) *has at least one bounded nontrivial solution.*

Theorem 1.2 *Assume that* $\lambda_k < c < \lambda_{k+1}$, g *satisfies* (g1)-(g4), $g(u)u - \mu G(u)$ *is not bounded and there exists a small* $\epsilon > 0$ *such that* $\int_{\Omega^-} a^-(x)\,dx < \epsilon$. *Then* (1.1) *has at least two solutions*, (i) *one of which is nontrivial and bounded, and* (ii) *the other of which has a large norm such that*

$$\max_{x\in\Omega} \big|u(x)\big| > M \quad \textit{for some } M.$$

The outline of Theorem 1.1 and Theorem 1.2 is as follows: In Section 2, we prove that the corresponding functional $I(u)$ of (1.1), which is introduced in (2.1), is continuous and

Fréchet differentiable and satisfies the (PS) condition. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2 by the variational method, the generalized mountain pass geometry and the critical point theory.

2 Palais-Smale condition

Any element u in $L^2(\Omega)$ can be written as

$$u = \sum h_k \phi_k \quad \text{with } \sum h_k^2 < \infty.$$

We define a subspace H of $L^2(\Omega)$ as follows:

$$H = \left\{ u \in L^2(\Omega) \mid \sum \left|\lambda_k(\lambda_k - c)\right| h_k^2 < \infty \right\}.$$

Then this is a Banach space with a norm

$$\|u\| = \left[\sum \left|\lambda_k(\lambda_k - c)\right| h_k^2 \right]^{\frac{1}{2}}.$$

Since $\lambda_k \to +\infty$ and c is fixed, we have

(i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$,

(ii) $\|u\| \geq C\|u\|_{L^2(\Omega)}$, for some $C > 0$,

(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $\|u\| = 0$,

which are proved in [6].

Let

$$H_+ = \left\{ u \in H \mid h_k = 0 \text{ if } \lambda_k(\lambda_k - c) < 0 \right\},$$
$$H_- = \left\{ u \in H \mid h_k = 0 \text{ if } \lambda_k(\lambda_k - c) > 0 \right\}.$$

Then $H = H_- \oplus H_+$. Let P_+ be the orthogonal projection on H_+ and P_- be the orthogonal projection on H_-.

We are looking for the weak solutions of (1.1). The weak solutions of (1.1) coincide with the critical points of the associated functional

$$\begin{aligned} I(u) &\in C^1(H, R), \\ I(u) &= \int_\Omega \left[\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \int_\Omega a(x)G(u) \right] dx \qquad (2.1) \\ &= \frac{1}{2}\left(\|P_+ u\|^2 - \|P_- u\|^2 \right) - \int_\Omega a(x)G(u)\, dx. \end{aligned}$$

By (g1) and (g2), I is well defined. By Proposition 2.1, $I \in C^1(H,R)$ and I is Fréchet differentiable in H.

Proposition 2.1 *Assume that* $\lambda_k < c < \lambda_{k+1}$, $k \geq 1$, *and that* g *satisfies* (g1)-(g4). *Then* $I(u)$ *is continuous and Fréchet differentiable in* H *with Fréchet derivative*

$$I'(u)h = \int_\Omega \left[\Delta u \cdot \Delta h - c\nabla u \cdot \nabla h - a(x)g(u)h \right] dx. \qquad (2.2)$$

If we set

$$K(u) = \int_\Omega a(x)G(u)\,dx,$$

then $K'(u)$ *is continuous with respect to weak convergence,* $K'(u)$ *is compact, and*

$$K'(u)h = \int_\Omega a(x)g(u)h\,dx \quad \textit{for all } h \in H.$$

This implies that $I \in C^1(H,R)$ *and* $K(u)$ *is weakly continuous.*

The proof of Proposition 2.1 is the same as that of Appendix B in [7].

Proposition 2.2 (Palais-Smale condition) *Assume that* $\lambda_k < c < \lambda_{k+1}$, $k \geq 1$, *and g satisfies* (g1)-(g4). *We also assume that* $g(u)u - \mu G(u)$ *is bounded or that there exists an* $\epsilon > 0$ *such that* $\int_{\Omega^-} a^-(x)\,dx < \epsilon$. *Then* $I(u)$ *satisfies the Palais-Smale condition.*

Proof Suppose that (u_m) is a sequence with $I(u_m) \leq M$ and $I'(u_m) \to 0$ as $m \to \infty$. Then by (g2), (g3), and the Hölder inequality and the Sobolev Embedding Theorem, for large m and $\mu > 2$ with $u = u_m$, we have

$$\begin{aligned}
M + \frac{1}{2}\|u\| \geq I(u) - \frac{1}{2}I'(u)u &= \int_\Omega \left[\frac{1}{2}a(x)g(u)u - a(x)G(u)\right] dx \\
&= \int_\Omega a^+(x)\left[\frac{1}{2}g(u)u - G(u)\right] dx - \int_\Omega a^-(x)\left[\frac{1}{2}g(u)u - G(u)\right] dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\mu \int_\Omega a^+(x) \cdot G(u)\,dx - \max_\Omega \left|\frac{1}{2}g(u)u - G(u)\right| \int_{\Omega^-} a^-(x)\,dx \\
&\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\mu \int_\Omega a^+(x) \cdot \left(a_5|u|^\mu - a_4\right) dx \\
&\quad - \max_\Omega \left|\frac{1}{2}g(u)u - G(u)\right| \int_{\Omega^-} a^-(x)\,dx.
\end{aligned}$$

Since $\frac{1}{2}g(u)u - G(u)$ is bounded or there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x) < \epsilon$; we have

$$1 + \|u\| \geq M_1 \int_\Omega |u|^\mu\,dx \geq M_2 \left(\int_\Omega |u|^2\,dx\right)^{\frac{1}{2}\cdot\mu}. \tag{2.3}$$

Moreover since

$$\left|I'(u_m)\varphi\right| \leq \|\varphi\| \tag{2.4}$$

for large m and all $\varphi \in H$, choosing $\varphi = P_+u_m \in H_+$ gives

$$\begin{aligned}
\|P_+u_m\|^2 &= \int_\Omega \left(\Delta^2 u_m + c\Delta u_m\right) \cdot P_+u_m\,dx \\
&= \int_\Omega a(x)g(u_m)P_+u_m\,dx \\
&\leq \int_\Omega \left|a(x)\right|\left|g(u_m)\right||u_m|\,dx
\end{aligned}$$

$$\begin{aligned}
&\le \|a\|_\infty \int_\Omega \bigl(a_1|u_m| + a_2|u_m|^\mu\bigr)\,dx \\
&\le C_1 \int_\Omega |u_m|^\mu\,dx + C_2\|u_m\|_{L^2(\Omega)} \\
&\le C_1 \int_\Omega |u_m|^\mu\,dx + C_2'\|u_m\|.
\end{aligned}$$

Taking $\varphi = -P_- u_m$ in (2.4) yields

$$\begin{aligned}
\|P_- u_m\|^2 &= \int_\Omega \bigl(\Delta^2 u_m + c\Delta u_m\bigr)\cdot(-P_- u_m)\,dx \\
&= \int_\Omega a(x)g(u_m)\cdot(-P_- u_m)\,dx \\
&\le \int_\Omega \bigl|a(x)\bigr|\bigl|g(u_m)\bigr||u_m|\,dx \\
&\le C_3 \int_\Omega |u_m|^\mu\,dx + C_4\|u_m\|.
\end{aligned}$$

Thus, by (2.3), we have

$$\begin{aligned}
\|u_m\|^2 &= \|P_+ u_m\|^2 + \|P_- u_m\|^2 \le M_3 \int_\Omega |u_m|^\mu\,dx + M_4\|u_m\| \\
&\le M_5\bigl(1 + \|u_m\|\bigr) + M_4\|u_m\| \le M_6\bigl(1 + \|u_m\|\bigr),
\end{aligned}$$

from which the boundedness of (u_m) follows. Thus (u_m) converges weakly in H. Since $P_\pm I'(u_m) = \pm P_\pm u_m + P_\pm \tilde{\mathcal{P}}(u_m)$ with $\tilde{\mathcal{P}}$ compact and the weak convergence of $P_\pm u_m$ imply the strong convergence of $P_\pm u_m$ and hence (PS) condition holds. □

3 Proof of Theorem 1.1

We shall show that $I(u)$ satisfies the generalized mountain pass geometrical assumptions.

We recall the generalized mountain pass geometry.

Let $H = V \oplus X$, where $V \neq \{0\}$ and is finite dimensional. Suppose that $I \in C^1(H, R)$, satisfies the Palais-Smale condition, and

(i) there are constants $\rho, \alpha > 0$ and a bounded neighborhood B_ρ of 0 such that $I|_{\partial B_\rho \cap X} \ge \alpha$, and

(ii) there is an $e \in \partial B_1 \cap X$ and $R > \rho$ such that if $Q = (\bar{B}_R \cap V) \oplus \{re \mid 0 < r < R\}$, then $I|_{\partial Q} \le 0$.

Then I possesses a critical value $b \ge \alpha$. Moreover b can be characterized as

$$b = \inf_{\gamma\in\Gamma} \max_{u\in Q} I\bigl(\gamma(u)\bigr),$$

where

$$\Gamma = \bigl\{\gamma \in C(\bar{Q}, H) \mid \gamma = id \text{ on } \partial Q\bigr\}.$$

Let $H_k = \operatorname{span}\{\phi_1, \dots, \phi_k\}$. Then H_k is a subspace of H such that

$$H = \bigoplus_{k\in N} H_k \quad \text{and} \quad H = H_k \oplus H_k^\perp.$$

Let

$$B_r = \{u \in H \mid \|u\| \leq r\}.$$

We have the following generalized mountain pass geometrical assumptions.

Lemma 3.1 *Assume that* $\lambda_k < c < \lambda_{k+1}$ *and* g *satisfies* (g1)-(g4). *Then*

(i) *there are constants* $\rho > 0$, $\alpha > 0$ *and a bounded neighborhood* B_ρ *of* 0 *such that* $I|_{\partial B_\rho \cap H_k^\perp} \geq \alpha$, *and*

(ii) *there is an* $e \in \partial B_1 \cap H_k^\perp$ *and* $R > \rho$ *such that if* $Q = (\bar{B}_R \cap H_k) \oplus \{re \mid 0 < r < R\}$, *then* $I|_{\partial Q} \leq 0$, *and*

(iii) *there exists* $u_0 \in H$ *such that* $\|u_0\| > \rho$ *and* $I(u_0) \leq 0$.

Proof (i) Let $u \in H_k^\perp$. Then

$$\int_\Omega \left(\Delta^2 u + c\Delta u\right) u\,dx \geq \lambda_{k+1}(\lambda_{k+1} - c)\|u\|^2_{L^2(\Omega)} > 0.$$

Thus by (g2), (g4), and the Hölder inequality, we have

$$\begin{aligned} I(u) &= \frac{1}{2}\|P_+ u\|^2 - \frac{1}{2}\|P_- u\|^2 - \int_\Omega a(x)G(u)\,dx \\ &\geq \frac{1}{2}\|P_+ u\|^2 - \|a\|_\infty \int_\Omega C_1|u|^\mu\,dx \\ &\geq \frac{1}{2}\|P_+ u\|^2 - \|a\|_\infty C_1'\|u\|^\mu \end{aligned}$$

for $C_1, C_1' > 0$. Since $\mu > 2$, there exist $\rho > 0$ and $\alpha > 0$ such that if $u \in \partial B_\rho$, then $I(u) \geq \alpha$.

(ii) Let B_r be a ball with radius $r > 0$, e be a fixed element in $\partial B_1 \cap H_k^\perp$ and $u \in (\bar{B}_r \cap H_k) \oplus \{re \mid 0 < r\}$. Then $u = v + w$, $v \in B_r \cap H_k$, $w = re$. We note that

$$\text{since } v \in H_k, \quad \int_\Omega \left(\Delta^2 v + c\Delta v\right) v\,dx \leq \lambda_k(\lambda_k - c)\|v\|^2_{L^2(\Omega)} < 0.$$

Thus we have

$$\begin{aligned} I(u) &= \frac{1}{2}r^2 - \frac{1}{2}\|P_- v\|^2 - \int_\Omega a(x)G(v + re)\,dx \\ &\leq \frac{1}{2}r^2 + \frac{1}{2}\left(\lambda_k(\lambda_k - c)\right)\|v\|^2_{L^2(\Omega)} - \int_{\Omega^+} a(x)\left(a_5|v + re|^\mu - a_4\right)dx. \end{aligned}$$

Since $\mu > 2$, there exists $R > 0$ such that if $u \in Q = (\bar{B}_R \cap H_k) \oplus \{re \mid 0 < r < R\}$, then $I(u) < 0$.

(iii) If we choose $\psi \in H$ such that $\|\psi\| = 1$, $\psi \geq 0$ in Ω and $\operatorname{supp}(\psi) \subset \Omega^+$, then we have

$$\begin{aligned} I(t\psi) &\leq \frac{1}{2}\left\|P_+(t\psi)\right\|^2 - \frac{1}{2}\left\|P_-(t\psi)\right\|^2 - \int_{\Omega^+} a(x)\left(a_5 t^\mu \psi^\mu - a_4\right)dx \\ &\leq \frac{1}{2}\|t\psi\|^2 - \int_{\Omega^+} a(x)\left(a_5 t^\mu \psi^\mu - a_4\right)dx \\ &= \frac{1}{2}t^2 - \int_{\Omega^+} a(x)\left(a_5 t^\mu \psi^\mu - a_4\right)dx \end{aligned}$$

for all $t > 0$. Since $\mu > 2$, for t_0 great enough, $u_0 = t_0\psi$ is such that $\|u_0\| > \rho$ and $I(u_0) \le 0$. □

Proof of Theorem 1.1 By Proposition 2.1 and Proposition 2.2, $I(u) \in C^1(H,R)$ and satisfies the Palais-Smale condition. By Lemma 3.1, there are constants $\rho > 0$, $\alpha > 0$ and a bounded neighborhood B_ρ of 0 such that $I|_{\partial B_\rho \cap H_m^\perp} \ge \alpha$, and there is an $e \in \partial B_1 \cap H_k^\perp$ and $R > \rho$ such that if $Q = (\bar{B}_R \cap H_k) \oplus \{re \mid 0 < r < R\}$, then $I|_{\partial Q} \le 0$, and there exists $u_0 \in H$ such that $\|u_0\| > \rho$ and $I(u_0) \le 0$. By the generalized mountain pass theorem, $I(u)$ has a critical value $b \ge \alpha$. Moreover, b can be characterized as

$$b = \inf_{\gamma\in\Gamma} \max_{u\in Q} I\big(\gamma(u)\big),$$

where

$$\Gamma = \big\{\gamma \in C(\bar{Q},H) \mid \gamma = id \text{ on } \partial Q\big\}.$$

We denote by $\tilde{u}$ a critical point of I such that $I(\tilde{u}) = b$. We claim that there exists a constant $C > 0$ such that

$$\big\|a^+(x)^{\frac{1}{\mu}}\tilde{u}\big\|_{L^2(\Omega)} \le C\left(1 + L\int_{\Omega^-} a^-(x)\,dx\right)^{\frac{1}{\mu}},$$

where $L = \max_\Omega |\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})|$.

In fact, we have

$$b \le \max_{0\le t\le 1} I(tu_0)$$

and

$$\begin{aligned} I(tu_0) &= t^2\left(\frac{1}{2}\|P_+u_0\|^2 - \frac{1}{2}\|P_-u_0\|^2\right) - \int_\Omega a(x)G(tu_0)\,dx \\ &\le t^2\|u_0\|^2 - \int_\Omega a^+(x)G(tu_0)\,dx + \int_\Omega a^-(x)G(tu_0)\,dx \\ &\le t^2\|u_0\|^2 - a_5t^\mu\int_\Omega a^+(x)u_0^\mu\,dx + a_4\int_\Omega a^+(x)\,dx + a_5t^\mu\int_\Omega a^-(x)u_0^\mu\,dx \\ &= Ct^2 - Ct^\mu + C + C't^\mu. \end{aligned}$$

Since $0 \le t \le 1$, b is bounded: $b < \tilde{C}$.

We can write

$$\begin{aligned} b &= I(\tilde{u}) - \frac{1}{2}I'(\tilde{u})\tilde{u} \\ &= \int_\Omega a(x)\left(\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})\right)dx \\ &= \int_\Omega a^+(x)\left(\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})\right)dx - \int_\Omega a^-(x)\left(\frac{1}{2}g(\tilde{u})\tilde{u} - G(\tilde{u})\right)dx \end{aligned}$$

$$\geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\int_\Omega a^+(x)g(\tilde{u})\tilde{u}\,dx - \max_\Omega\left|\frac{1}{2}g(\tilde{u})\tilde{u}-G(\tilde{u})\right|\int_{\Omega^-}a^-(x)\,dx$$
$$\geq \left(\frac{1}{2}-\frac{1}{\mu}\right)\mu\int_\Omega a^+(x)\bigl(a_3|\tilde{u}|^\mu - a_4\bigr)\,dx - L\int_{\Omega^-}a^-(x)\,dx,$$

where $L = \max_\Omega|\frac{1}{2}g(\tilde{u})\tilde{u}-G(\tilde{u})|$. Thus we have

$$C'\left(1+L\int_{\Omega^-}a^-(x)\,dx\right) \geq \int_\Omega a^+(x)|\tilde{u}|^\mu\,dx \geq C''\left[\int_\Omega\bigl(a^+(x)^{\frac{1}{\mu}}|\tilde{u}|\bigr)^2\,dx\right]^{\frac{\mu}{2}} \tag{3.1}$$

for some constants $C', C'' > 0$, from which we conclude that $\tilde{u}$ is bounded and the proof of Theorem 1.1 is complete. □

4 Proof of Theorem 1.2

Assume that $\frac{1}{2}g(u)u - G(u)$ is not bounded and there exists an $\epsilon > 0$ such that $\int_{\Omega^-}a^-(x, t)\,dx < \epsilon$. By Proposition 2.1 and Proposition 2.2, $I \in C^1(H,R)$ and satisfies the Palais-Smale condition. By Lemma 3.1 and the generalized mountain pass theorem, $I(u)$ has a critical value b with critical point $\tilde{u}$ such that $I(\tilde{u}) = b$. If $\int_{\Omega^-}a^-(x)\,dx$ is sufficiently small, by (3.1), we have

$$\int_\Omega a^+(x)|\tilde{u}|^\mu\,dx \leq C$$

for $C > 0$, from which we can conclude that $\tilde{u}$ is bounded and the proof of Theorem 1.2(i) is complete.

Next we shall prove Theorem 1.2(ii). We may assume that $R_n < R_{n+1}$ for all $n \in N$. Let us set $D_n = B_{R_n} \cap H_n$, $\partial D_n = \partial B_{R_n} \cap H_n$.

Lemma 4.1 *Assume that g satisfies* (g1)-(g4), $\frac{1}{2}g(u)u - G(u)$ *is not bounded and there exists an* $\epsilon > 0$ *such that* $\int_{\Omega^-}a^-(x,t) < \epsilon$. *Then there exists an* $R_n > 0$ *such that*

$$I(u) \leq 0 \quad \textit{for } u \in H_n \backslash B_{R_n}. \tag{4.1}$$

Proof Let us choose $\psi \in H$ such that $\|\psi\| = 1$, $\psi \geq 0$ in Ω and $\operatorname{supp}(\psi) \subset \Omega^+$. Then, by (g2), (g4), and the Hölder inequality, we have

$$\begin{aligned}
I(t\psi) &= \frac{1}{2}\|P_+t\psi\|^2 - \frac{1}{2}\|P_-t\psi\|^2 - \int_\Omega a(x)G(t\psi)\,dx \\
&= \frac{1}{2}\|P_+t\psi\|^2 - \frac{1}{2}\|P_-t\psi\|^2 - \int_{\Omega^+}a^+(x)G(t\psi)\,dx + \int_{\Omega^-}a^-(x)G(t\psi)\,dx \\
&\leq \frac{1}{2}\|P_+t\psi\|^2 - \frac{1}{2}\|P_-t\psi\|^2 - \int_{\Omega^+}a^+(x)\bigl(a_5t^\mu\psi^\mu - a_4\bigr)\,dx \\
&\quad + \bigl\|G(t\psi)\bigr\|_\infty\int_{\Omega^-}a^-(x)\,dx \\
&\leq \frac{1}{2}t^2 - \int_{\Omega^+}a^+(x)\bigl(a_5t^\mu\psi^\mu - a_4\bigr)\,dx + \epsilon'
\end{aligned}$$

for small $\epsilon' > 0$. Since $\mu > 2$, there exist t_n great enough for each n and an $R_n > 0$ such that $u_n = t_n\psi$ and $I(u_n) < 0$ if $u_n \in H_n\backslash B_{R_n}$ and $\|u_n\| > R_n$, so the lemma is proved. □

Let us set

$$\Gamma_n = \{\gamma \in C([0,1],H) \mid \gamma(0) = 0 \text{ and } \gamma(1) = u_n\}$$

and

$$b_n = \inf_{\gamma\in\Gamma_n} \max_{[0,1]} I(\gamma(u)), \quad n \in N.$$

Proof of Theorem 1.2(ii) We assume that $g(u)u - \mu G(u)$ is not bounded and there exists an $\epsilon > 0$ such that $\int_{\Omega^-} a^-(x)\,dx < \epsilon$. By Proposition 2.1 and Proposition 2.2, $I \in C^1(H,R)$ and satisfies the Palais-Smale condition. By Lemma 4.1, there exists an $R_n > 0$ such that $I(u_n) \leq 0$ for $u_n \in H_n \backslash B_{R_n}$. We note that $I(0) = 0$. By Lemma 4.1 and the generalized mountain pass theorem, for n large enough $b_n > 0$ is a critical value of I and $\lim_{n\to\infty} b_n = +\infty$. Let $\tilde{u}_n$ be a critical point of I such that $I(\tilde{u}_n) = b_n$. Then for each real number M, $\max_\Omega |\tilde{u}_n(x)| \geq M$. In fact, by contradiction, $\Delta^2 u + c\Delta u = a(x)g(u)$ and $\max_\Omega |\tilde{u}_n(x)| \leq K$ imply that

$$I(\tilde{u}_n) \leq \max_{|\tilde{u}_n|\leq K} \left(\frac{1}{2} g(\tilde{u}_n)\tilde{u}_n - G(\tilde{u}_n)\right) \int_\Omega \left|a(x)\right| dx,$$

which means that b_n is bounded. This is absurd because of the fact that $\lim_{n\to\infty} b_n = +\infty$. Thus we complete the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
TJ and Q-HC participated in the sequence alignment and drafted the manuscripted. Both authors read and approved the final manuscript.

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Acknowledgements
This work (Choi) was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (KRF-2013010343).

Received: 28 June 2013 Accepted: 8 January 2014 Published: 04 Feb 2014

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10.1186/1687-2770-2014-30
Cite this article as: Jung and Choi: Nonlinear biharmonic boundary value problem. *Boundary Value Problems* 2014, 2014:30