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Positive solutions for the singular nonlocal boundary value problems involving nonlinear integral conditions

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Abstract

In this paper, using the theory of fixed point index on a cone and the Leray-Schauder fixed point theorem, we present the multiplicity of positive solutions for the singular nonlocal boundary-value problems involving nonlinear integral conditions and the existence of at least one positive solution for the singular nonlocal boundary-value problems with sign-changed nonlinearities.

MSC: 34B10; 34B15; 34B18

Keywords: nonlocal boundary conditions; positive solution; fixed point index

1 Introduction

Nonlocal boundary-value problems with linear and nonlinear integral conditions have seen a great deal of study lately (see [1–16], and references therein) because of their interesting theory and their applications to various problems, such as heat flow in a bar of finite length [4, 11]. In this paper, we consider the existence of positive solutions of the nonlinear boundary-value problem (BVP) of the form

$$-y'' = q(t)f\bigl(t, y(t)\bigr), \quad t \in (0,1) \tag{1.1}$$

with integral boundary conditions

$$y(0) = H\bigl(\phi(y)\bigr), \qquad y(1) = 0, \tag{1.2}$$

where $\phi(y)$ is a linear functional on $C[0,1]$ given by

$$\phi(y) = \int_0^1 y(s)\,d\alpha(s)$$

involving a Stieltjes integral with a signed measure.

In [2], Goodrich considered the following problem:

$$-y'' = \lambda g\bigl(t, y(t)\bigr), \quad t \in (0,1) \tag{1.3}$$

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with integral boundary conditions

$$y(0) = H\big(\phi(y)\big), \qquad y(1) = 0 \tag{1.4}$$

and deduced the existence of at least one positive solution to the BVP (1.3)-(1.4) in which $H(\phi(y))$ has either asymptotically sublinear or asymptotically superlinear growth, and in [3] Goodrich demonstrated that if the nonlinear functional $H(\phi(y))$ satisfies a certain asymptotic behavior, then the BVP (1.3)-(1.4) possesses at least one positive solution. For the case that H is linear and $\phi(y) = \int_0^1 y(s)\,d\alpha(s)$ involves a signed measure, Webb and Infante discussed the multiplicity of positive solutions for nonlocal boundary-value problems [12–14]. For the case that H is linear and the Borel measure associated with the Lebesgue-Stieltjes integral is positive, we can find some results on the existence of positive solutions [7, 8, 16, 17]. The results in the above literature are obtained under the condition that $f(t,x)$ is continuous on $(0,1) \times [0,+\infty)$, *i.e.*, f has no singularity at $x = 0$. And it is well known that study of singular two-point boundary-value problems for the second-order differential equation (1.1) (singular in the dependent variable) is very important and there are many results on the existence of positive solutions [15, 18–24]. But there are fewer results on the existence of positive solutions for the singular BVP (1.1)-(1.2) [5, 6]. One goal in this paper is to consider the existence of positive solutions under the condition that $f(t,x)$ is singular at $x = 0$. Our paper has the following features.

Firstly, in order to overcome the difficulties of the singularity of f we establish a new cone and get the new condition (3.13) which is different from that in [5, 6]. Moreover, we get a multiplicity of positive solutions for BVP (1.1)-(1.2) different from that in [2, 3, 12–14] under the condition that $H(y)$ or $f(t,y)$ is superlinear at $y = +\infty$.

Secondly, when f is singular and sign-changed, we get the existence of at least one positive solution to the BVP (1.1)-(1.2) which is different from that in [2, 3, 5, 6, 12–14] where f is nonnegative and continuous at $x = 0$. Moreover, the results are different from that in [7, 8, 16, 17] where integral boundary conditions are linear and the Borel measure is positive.

Our paper is organized as follows. In Section 2, we present some lemmas and preliminaries. Section 3 discusses the existence of multiple positive solutions for the BVP (1.1)-(1.2) when f is positive. In Section 4, we discuss the existence of at least one positive solution of BVP (1.1)-(1.2) when f is singular and sign-changed.

2 Preliminaries

In this paper, the following lemmas are needed.

Lemma 2.1 (see [25]) *Let Ω be a bounded open set in real Banach space E, P a cone of $E, \theta \in \Omega$ and $A : \overline{\Omega} \cap P \to P$ continuous and compact. Suppose $\lambda Ax \neq x$, $\forall x \in \partial\Omega \cap P$, $\lambda \in (0,1]$. Then*

$$i(A, \Omega \cap P, P) = 1.$$

Lemma 2.2 (see [25]) *Let Ω be a bounded open set in real Banach space E, P a cone of $E, \theta \in \Omega$ and $A : \overline{\Omega} \cap P \to P$ continuous and compact. Suppose $Ax \nleq x$, $\forall x \in \partial\Omega \cap P$. Then*

$$i(A, \Omega \cap P, P) = 0.$$

Lemma 2.3 (see [25, 26]) *Let E be a Banach space, $R > 0$, $B_R = \{x \in E : \|x\| \leq R\}$, and $F : B_R \to E$ a continuous compact operator. If $x \neq \lambda F(x)$ for any $x \in E$ with $\|x\| = R$ and $0 < \lambda < 1$, then F has a fixed point in B_R.*

Let us begin by stating the hypotheses which we shall impose on the BVP (1.1)-(1.2).

(C_1) Assume that there are three linear functionals $\phi, \phi_1, \phi_2 : C([0,1]) \to R$ such that

$$\phi(y) = \phi_1(y) + \phi_2(y).$$

Moreover, assume that there exists a constant $\varepsilon_0 > 0$ such that

$$\phi_2(y) \geq \varepsilon_0 \|y\|$$

holds for each $y \in P$, where P is the cone introduced in (2.1) below [2].

(C_2) The functionals $\phi_1(y)$ and $\phi_2(y)$ are linear and, in particular, have the form

$$\phi_1(y) := \int_0^1 y(t)\, d\alpha_1(t), \qquad \phi_2(y) := \int_0^1 y(t)\, d\alpha_2(t),$$

where $\alpha_1, \alpha_2 : [0,1] \to R$ satisfy $\alpha_1, \alpha_2 \in BV([0,1])$ with

$$\int_0^1 (1-t)\, d\alpha_1(t) \geq 0, \qquad \int_0^1 (1-t)\, d\alpha_2(t) \geq 0$$

and

$$\int_0^1 k(t,s)\, d\alpha_1(t) \geq 0, \qquad \int_0^1 k(t,s)\, d\alpha_2(t) \geq 0$$

hold, where the latter holds for each $s \in [0,1]$ and $k(t,s)$ is defined in (3.2) below [2].

(C_3) Let $H : R \to R$ be a real-valued, continuous function. Moreover, $H : (0,+\infty) \to (0,+\infty)$.

(C_4)

$$\begin{cases} f : [0,1] \times (0,\infty) \to (0,\infty) \text{ is continuous} \\ \text{and there exists a function } \psi_1 \\ \text{continuous on } [0,1] \text{ and positive on } (0,1) \text{ such that} \\ f(t,y) \geq \psi_1(t) \text{ on } (0,1) \times (0,1]. \end{cases}$$

(C_5)

$$q \in C(0,1), \qquad q > 0 \quad \text{on } (0,1) \quad \text{and} \quad \int_0^1 (1-t)q(t)\, dt < \infty.$$

Let $C[0,1] = \{y : [0,1] \to R : y(t) \text{ is continuous on } [0,1]\}$ with norm $\|y\| = \max_{t\in[0,1]} |y(t)|$. It is easy to see that $C[0,1]$ is a Banach space.

Assume that (C_2) hold. Define

$$P = \{ y \in C[0,1] : y \text{ is concave on } [0,1] \text{ with } y(t) \ge 0 \text{ for all } t \in [0,1], \\ \phi_1(y) \ge 0, \phi_2(y) \ge 0 \}. \tag{2.1}$$

It is easy to prove P is a cone of $C[0,1]$ and we have the following lemma.

Lemma 2.4 (see [20]) *Let $y \in P$ (defined in (2.1)). Then*

$$y(t) \ge t(1-t)\|y\| \quad \textit{for } t \in [0,1].$$

3 Multiplicity of positive solutions for the singular boundary-value problems with positive nonlinearities

In this section, we consider the existence of multiple positive solutions for the BVP (1.1)-(1.2). To show that the BVP (1.1)-(1.2) has a solution, for $y \in P$, we define

$$(T_\epsilon y)(t) = (1-t)H(\phi(y)) + \int_0^1 k(t,s)q(s)f(s, \max\{\epsilon, y(s)\})\,ds, \\ t \in [0,1], 1 \ge \epsilon > 0, \tag{3.1}$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ (1-s)t, & 0 \le t \le s \le 1. \end{cases} \tag{3.2}$$

Lemma 3.1 *Suppose* (C_1)-(C_5) *hold. Then $T_\epsilon : P \to P$ is continuous and compact for all* $1 \ge \epsilon > 0$.

Proof It is easy to prove that T_ϵ is well defined and $(T_\epsilon y)(t) \ge 0$ for all $t \in P$. For $y \in P$, we have

$$\begin{cases} (T_\epsilon y)''(t) \le 0 \quad \text{on } (0,1), \\ (T_\epsilon y)(0) = H(\phi(y)), \qquad (T_\epsilon y)(1) = 0, \end{cases}$$

and so

$$(T_\epsilon y)(t) \text{ is concave on } [0,1]. \tag{3.3}$$

Moreover, from (C_2), we may estimate

$$\begin{aligned} \phi_1(T_\epsilon y) &= \int_0^1 (1-t)\,d\alpha_1(t)H(\phi(y)) + \int_0^1 \int_0^1 k(t,s)q(s)f(s, \max\{\epsilon, y(s)\})\,ds\,d\alpha_1(t) \\ &= \int_0^1 (1-t)\,d\alpha_1(t)H(\phi(y)) + \int_0^1 q(s)f(s, \max\{\epsilon, y(s)\}) \int_0^1 k(t,s)\,d\alpha_1(t)\,ds \\ &\ge 0 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\phi_2(T_\epsilon y) &= \int_0^1 (1-t)\,d\alpha_2(t)H\big(\phi(y)\big) + \int_0^1\int_0^1 k(t,s)q(s)f\big(s,\max\{\epsilon,y(s)\}\big)\,ds\,d\alpha_2(t)\\ &= \int_0^1 (1-t)\,d\alpha_2(t)H\big(\phi(y)\big) + \int_0^1 q(s)f\big(s,\max\{\epsilon,y(s)\}\big)\int_0^1 k(t,s)\,d\alpha_2(t)\,ds\\ &\geq 0. \end{aligned} \tag{3.5}$$

Combining (3.3), (3.4), and (3.5), $T_\epsilon : P \to P$. A standard argument shows that $T_\epsilon : P \to P$ is continuous and compact [9, 18, 26]. □

Define

$$\Phi_r := \big\{x \in P \cap C^2\big((0,1),R\big) : \|x\| \leq r \text{ and } x \text{ satisfies}$$
$$x''(t) + q(t)f\big(t,\max\{\epsilon,x(t)\}\big) = 0, 0 < t < 1, x(0) = H\big(\phi(x)\big), x(1) = 0,\ \forall 1 \geq \epsilon > 0\big\}.$$

Lemma 3.2 *If $\Phi_r \neq \emptyset$ and* (C_2) *hold, there exists a $\delta_r > 0$ such that*

$$x(0) \geq \delta_r t(1-t), \quad \forall t \in [0,1], x \in \Phi_r.$$

Proof Suppose $x \in \Phi_r$. There are two cases to consider.
(1) $\|x\| > 1$. Lemma 2.4 implies that

$$x(t) \geq t(1-t)\|x\| \geq t(1-t), \quad t \in [0,1]. \tag{3.6}$$

(2) $0 < \|x\| \leq 1$. Condition (C_4) guarantees that

$$\begin{aligned} x(t) &= (1-t)H\big(\phi(x)\big) + \int_0^1 k(t,s)q(s)f\big(s,\max\{\epsilon,x(s)\}\big)\,ds\\ &\geq \int_0^1 k(t,s)q(s)\psi_1(s)\,ds := \gamma_0(t), \quad t \in [0,1]. \end{aligned}$$

Since $\gamma_0''(t) \geq 0$, $\gamma_0(0) = 0$, and $\gamma_0(1) = 0$, we know that γ_0 is concave on $[0,1]$ and $\gamma_0(t) \geq 0$ for all $t \in [0,1]$. And from (C_2), a similar argument as (3.4) and (3.5) shows that $\phi_1(\gamma_0) \geq 0$ and $\phi_2(\gamma_0) \geq 0$. Then $\gamma_0 \in P$ and Lemma 2.4 implies that

$$\gamma_0(t) \geq t(1-t)\|\gamma_0\|, \quad \forall t \in [0,1]. \tag{3.7}$$

Let $\delta_1 = \min\{1, \|\gamma_0\|\}$. From (3.6) and (3.7), one has

$$x(t) \geq \delta_1 t(1-t), \quad \forall t \in [0,1],$$

which means that

$$r \geq \|x\| \geq \delta_1.$$

Thus

$$\phi(x) = \int_0^1 x(s)\,d\alpha_1(s) + \int_0^1 x(s)\,d\alpha_2(s) \le c_0\|x\| \le c_0 r,$$

where

$$c_0 \stackrel{\text{def.}}{=} \int_0^1 \left|d\alpha_1(s)\right| + \int_0^1 \left|d\alpha_2(s)\right|$$

and (C_1) guarantees that

$$\phi(x) \ge \phi_2(x) \ge \varepsilon_0\|x\|.$$

And so

$$x(0) = H\bigl(\phi(x)\bigr) \ge \min_{s\in[\varepsilon_0\delta_1, c_0 r]} H(s) := \delta_r > 0.$$

The concavity $x(t)$ yields

$$x(t) \ge \delta_r(1-t) > 0, \quad \forall t \in [0,1], x \in \Phi_r.$$

The proof is complete. □

For $R > 0$, let

$$\Omega_R = \bigl\{x \in C[0,1] : \|x\| < R\bigr\}.$$

We have the following lemmas.

Lemma 3.3 *Suppose that* (C_1)-(C_5) *hold and there exists an* $a \in (0, \frac{1}{2})$ *such that*

$$\lim_{y\to+\infty} \frac{f(t,y)}{y} = +\infty \tag{3.8}$$

uniformly on $[a, 1-a]$. *Then, there exists an* $R' > 1$ *such that for all* $R \ge R'$

$$i(T_\epsilon, \Omega_R \cap P, P) = 0, \quad \forall 0 < \epsilon \le 1.$$

Proof From (3.8), there exists an $R_1 > 1$ such that

$$f(t,y) \ge N^* y, \quad \forall y \ge R_1, \tag{3.9}$$

where

$$N^* > \frac{2}{a^2 \int_a^{1-a} k(a,s)q(s)\,ds}.$$

Let $R' = \frac{R_1}{a^2}$ and

$$\Omega_R := \{x \in C[0,1] : \|x\| < R\}, \quad \forall R \geq R'.$$

Now we show

$$T_\epsilon y \nleq y \quad \text{for } y \in P \cap \partial\Omega_R, \forall 0 < \epsilon \leq 1. \tag{3.10}$$

Suppose that there exists a $y_0 \in P \cap \partial\Omega_R$ with $T_\epsilon y_0 \leq y_0$. Then, $\|y_0\| = R$. Since $y_0(t)$ is concave on $[0,1]$ (since $y_0 \in P$) we find from Lemma 2.4 that $y_0(t) \geq t(1-t)\|y_0\| \geq t(1-t)R$ for $t \in [0,1]$. For $t \in [a, 1-a]$, one has

$$y_0(t) \geq a^2 R \geq a^2 R' = R_1, \quad \forall t \in [a, 1-a],$$

which together with (3.9) yields

$$f(t, \max\{\epsilon, y_0(t)\}) = f(t, y_0(t)) \geq N^* y_0(t) \geq N^* a^2 R, \quad \forall t \in [a, 1-a]. \tag{3.11}$$

Then we have, using (3.11),

$$\begin{aligned} y_0(a) \geq T_\epsilon y_0(a) &= (1-a)H(\phi(y_0)) + \int_0^1 k(a,s)q(s)f(s, \max\{\epsilon, y_0(s)\})\,ds \\ &\geq \int_a^{1-a} k(a,s)q(s)f(s, \max\{\epsilon, y_0(s)\})\,ds \\ &= \int_a^{1-a} k(a,s)q(s)f(s, y_0(s))\,ds \\ &\geq N^* R a^2 \int_a^{1-a} k(a,s)q(s)\,ds \\ &> R = \|y_0\|, \end{aligned}$$

which is a contradiction. Hence equation (3.10) is true. Lemma 2.2 guarantees that

$$i(T_\epsilon, \Omega_R \cap P, P) = 0, \quad \forall 0 < \epsilon \leq 1.$$

The proof is complete. □

Lemma 3.4 *Suppose that* (C_1)-(C_5) *hold and*

$$\lim_{s \to +\infty} \frac{H(s)}{s} = +\infty. \tag{3.12}$$

Then, there exists an $R' > 1$ *such that for all* $R \geq R'$

$$i(T_\epsilon, \Omega_R \cap P, P) = 0, \quad \forall 0 < \epsilon \leq 1.$$

Proof From equation (3.12), there exists an $R_1 > 1$ such that

$$H(s) \geq N^* s, \quad \forall s \geq R_1, \tag{3.13}$$

where

$$N^* > \frac{2}{\varepsilon_0} \quad \bigl(\varepsilon_0 \text{ defined in } (\mathrm{C}_1)\bigr).$$

Let $R' = \frac{R_1}{\varepsilon_0}$ and

$$\Omega_R = \bigl\{x \in C[0,1] : \|x\| < R\bigr\}, \quad \forall R \geq R'.$$

Now we show

$$T_\epsilon y \nleq y \quad \text{for } y \in P \cap \partial\Omega_R, \forall 0 < \epsilon \leq 1. \tag{3.14}$$

Suppose that there exists a $y_0 \in P \cap \partial\Omega_R$ with $T_\epsilon y_0 \leq y_0$. Then, $\|y_0\| = R$. Now (C_1) guarantees that

$$\phi(y_0) = \phi_1(y_0) + \phi_2(y_0) \geq \varepsilon_0 \|y_0\| = \varepsilon_0 R \geq R_1,$$

which together with equation (3.13) implies that

$$y_0(0) \geq T_\epsilon y_0(0) = H\bigl(\phi(y_0)\bigr) \geq N^* \phi(y_0) > \frac{2}{\varepsilon_0}\varepsilon_0 \|y_0\| > \|y_0\|.$$

This is a contradiction. Hence (3.14) is true. Lemma 2.2 guarantees that

$$i(T_\epsilon, \Omega_R \cap P, P) = 0, \quad \forall 0 < \epsilon \leq 1.$$

The proof is complete. □

Theorem 3.1 *Suppose* (C_1)-(C_5) *hold and the following conditions are satisfied*:

$$\begin{cases} 0 \leq f(t,y) \leq g(y) + h(y) \textit{ on } [0,1] \times (0,\infty) \textit{ with} \\ g > 0 \textit{ continuous and nonincreasing on } (0,\infty), \\ h \geq 0 \textit{ continuous on } [0,\infty), \textit{ and} \\ \frac{h}{g} \textit{ nondecreasing on } (0,\infty) \end{cases} \tag{3.15}$$

and

$$\sup_{r \in (0,+\infty)} \min\left\{ \frac{1}{1 + \frac{h(r)}{g(r)}} \int_0^r \frac{dy}{g(y)}, \frac{r}{\max_{y \in [0, c_0 r]} H(y)} \right\} > \max\{1, b_0\} \tag{3.16}$$

hold; *here*

$$b_0 = \int_0^1 (1-s) q(s)\, ds, \qquad c_0 = \int_0^1 \bigl|d\alpha_1(s)\bigr| + \int_0^1 \bigl|d\alpha_2(s)\bigr|.$$

Then the BVP (1.1)-(1.2) *has at least one positive solution*.

Proof From equation (3.16), choose $\epsilon > 0$ and $r > 0$ with $\epsilon < \min\{1, r\}$ such that

$$\min\left\{\frac{1}{1+\frac{h(r)}{g(r)}}\int_0^r \frac{dy}{g(y)}, \frac{r}{\max_{y\in[0,c_0 r]} H(y)}\right\} > \max\{1, b_0\}. \tag{3.17}$$

Let

$$\Omega_1 = \bigl\{y \in C[0,1] : \|y\| < r\bigr\},$$

and $n_0 > \frac{1}{\epsilon}$. For $n \in \{n_0, n_0+1, \ldots\}$, we define $T_{\frac{1}{n}}$ as in equation (3.1). Lemma 3.1 guarantees that $T_{\frac{1}{n}} : P \to P$ is continuous and compact.

Now we show that

$$y \neq \lambda T_{\frac{1}{n}} y, \quad \forall y \in \partial\Omega_1 \cap P, \lambda \in (0,1]. \tag{3.18}$$

Suppose that there is a $y_0 \in \partial\Omega_1 \cap P$ and $\lambda_0 \in [0,1]$ with $y_0 = \lambda_0 T_{\frac{1}{n}} y_0$, *i.e.*, y_0 satisfies

$$\begin{cases} y_0''(t) + \lambda_0 q(t) f(t, \max\{\frac{1}{n}, y_0(t)\}) = 0, & 0 < t < 1, \\ y_0(0) = \lambda_0 H(\phi(y)), & y_0(1) = 0. \end{cases} \tag{3.19}$$

Then $y_0''(t) \leq 0$ on $(0,1)$. From equation (3.17), we have $y_0(0) = \lambda_0 H(\phi(y_0)) \leq \max_{y\in[0,c_0 r]} H(y) < r$, which together with $y_0(1) = 0 < r$ implies that there exists a $t_0 \in (0,1)$ with $y_0(t_0) = \|y_0\| = r$, $y_0'(t_0) = 0$ and $y_0'(t) \leq 0$ for all $t \in (t_0, 1)$. For $t \in (0,1)$, from equations (3.15) and (3.19), we have

$$\begin{aligned} -y_0''(t) &\leq g\left(\max\left\{\frac{1}{n}, y_0(t)\right\}\right)\left\{1 + \frac{h(\max\{\frac{1}{n}, y_0(t)\})}{g(\max\{\frac{1}{n}, y_0(t)\})}\right\} q(t) \\ &\leq g\left(\max\left\{\frac{1}{n}, y_0(t)\right\}\right)\left\{1 + \frac{h(r)}{g(r)}\right\} q(t). \end{aligned} \tag{3.20}$$

We integrate equation (3.20) from t_0 $(t_0 < t)$ to t to obtain

$$\begin{aligned} -y_0'(t) &\leq g\left(\max\left\{\frac{1}{n}, y_0(t)\right\}\right)\left\{1 + \frac{h(r)}{g(r)}\right\}\int_{t_0}^t q(s)\,ds \\ &\leq g\bigl(y_0(t)\bigr)\left\{1 + \frac{h(r)}{g(r)}\right\}\int_{t_0}^t q(s)\,ds \end{aligned} \tag{3.21}$$

and then integrate equation (3.21) from t_0 to 1 to obtain

$$\begin{aligned} \int_{y_0(1)}^{y_0(t_0)} \frac{dy}{g(y)} &\leq \left\{1 + \frac{h(r)}{g(r)}\right\}\int_{t_0}^1 \int_{t_0}^s q(\tau)\,d\tau\,ds \\ &= \left\{1 + \frac{h(r)}{g(r)}\right\}\int_{t_0}^1 (1-s)q(s)\,ds \\ &\leq \left\{1 + \frac{h(r)}{g(r)}\right\}\int_0^1 (1-s)q(s)\,ds, \end{aligned}$$

i.e.,

$$\int_0^r \frac{dy}{g(y)} \le \left\{1 + \frac{h(r)}{g(r)}\right\} \int_0^1 (1-s)q(s)\,ds,$$

which contradicts equation (3.17). Therefore, equation (3.18) is true. Lemma 2.1 implies that

$$i(T_{\frac{1}{n}}, \Omega_1 \cap P, P) = 1,$$

which yields the result that there exists a $y_n \in \Omega_1 \cap P$ such that

$$T_{\frac{1}{n}} y_n = y_n,$$

i.e., $\Phi_r \neq \emptyset$ in Lemma 3.2. Now Lemma 3.2 guarantees that there exists a $\delta_r > 0$ such that

$$y_n(0) \ge \delta_r, \qquad y_n(t) \ge \delta_r(1-t), \quad \forall t \in [0,1], x \in \{n_0, n_0+1, \ldots\}. \tag{3.22}$$

Now we consider the set $\{y_n\}_{n=n_0}^{\infty}$. Obviously, $\|y_n\| \le r$ means that

$$\text{the functions belonging to } \{y_n(t)\} \text{ are uniformly bounded on } [0,1]. \tag{3.23}$$

Now we show that

$$\text{the functions belonging to } \{y_n(t)\} \text{ are equicontinuous on } [0,1]. \tag{3.24}$$

There are two cases to consider.

(1) There exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ with $y_{n_i}(0) = H(\phi(y_{n_i})) < \|y_{n_i}\|$. Without loss of generality, we assume that $y_n(0) = H(\phi(y_n)) < \|y_n\|$, $n \in \{n_0, n_0+1, \ldots\}$, which together with $y_n(1) = 0$ implies that there exists a t_n satisfying that $y_n'(t_n) = 0$ with $y_n'(t) \ge 0$ for $t \in (0, t_n)$ and $y_n'(t) \le 0$ for $t \in (t_n, 1)$. Let $t' = \sup\{t_n, n \ge n_0\}$. Now we show that $t' < 1$. To the contrary, suppose that $t' = 1$. Then there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} \to 1$ as $n_i \to +\infty$. From equation (3.21), using y_n in place of y_0, we have

$$\int_0^{y_{n_i}(t_{n_i})} \frac{1}{g(y)}\,dy \le \left(1 + \frac{h(r)}{g(r)}\right) \int_{t_{n_i}}^1 (1-s)q(s)\,ds,$$

which implies that

$$y_{n_i}(t_{n_i}) \to 0, \quad \text{as } n_i \to +\infty.$$

This contradicts $y_{n_i}(t) \ge \delta_r(1-t)$ for all $t \in [0,1]$.

Let $t_0 \in (t', 1)$. From equation (3.22), we have

$$y_n(t) \ge k_0 := \min_{t \in [0,t_0]} \delta_r(1-t), \quad t \in [0, t_0].$$

Similarly as the proof in equation (3.21), one has

$$y_n'(t) \le g(k_0)\left(1 + \frac{h(r)}{g(r)}\right) \int_0^1 q(s)\,ds,$$

which means that

$$\text{the functions belonging to } \{y_n(t)\} \text{ are equicontinuous on } [0, t_0]. \tag{3.25}$$

For $t_1, t_2 \in [t_0, 1)$, from equation (3.21), using y_n in place of y_0, we have

$$\left| \int_{y_n(t_1)}^{y_n(t_2)} \frac{1}{g(y)}\,dy \right| \le \left(1 + \frac{h(r)}{g(r)}\right) \int_0^1 q(s)\,ds |t_1 - t_2|,$$

which yields

$$\text{the functions belonging to } \{y_n(t)\} \text{ are equicontinuous on } [t_0, 1]. \tag{3.26}$$

Combining equations (3.25) and (3.26), we find that equation (3.24) holds.

(2) There exists a $k_1 > 0$ such that $y_n(0) = \|y_n\|$ and $y_n(t)$ is nonincreasing on $[0,1]$ for all $n > k_1$. From $y_n(0) = H(\phi(y_n)) = \|y_n\|$ and $y_n(1) = 0$, there exists $t_n \in (0,1)$ such that $y_n'(t_n) = -H(\phi(y_n))$. Now $y_n''(t) \le 0$ implies that $y_n'(0) \ge y_n'(t_n) = -H(\phi(y_n))$. Hence, from equation (3.20), using y_n in place of y_0, we have

$$-y_n'(t) + y_n'(0) \le g\big(y_n(t)\big)\left(1 + \frac{h(r)}{g(r)}\right) \int_0^t q(s)\,ds, \quad t \in (0,1)$$

and so

$$\begin{aligned} -\frac{y_n'(t)}{g(y_n(t))} &\le \left(1 + \frac{h(r)}{g(r)}\right) \int_0^t q(s)\,ds - \frac{y_0'(0)}{g(y_n(t))} \\ &\le \left(1 + \frac{h(r)}{g(r)}\right) \int_0^t q(s)\,ds + \frac{H(\phi(y_n))}{g(y_n(t))} \\ &\le \left(1 + \frac{h(r)}{g(r)}\right) \int_0^t q(s)\,ds + \frac{1}{g(r)} \max_{s \in [0, c_0 r]} H(r), \quad t \in (0,1). \end{aligned}$$

Then

$$\begin{aligned} \left| \int_{y_n(t_1)}^{y_n(t_2)} \frac{1}{g(y)}\,dy \right| &= \left| \int_{t_1}^{t_2} \frac{y_n'(s)}{g(y_n(s))}\,ds \right| \\ &\le \left(1 + \frac{h(r)}{g(r)}\right) \left| \int_{t_1}^{t_2} \int_0^s q(\tau)\,d\tau\,ds \right| + \frac{1}{g(r)} \max_{s \in [0, c_0 r]} H(r) |t_1 - t_2|, \\ &\quad \forall t_1, t_2 \in [0,1], \end{aligned}$$

which implies that (3.24) hold.

Now Arzela-Ascoli theorem guarantees that $\{y_n(t)\}$ has a convergent subsequence. Without loss of generality, we assume that there is a $y_* \in C[0,1]$ such that

$$\lim_{n \to +\infty} y_n = y_*,$$

which together with equation (3.22) and $y_n(1) = 0$ implies that

$$y_*(1) = 0, \qquad y_*(t) \geq \delta_r(1-t), \quad \forall t \in [0,1]. \tag{3.27}$$

Since y_n ($n \in \mathbb{N}$) satisfies $y_n = T_{\frac{1}{n}} y_n$, we have

$$y_n''(t) = -q(t)f\left(t, \max\left\{\frac{1}{n}, y_n(t)\right\}\right) = 0, \quad 0 < t < 1.$$

We integrate the above equation from $\frac{1}{2}$ to t to yield

$$y_n'(t) = y_n'\left(\frac{1}{2}\right) - \int_{\frac{1}{2}}^{t} q(s)f\left(s, \max\left\{\frac{1}{n}, y_n(s)\right\}\right) ds,$$

and so

$$\begin{aligned} y_n(t) &= y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) - \int_{\frac{1}{2}}^{t}\int_{\frac{1}{2}}^{s} q(\tau)f\left(\tau, \max\left\{\frac{1}{n}, y_n(\tau)\right\}\right) d\tau\, ds \\ &= y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{\frac{1}{2}}^{t} (s-t)q(s)f\left(s, \max\left\{\frac{1}{n}, y_n(s)\right\}\right) ds \end{aligned}$$

for $t \in (0,1)$ and

$$y_n(0) = H\bigl(\phi(y_n)\bigr) = H\left(\int_0^1 y_n(s)\, d\alpha_1(s) + \int_0^1 y_n(s)\, d\alpha_2(s)\right),$$

and the Lebesgue Dominated Convergent theorem together with equation (3.27) implies that

$$\begin{aligned} y_*(t) &= \lim_{n\to+\infty} y_n(t) \\ &= \lim_{n\to+\infty}\left[y_n\left(\frac{1}{2}\right) + y_n'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{\frac{1}{2}}^{t} (s-t)q(s)f\left(s, \max\left\{\frac{1}{n}, y_n(s)\right\}\right) ds\right] \\ &= y_*\left(\frac{1}{2}\right) + y_*'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) + \int_{\frac{1}{2}}^{t} (s-t)q(s)f\bigl(s, y_*(s)\bigr) ds \end{aligned} \tag{3.28}$$

for $t \in (0,1)$ and

$$\begin{aligned} y_*(0) &= \lim_{n\to+\infty} y_n(0) \\ &= \lim_{n\to+\infty} H\bigl(\phi(y_n)\bigr) \\ &= \lim_{n\to+\infty} H\left(\int_0^1 y_n(s)\, d\alpha_1(s) + \int_0^1 y_n(s)\, d\alpha_2(s)\right) \\ &= H\bigl(\phi_1(y_*) + \phi_2(y_*)\bigr) \\ &= H\bigl(\phi(y_*)\bigr). \end{aligned} \tag{3.29}$$

We differentiate equation (3.28) to get

$$y_*''(t) + q(t)f\big(t, y_*(t)\big) = 0, \quad t \in (0,1),$$

which together with equations (3.27) and (3.29) means that the BVP (1.1)-(1.2) has at least one positive solution. The proof is complete. □

Theorem 3.2 *Suppose the conditions of Theorem* 3.1 *hold and there exists an* $a \in (0, \frac{1}{2})$ *such that*

$$\lim_{y \to +\infty} \frac{f(t,y)}{y} = +\infty$$

uniformly on $[a, 1-a]$. *Then the BVP* (1.1)-(1.2) *has at least two positive solutions*.

Proof Choose $r > 0$ as in (3.17), $n_0 > 0$ with $\frac{1}{n_0} < \min\{1, r\}$, and $R > \max\{r, R'\}$ in Lemma 3.3. Set $\mathbb{N}_{n_0} = \{n_0, n_0 + 1, \ldots\}$, and

$$\Omega_1 = \big\{y \in C[0,1] : \|y\| < r\big\},$$
$$\Omega_2 = \big\{y \in C[0,1] : \|y\| < R\big\}.$$

By the proof of Theorem 3.1 and Lemma 3.3, we have

$$i(T_{\frac{1}{n}}, \Omega_1 \cap P, P) = 1$$

and

$$i(T_{\frac{1}{n}}, \Omega_2 \cap P, P) = 0,$$

which implies that

$$i\big(T_{\frac{1}{n}}, (\Omega_2 - \overline{\Omega}_1) \cap P, P\big) = -1.$$

Then, there exist $x_{1,n} \in \Omega_1 \cap P$ and $x_{2,n} \in (\Omega_2 - \overline{\Omega}_1) \cap P$ such that

$$T_{\frac{1}{n}} x_{1,n} = x_{1,n}, \qquad T_{\frac{1}{n}} x_{2,n} = x_{2,n}.$$

By the proof of Theorem 3.1, there exist a subsequence $\{x_{1,n_i}\}$ of $\{x_{1,n}\}$ and $x_1 \in P$ such that

$$\lim_{n_i \to +\infty} x_{1,n_i}(t) = x_1(t), \quad t \in [0,1].$$

And moreover, $x_1(t)$ is a positive solution to the BVP (1.1)-(1.2) with $r > x_1(t) \geq \delta_r(1-t)$, $\forall t \in [0,1]$.

A similar argument shows that there exist a subsequence $\{x_{2,n_j}\}$ of $\{x_{2,n}\}$ and $x_2 \in P \cap (\Omega_2 - \overline{\Omega}_1)$ such that

$$\lim_{n_i \to +\infty} x_{2,n_j}(t) = x_2(t), \quad t \in [0,1].$$

And moreover, $x_2(t)$ is a positive solution to the BVP (1.1)-(1.2) and equation (3.18) guarantees that $\|x_2\| > r$. Hence, $x_1(t)$ and $x_2(t)$ are two positive solutions for the BVP (1.1)-(1.2). The proof is complete. □

Theorem 3.3 *Suppose the conditions of Theorem* 3.1 *hold and*

$$\lim_{s\to+\infty}\frac{H(s)}{s}=+\infty.$$

Then the BVP (1.1)-(1.2) *has at least two positive solutions.*

Proof Choose $r > 0$ as in (3.17), $n_0 > 0$ with $\frac{1}{n_0} < \min\{1, r\}$, and $R > \max\{r, R'\}$ in Lemma 3.4. Set $\mathbb{N}_{n_0} = \{n_0, n_0 + 1, \dots\}$, and

$$\Omega_1=\bigl\{y\in C[0,1]:\|y\|<r\bigr\},$$
$$\Omega_2=\bigl\{y\in C[0,1]:\|y\|<R\bigr\}.$$

By the proof of Theorem 3.1 and Lemma 3.4, we have

$$i(T_{\frac{1}{n}},\Omega_1\cap P,P)=1$$

and

$$i(T_{\frac{1}{n}},\Omega_2\cap P,P)=0,$$

which implies that

$$i\bigl(T_{\frac{1}{n}},(\Omega_2-\overline{\Omega}_1)\cap P,P\bigr)=-1.$$

Then, there exist $x_{1,n} \in \Omega_1 \cap P$ and $x_{2,n} \in (\Omega_2 - \overline{\Omega}_1) \cap P$ such that

$$T_{\frac{1}{n}}x_{1,n}=x_{1,n},\qquad T_{\frac{1}{n}}x_{2,n}=x_{2,n}.$$

A similar argument to that in Theorem 3.2 shows that the BVP (1.1)-(1.2) has at least two positive solutions. The proof is complete. □

Example 3.1 Consider

$$y''(t)+\mu\frac{1}{\sqrt{1-t}}\left(\frac{1}{200}+\frac{1}{300}\sin t^2+\frac{1}{100}y^{-\delta_1}(t)+\frac{1}{100}y^{\delta_2}(t)\right)=0,\quad 0<t<1,\quad (3.30)$$

with

$$y(0)=H\bigl(\phi(y)\bigr),\qquad y(1)=0,\qquad (3.31)$$

where

$$H(t)=\frac{1}{2}t+\frac{1}{3}t^{\frac{1}{3}},\qquad \phi(y)=\phi_1(y)+\phi_2(y)=\int_0^1 y(s)\,d\alpha_1(s)+\int_0^1 y(s)\,d\alpha_2(s),$$

with

$$d\alpha_1(s) = \frac{1}{8}\cos 2\pi s\,ds, \qquad d\alpha_2(s) = \frac{1}{8}\,de^s, \\ \delta_1 > 0, \qquad \delta_2 > 1, \qquad \frac{100}{(\delta_1+1)3} > 1. \tag{3.32}$$

Then equations (3.30)-(3.31) have at least two positive solutions.

To prove that the BVP (3.30)-(3.31) has at least two positive solutions, we use Theorem 3.2. Let $q(t) = \mu\frac{1}{\sqrt{1-t}}$, $f(t,y) = \frac{1}{200} + \frac{1}{300}\sin t^2 + \frac{1}{100}y^{-\delta_1} + \frac{1}{100}y^{\delta_2}$, $g(y) = \frac{1}{100}y^{-\delta_1}$, $h(y) = \frac{1}{100} + \frac{1}{100}y^{\delta_2}$, $c_0 = \int_0^1 |d\alpha_1(s)| + \int_0^1 |d\alpha_2(s)| = \frac{1}{4\pi} + \frac{e-1}{8}$, $b_0 = \frac{2}{3}\mu$. For $y \in P$ (defined in (2.1)), we have

$$\phi_2(y) = \int_0^1 y(t)\frac{1}{8}e^s\,ds \ge \|y\|\int_0^1 s(1-s)\frac{1}{8}e^s\,ds,$$

which means that (C_1) holds. Since

$$\int_0^1 (1-t)\,d\alpha_1(t) = 0, \qquad \int_0^1 (1-t)\,d\alpha_2(t) > 0,$$
$$\int_0^1 k(t,s)\,d\alpha_1(t) = (1-s)\int_0^s t\,d\alpha_1(t) + s\int_s^1 (1-t)\,d\alpha_1(t) = \frac{1-\cos 2\pi s}{32\pi^2} \ge 0,$$

and

$$\int_0^1 k(t,s)\,d\alpha_2(t) = (1-s)\int_0^s t\,d\alpha_2(t) + s\int_s^1 (1-t)\,d\alpha_2(t) \ge 0,$$

(C_2) is true. Since $c_0 < 1$, we have $\max_{y\in[0,c_0 r]} H(y) = \frac{1}{2}c_0 r + \frac{1}{3}(c_0 r)^{\frac{1}{3}} \le \frac{1}{2}r + \frac{1}{3}r^{\frac{1}{3}}$. Then

$$\frac{1}{\max_{y\in[0,c_0 1]} H(y)} = \frac{1}{\frac{1}{2}c_0 1 + \frac{1}{3}(c_0 1)^{\frac{1}{3}}} > 1.$$

Equation (3.32) guarantees that

$$\frac{1}{1+\frac{h(1)}{g(1)}}\int_0^1 \frac{1}{g(y)}\,dy = \frac{100}{3(1+\delta_1)} > 1.$$

Letting $\mu_0 < 3$, we have

$$\sup_{r\in(0,+\infty)} \min\left\{\frac{1}{1+\frac{h(r)}{g(r)}}\int_0^r \frac{dy}{g(y)}, \frac{r}{\max_{y\in[0,c_0 r]} H(y)}\right\} > \max\{1, b_0\},$$

for all $\mu \le \mu_0$, which means that equations (3.15)-(3.16) hold. Since

$$f(t,x) \ge \frac{1}{200} + \frac{1}{300}\sin t^2, \quad \forall (t,x) \in [0,1]\times(0,1],$$

we get (C_4). Moreover, since

$$\lim_{y\to+\infty} \frac{f(t,y)}{y} = +\infty$$

uniformly on $[0,1]$, all conditions of Theorem 3.2 hold, which implies that equations (3.30)-(3.31) have at least two positive solutions.

Example 3.2 Consider

$$y''(t) + \mu y^{-\delta_1}(t) = 0, \quad 0 < t < 1, \tag{3.33}$$

with

$$y(0) = H\big(\phi(y)\big), \qquad y(1) = 0, \tag{3.34}$$

where

$$H(t) = \frac{1}{2}t^3 + \frac{1}{3}t^{\frac{1}{3}}, \qquad \phi(y) = \phi_1(y) + \phi_2(y) = \int_0^1 y(s)\, d\alpha_1(s) + \int_0^1 y(s)\, d\alpha_2(s),$$

with

$$d\alpha_1(s) = \frac{1}{8}\cos 2\pi s\, ds, \qquad d\alpha_2(s) = \frac{1}{8}\, de^s, \qquad \delta_1 > 0.$$

Then equations (3.33)-(3.34) have at least two positive solutions.

To prove that the BVP (3.33)-(3.34) has at least two positive solutions, we use Theorem 3.3. Let $q(t) = \mu, f(t,y) = y^{-\delta_1}, g(y) = y^{-\delta_1}, h(y) = 0$, $c_0 = \frac{1}{4\pi} + \frac{e-1}{8}$, $b_0 = \frac{1}{2}\mu$. Since $c_0 < 1$, we have $\max_{y\in[0,c_0 r]} H(y) = \frac{1}{2}(c_0 r)^3 + \frac{1}{3}(c_0 r)^{\frac{1}{3}} \leq \frac{1}{2}r^3 + \frac{1}{3}r^{\frac{1}{3}}$. Then

$$\frac{1}{\max_{y\in[0,c_0 1]} H(y)} = \frac{1}{\frac{1}{2}(c_0 1)^3 + \frac{1}{3}(c_0 1)^{\frac{1}{3}}} > 1.$$

Also we have

$$\lim_{r\to+\infty} \int_0^r \frac{dy}{g(y)} \left(1 + \frac{h(r)}{g(r)}\right)^{-1} = +\infty.$$

Then, letting $\mu_0 \leq 2$, we get

$$\sup_{r\in(0,+\infty)} \min\left\{\frac{1}{1+\frac{h(r)}{g(r)}} \int_0^r \frac{dy}{g(y)}, \frac{r}{\max_{y\in[0,c_0 r]} H(y)}\right\} > \max\{1, b_0\},$$

for all $\mu \leq \mu_0$, which means that equations (3.15)-(3.16) hold. Since

$$f(t,x) \geq 1, \quad \forall (t,x) \in [0,1] \times (0,1],$$

we get (C_4). Obviously, (C_1)-(C_3), and (C_5) hold. Moreover, since

$$\lim_{y\to+\infty} \frac{H(s)}{s} = +\infty$$

uniformly on $[0,1]$, all conditions of Theorem 3.3 hold, which implies that equations (3.30)-(3.31) have at least two positive solutions.

4 Positive solutions for singular boundary-value problems with sign-changing nonlinearities

(H_1) Assume that there are three linear functionals $\phi, \phi_1, \phi_2 : C([0,1]) \to R$

$$\phi(y) = \phi_1(y) + \phi_2(y), \qquad \phi_1(y) := \int_0^1 y(t)\,d\alpha_1(t), \qquad \phi_2(y) := \int_0^1 y(t)\,d\alpha_2(t),$$

where $\alpha_1, \alpha_2 : [0,1] \to R$ satisfy $\alpha_1, \alpha_2 \in BV([0,1])$;

(H_2) $a(t) \in C([0,1], (0,+\infty))$, $(1-t)q(t) \in L^1((0,1])$;

(H_3) Let $H : R \to [0,+\infty)$ be a real-valued, continuous function. Moreover, $H : (0,+\infty) \to (0,+\infty)$;

(H_4) $f(t,y) \in C([0,1] \times (0,+\infty), (-\infty,+\infty))$, there exists a decreasing function $F(y) \in C((0,+\infty),(0,+\infty))$, and a nonnegative function $G(y) \in C([0,+\infty),[0,+\infty))$ such that $f(t,y) \le F(y) + G(y)$ and there exists a $b \in C((0,1),(0,+\infty))$ such that

$$f(t,y) \ge a(t), \quad \forall 0 < y \le b(t), t \in (0,1);$$

(H_5) there exist $R > 1$ such that

$$\int_0^R \frac{dy}{F(y)} \cdot \left(1 + \frac{\bar{G}(R)}{F(R)}\right)^{-1} > \int_0^1 (1-s)q(s)\,ds$$

and

$$\max_{y\in[0,rc_0]} H(y) < r, \quad \forall R \ge r > 0, \text{where } c_0 = \int_0^1 \left|d\alpha_1(s)\right| + \int_0^1 \left|d\alpha_2(s)\right|,$$

where $\bar{G}(R) = \max_{s\in[0,R]} G(s)$.

For $n > 3$, let $b_n = \min\{\frac{1}{n}, \min_{t\in[\frac{1}{n},1-\frac{1}{n}]} b(t)\}$. Obviously, $b_n > 0$. For $y \in C_n = C[\frac{1}{n}, 1-\frac{1}{n}]$, we define T_n as

$$(T_n y)(t) = \left(1 - \frac{1}{n} - t\right)H\bigl(\phi_n(y)\bigr) + b_n + \int_{\frac{1}{n}}^{1-\frac{1}{n}} k_n(t,s)q(s)f\bigl(s, \max\{b_n, y(s)\}\bigr)\,ds,$$
$$t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right],$$

where

$$k_n(t,s) = \begin{cases} (s-\frac{1}{n})(1-\frac{1}{n}-t), & \frac{1}{n} \le s \le t \le 1-\frac{1}{n}, \\ (t-\frac{1}{n})(1-\frac{1}{n}-s), & \frac{1}{n} \le t \le s \le 1-\frac{1}{n} \end{cases}$$

and

$$\phi_n(y) = \int_{\frac{1}{n}}^{1-\frac{1}{n}} y(s)\,d\alpha_1(s) + \int_{\frac{1}{n}}^{1-\frac{1}{n}} y(s)\,d\alpha_2(s).$$

From a standard argument (see [18, 25, 26]), we have the following result.

Lemma 4.1 *Suppose* (H_1)-(H_4) *hold. Then the operator* T_n *is continuous and compact from* C_n *to* C_n.

From (H_3) and (H_5), there exists $\epsilon_0 > 0$ such that

$$\begin{aligned}&\int_{\epsilon_0}^{R} \frac{dy}{F(y)} \cdot \left(1 + \frac{\bar{G}(R)}{F(R)}\right)^{-1} > \int_0^1 (1-s)q(s)\,ds,\\&\max_{y\in[0,c_0R]} H(y) + \epsilon_0 < R.\end{aligned} \tag{4.1}$$

Choose $n_0 > 3$ with $\frac{1}{n_0} < \epsilon_0$ and let $\mathbb{N}_{n_0} = \{n_0, n_0+1, \ldots\}$. Now we have the following lemmas.

Lemma 4.2 *Suppose* (H_1)-(H_5) *hold. Then, for* $n \in \mathbb{N}_0$, *there exists a* $x_n \in C_n$ *with* $b_n \le x_n(t) \le R$ *such that*

$$x_n(t) = \left(1 - \frac{1}{n} - t\right)H\big(\phi_n(x_n)\big) + b_n + \int_{\frac{1}{n}}^{1-\frac{1}{n}} k_n(t,s)q(s)f\big(s, x_n(s)\big)\,ds, \quad t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right].$$

Proof Let $\Omega = \{y \in C_n : \|y\| < R\}$. For $y \in \partial\Omega$, we now prove that

$$\begin{aligned}y(t) \ne \lambda(T_n y)(t) &= \lambda\left(\left(1 - \frac{1}{n} - t\right)H\big(\phi_n(y)\big) + b_n\right)\\&\quad + \lambda\int_{\frac{1}{n}}^{1-\frac{1}{n}} k_n(t,s)q(s)f\big(s, \max\{b_n, y(s)\}\big)\,ds, \quad t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right]\end{aligned} \tag{4.2}$$

for any $\lambda \in (0,1]$.

Suppose equation (4.2) is not true. Then there exists $y \in C[\frac{1}{n}, 1 - \frac{1}{n}]$ with $\|y\| = R$ and $0 < \lambda < 1$ such that

$$\begin{aligned}y(t) = \lambda(Ty)(t) &= \lambda\left(\left(1 - \frac{1}{n} - t\right)H\big(\phi_n(y)\big) + b_n\right)\\&\quad + \lambda\int_{\frac{1}{n}}^{1-\frac{1}{n}} k_n(t,s)q(s)f\big(s, \max\{b_n, y(s)\}\big)\,ds, \quad t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right].\end{aligned} \tag{4.3}$$

We first claim that $y(t) \ge \lambda b_n$ for any $t \in [\frac{1}{n}, 1 - \frac{1}{n}]$.

Suppose there exists a $\eta \in (0,1)$ with $y(\eta) < \lambda b_n$. Let $\gamma_0 = \inf\{t_1 : y(s) < \lambda b_n, \forall s \in [t_1, \eta]\}$ and $\gamma_1 = \sup\{t_1 : y(s) < \lambda b_n, \forall s \in [\eta, t_1]\}$. Since $y(\frac{1}{n}) \ge \lambda b_n$ and $y(1 - \frac{1}{n}) = \lambda b_n$, we have $\gamma_0 \ge \frac{1}{n}$, $\gamma_1 \le 1 - \frac{1}{n}$, $y(\gamma_0) = y(\gamma_1) = \lambda b_n$, and $y(t) < \lambda b_n$ for all $t \in (\gamma_0, \gamma_1)$, which implies that

$$y''(t) = -\lambda q(t)f(t, b_n) < 0, \quad t \in (\gamma_0, \gamma_1)$$

and so $y(t)$ is concave down on $[\gamma_0, \gamma_1]$. This is a contradiction.

Now (H_5) guarantees that

$$y\left(\frac{1}{n}\right) = \lambda\left(\left(1 - \frac{2}{n}\right)H\big(\phi_n(y)\big) + b_n\right) \le \max_{r\in[0,c_0R]} h(r) + \epsilon_0 < R,$$

which together with $y(1-\frac{1}{n}) = \lambda b_n < R$ means that there is a $t \in (\frac{1}{n}, 1-\frac{1}{n})$ with $y'(t) = 0$ and $y(t) = R$. Let $t^* = \sup\{t : y(t) = R, y'(t) = 0\}$ and $t_* = \inf\{t : y(t) = R, y'(t) = 0\}$. Obviously, $\frac{1}{n} < t_* \leq t^* < 1-\frac{1}{n}$, $y(t_*) = R$, $y'(t_*) = 0$, $y(t^*) = R$, $y'(t^*) = 0$, $y(t) < R$ for all $t \in (t^*, 1-\frac{1}{n}]$ and $y(t) < R$ for all $t \in (\frac{1}{n}, t_*]$. Let $t_1 = \inf\{t^* < t \leq 1-\frac{1}{n} : y(t) = \lambda y(1-\frac{1}{n})\}$ and $t_1' = \sup\{t < t_* \leq 1-\frac{1}{n} : y(t) = \lambda y(\frac{1}{n})\}$. It is easy to see that $t^* < t_1 \leq 1-\frac{1}{n}$, $y(t) > y(t_1)$ for all $t \in (t^*, t_1)$, $t_1' < t_*$ and $y(t) > y(t_1')$ for all $t \in (t_1', t_*)$.

Now we consider the properties of y on (t^*, t_1). We get a countable set $\{t_i\}$ of $(t^*, t_1]$ such that

1. $t^* > \cdots \geq t_{2m} > t_{2m-1} > \cdots > t_5 \geq t_4 > t_3 \geq t_2 > t_1 = t_1$, $t_{2m} \to t^*$,
2. $y(t_{2i}) = y(t_{2i+1})$, $y'(t_{2i}) = 0$, $i = 1, 2, 3, \ldots$,
3. $y(t)$ is strictly decreasing in $[t_{2i}, t_{2i-1}]$, $i = 1, 2, 3, \ldots$ (if $y(t)$ is strictly decreasing in $[t^*, t_1]$, put $m = 1$; *i.e*, $[t_2, t_1] = [t^*, t_1]$).

Differentiating equation (4.3) and using the assumptions (H_2) and (H_4), we obtain

$$\begin{aligned} -y''(t) &= \lambda q(t) f\bigl(t, \max\{b_n, y(t)\}\bigr) \\ &\leq \lambda q(t)\bigl(F\bigl(\max\{b_n, y(t)\}\bigr) + G\bigl(\max\{b_n, y(t)\}\bigr)\bigr) \\ &= \lambda q(t) F\bigl(\max\{b_n, y(t)\}\bigr)\left(1 + \frac{G(\max\{b_n, y(t)\})}{F(\max\{b_n, y(t)\})}\right) \\ &< q(t) F\bigl(\max\{b_n, y(t)\}\bigr)\left(1 + \frac{\bar{G}(R)}{F(R)}\right) \\ &\leq q(t) F\bigl(y(t)\bigr)\left(1 + \frac{\bar{G}(R)}{F(R)}\right), \quad t \in [t_{2i}, t_{2i-1}), i = 1, 2, 3, \ldots. \end{aligned} \tag{4.4}$$

Integrating (4.4) from t_{2i} to t, we have, by the decreasing property of $F(y)$,

$$-\int_{t_{2i}}^t y''(s)\,ds \leq \left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{t_{2i}}^t q(s) F\bigl(y(s)\bigr)\,ds \leq F\bigl(y(t)\bigr)\left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{t_{2i}}^t q(s)\,ds,$$

for $t \in [t_{2i}, t_{2i-1})$, $i = 1, 2, 3, \ldots$; that is to say,

$$-y'(t) \leq F\bigl(y(t)\bigr)\left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{t_{2i}}^t q(s)\,ds, \quad t \in [t_{2i}, t_{2i-1}), i = 1, 2, 3, \ldots. \tag{4.5}$$

It follows from equation (4.5) that

$$-\frac{y'(t)}{F(y(t))} \leq \left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{t_{2i}}^t q(s)\,ds \leq \left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_0^t q(s)\,ds, \tag{4.6}$$

for $t \in [t_{2i}, t_{2i-1})$, $i = 1, 2, 3, \ldots$.

On the other hand, for any $z \in (\frac{1}{n}, 1-\frac{1}{n})$ with $y(z) > \lambda b_n$, we can choose i_0 and $z' \in (t^*, t_1)$ such that $z' \in [t_{2i_0}, t_{2i_0-1})$, $y(z') = y(z)$ and $z \leq z'$. Integrating equation (4.6) from t_{2i} to t_{2i-1}, $i = 1, 2, 3, \ldots, i_0 - 1$ and from t_{2i_0} to z', we have

$$\int_{y(t_{2i-1})}^{y(t_{2i})} \frac{dy}{F(y)} \leq \left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{t_{2i}}^{t_{2i-1}} \int_0^t q(s)\,ds\,dt, \quad i = 1, 2, 3, \ldots, i_0 - 1, \tag{4.7}$$

and

$$\int_{y(t_{2i_0})}^{y(z')} \frac{dy}{F(y)} \le \left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{z'}^{t_{2i_0}} \int_0^t q(s)\,ds\,dt. \tag{4.8}$$

Summing equation (4.7) from 1 to $i_0 - 1$, we have by equation (4.8) and $y(t_{2i}) = y(t_{2i+1})$

$$\int_{y(t_1)}^{y(z')} \frac{dy}{F(y)} \le \left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{z'}^{t_1} \int_0^t q(s)\,ds\,dt \le \left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{z}^{t_1} \int_0^t q(s)\,ds\,dt.$$

Since $y(z) = y(z')$,

$$\int_{y(t_1)}^{y(z)} \frac{dy}{F(y)} \le \left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{z}^{t_1} \int_0^t q(s)\,ds\,dt. \tag{4.9}$$

For the properties of y on (t_1', t_*), a similar argument shows that for any $z > t_1'$

$$\int_{y(t_1')}^{y(z)} \frac{dy}{F(y)} \le \left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{t_1'}^{z} \int_0^t q(s)\,ds\,dt. \tag{4.10}$$

Letting $z \to t^*$ in (4.9), we have

$$\begin{aligned}
\int_{\epsilon_0}^{R} \frac{dy}{F(y)} \le \int_{y(t_1)}^{R} \frac{dy}{F(y)} &\le \left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{t^*}^{t_1} \int_0^t q(s)\,ds\,dt \\
&\le \left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{0}^{1} \int_0^t q(s)\,ds\,dt \\
&= \left(1 + \frac{\bar{G}(R)}{F(R)}\right) \int_{0}^{1} (1-s)q(s)\,ds,
\end{aligned}$$

which contradicts equation (4.1). Hence equation (4.2) holds.

It follows from Lemma 3.2 that T_n has a fixed point x_n in C_n. Using x_n and 1 in place of y and λ in (4.3), we obtain easily $b_n \le x_n(t) \le R$, $t \in [\frac{1}{n}, 1 - \frac{1}{n}]$. And x_n satisfies

$$\begin{aligned}
x_n(t) &= \left(1 - \frac{1}{n} - t\right) H\big(\phi_n(x_n)\big) + b_n + \int_0^1 k_n(t,s) q(s) f\big(s, x_n(s)\big)\,ds, \\
&\quad t \in \left[\frac{1}{n}, 1 - \frac{1}{n}\right].
\end{aligned} \tag{4.11}$$

The proof is complete. □

Lemma 4.3 *Suppose that all conditions of Lemma* 4.2 *hold and* x_n *satisfies* (4.11). *For a fixed* $h \in (0, \min\{\frac{1}{2}, \eta\})$, *let* $m_{n,h} = \min\{x_n(t), t \in [h, 1-h]\}$. *Then* $m_h = \inf\{m_{n,h}\} > 0$.

Proof Since $x_n(t) \ge b_n > 0$, we get $m_h \ge 0$. For any fixed natural number n ($n > n_0$ defined in Lemma 4.2), let $t_n \in [h, 1-h]$ such that $x_n(t_n) = \min\{x_n(t), t \in [h, 1-h]\}$. If $m_h = 0$, there exists a countable set $\{n_i\}$ such that

$$\lim_{n_i \to +\infty} x_{n_i}(t_{n_i}) = 0.$$

So there exists N_0 such that $x_{n_i}(t_{n_i}) \leq \min\{b(t), t \in [\frac{h}{2}, 1-h]\}$, $n_i > N_0$. Let $\overline{\mathbb{N}}_0 = \{n_0 > N_0 : n \in \mathbb{N}_0$ with $\lim_{n_i \to +\infty} x_{n_i}(t_{n_i}) = 0\}$. Then we have two cases.

Case 1. There exist $n_k \in \overline{\mathbb{N}}_0$ and $t^*_{n_k} \in [\frac{h}{2}, h]$ such that $x_{n_k}(t^*_{n_k}) \geq x_{n_k}(t_{n_k})$. By the same argument in Lemma 4.2, we can get $t'_{n_k}, t''_{n_k} \in [\frac{h}{2}, 1]$, $t'_{n_k} < t''_{n_k}$ such that

$$\begin{aligned} &x_{n_k}(t) \leq \min\left\{b(t), t \in \left[\frac{h}{2}, 1\right]\right\}, \quad t \in \left[t'_{n_k}, t''_{n_k}\right], \\ &x_{n_k}(t) \leq x_{n_k}\left(t'_{n_k}\right), \qquad x_{n_k}(t) \leq x_{n_k}\left(t''_{n_k}\right), \quad t \in \left(t'_{n_k}, t''_{n_k}\right), \end{aligned} \tag{4.12}$$

and

$$x''_{n_k}(t) = -q(t)f\left(t, x_{n_k}(t)\right) < 0, \quad t \in \left(t'_{n_k}, t''_{n_k}\right). \tag{4.13}$$

The inequality (4.13) shows that $x_{n_k}(t)$ is concave down in $[t'_{n_k}, t''_{n_k}]$, which contradicts equation (4.12).

Case 2. $x_{n_i}(t) < x_{n_i}(t_{n_i})$, $t \in [\frac{h}{2}, h]$ for any $n_i \in \overline{\mathbb{N}}_0$. And so we have

$$\lim_{n_i \to +\infty} x_{n_i}(t) = 0, \quad t \in \left[\frac{h}{2}, h\right]. \tag{4.14}$$

On the other hand, for any $t \in [\frac{h}{2}, h]$,

$$\begin{aligned} x_{n_i}(t) &= \frac{2}{h}\int_{\frac{h}{2}}^{t}\left(s - \frac{h}{2}\right)(h-t)q(s)f\left(s, x_{n_i}(s)\right)ds \\ &\quad + \frac{2}{h}\int_{t}^{h}\left(t - \frac{h}{2}\right)(h-s)q(s)f\left(s, x_{n_i}(s)\right)ds + x_{n_i}\left(\frac{h}{2}\right) + x_{n_i}(h) \\ &\geq \frac{2}{h}\left[\int_{\frac{h}{2}}^{t}\left(s - \frac{h}{2}\right)(h-t)a(s)\,ds + \int_{t}^{h}\left(t - \frac{h}{2}\right)(h-s)a(s)\,ds\right] > 0, \end{aligned}$$

which contradicts equation (4.14). Hence, $m_h > 0$. The proof is complete. □

Theorem 4.1 *If* (H_1)-(H_5) *hold, then BVP* (1.1)-(1.2) *has at least one positive solution.*

Proof For any natural number $n \in \mathbb{N}$ (defined in Lemma 4.2), it follows from Lemma 4.2 that there exist $x_n \in C_n$, $b_n \leq x_n(t) \leq R$ for all $t \in [\frac{1}{n}, 1 - \frac{1}{n}]$ satisfying (4.11). Now we divide the proof into three steps.

Step 1. There exists a convergent subsequence of $\{x_n\}$ in $(0,1)$. For a natural number $k \geq n_0$ in Lemma 4.2, it follows from Lemma 4.3 that $0 < m_{\frac{1}{k}} \leq x_n(t) \leq R$, $t \in [\frac{1}{k}, 1 - \frac{1}{k}]$ for any natural numbers $n \in N$; *i.e.*, $\{x_n\}$ is uniformly bounded in $[\frac{1}{k}, 1 - \frac{1}{k}]$. Since x_n also satisfies

$$\begin{aligned} x_n(t) &= \frac{1}{1 - \frac{2}{k}}\int_{\frac{1}{k}}^{t}\left(s - \frac{1}{k}\right)\left(1 - \frac{1}{k} - t\right)q(s)f\left(s, x_n(s)\right)ds \\ &\quad + \frac{1}{1 - \frac{2}{k}}\int_{t}^{1-\frac{1}{k}}\left(t - \frac{1}{k}\right)\left(1 - \frac{1}{k} - s\right)q(s)f\left(s, x_n(s)\right)ds + x_n\left(\frac{1}{k}\right) + x_n\left(1 - \frac{1}{k}\right), \end{aligned}$$

we have

$$\begin{aligned} x_n'(t) = &- \frac{1}{1-\frac{2}{k}} \int_{\frac{1}{k}}^{t} \left(s - \frac{1}{k}\right) q(s) f\bigl(s, x_n(s)\bigr)\, ds \\ &+ \frac{1}{1-\frac{2}{k}} \int_{t}^{1-\frac{1}{k}} \left(1 - \frac{1}{k} - s\right) q(s) f\bigl(s, x_n(s)\bigr)\, ds. \end{aligned}$$

Obviously

$$\bigl|x_n'(t)\bigr| \le 2\left(1 - \frac{2}{k}\right) \max\left\{ q(t)\bigl|f\bigl(t, x_n(t)\bigr)\bigr| : (t,x) \in \left[\frac{1}{k}, 1 - \frac{1}{k}\right] \times [m_{\frac{1}{k}}, R]\right\}, \tag{4.15}$$

for $t \in [\frac{1}{k}, 1 - \frac{1}{k}]$. It follows from inequality (4.15) that $\{x_n\}$ is equicontinuous in $[\frac{1}{k}, 1 - \frac{1}{k}]$. The Ascoli-Arzela theorem guarantees that there exists a subsequence of $\{x_n(t)\}$ which converges uniformly on $[\frac{1}{k}, 1 - \frac{1}{k}]$. Then, for $k = n_0$, we choose a convergent subsequence of $\{x_n\}$ on $[\frac{1}{n_0}, 1 - \frac{1}{n_0}]$,

$$x_{n_1(n_0)}(t), x_{n_2(n_0)}(t), x_{n_3(n_0)}(t), \ldots, x_{n_k(n_0)}(t), \ldots;$$

for $k = n_0 + 1$, we choose a convergent subsequence of $\{x_{n_k(n_0)}\}$ on $[\frac{1}{n_0+1}, 1 - \frac{1}{n_0+1}]$,

$$x_{n_1(n_0+1)}(t), x_{n_2(n_0+1)}(t), x_{n_3(n_0+1)}(t), \ldots, x_{n_k(n_0+1)}(t), \ldots;$$

for $k = n_0 + 2$, we choose a convergent subsequence of $\{x_{n_k(n_0+1)}\}$ on $[\frac{1}{n_0+2}, 1 - \frac{1}{n_0+2}]$,

$$x_{n_1(n_0+2)}(t), x_{n_2(n_0+2)}(t), x_{n_3(n_0+2)}(t), \ldots, x_{n_k(n_0+2)}(t), \ldots;$$

$$\ldots, \ldots, \ldots, \ldots;$$

for $k = n_0 + j$, we choose a convergent subsequence of $\{x_{n_k(n_0+j-1)}\}$ on $[\frac{1}{n_0+j}, 1 - \frac{1}{n_0+j}]$,

$$x_{n_1(n_0+j)}(t), x_{n_2(n_0+j)}(t), x_{n_3(n_0+j)}(t), \ldots, x_{n_k(n_0+j)}(t), \ldots;$$

$$\ldots, \ldots, \ldots, \ldots.$$

We may choose the diagonal sequence $\{x_{n_{k+1}(n_0+k)}(t)\}$ which converges everywhere in $(0,1)$ and it is easy to verify that $\{x_{n_{k+1}(n_0+k)}(t)\}$ converges uniformly on any interval $[c,d] \subseteq (0,1)$. Without loss of generality, let $\{x_{n_{k+1}(n_0+k)}(t)\}$ be $\{x_n(t)\}$ in the rest. Putting $x(t) = \lim_{n\to+\infty} x_n(t)$, $t \in (0,1)$, we have $x(t)$ continuous in $(0,1)$ and $x(t) \ge m_h > 0$, $t \in [h, 1-h]$ for any $h \in (0, \frac{1}{2})$ by Lemma 4.3.

Step 2. $x(t)$ satisfies equation (1.1). Fixed $t \in (0,1)$, we may choose $h \in (0, \frac{1}{2})$ such that $t \in (h, 1-h)$ and

$$\begin{aligned} x_n(t) = &\frac{1}{1-2h} \int_h^t (s-h)(1-h-t) q(s) f\bigl(s, x_n(s)\bigr)\, ds \\ &+ \frac{1}{1-2h} \int_t^{1-h} (t-h)(1-h-s) q(s) f\bigl(s, x_n(s)\bigr)\, ds + x_n(h) + x_n(1-h). \end{aligned}$$

Letting $n \to +\infty$ in above equation, we have

$$x(t) = \frac{1}{1-2h}\int_h^t (s-h)(1-h-t)q(s)f\bigl(s,x(s)\bigr)\,ds + \frac{1}{1-2h}\int_t^{1-h}(t-h)(1-h-s)q(s)f\bigl(s,x(s)\bigr)\,ds + x(h) + x(1-h). \tag{4.16}$$

Differentiating equation (4.16), we get the desired result.

Step 3. $x(t)$ satisfies equation (1.2). Let

$$t_n = \sup\left\{t : x_n(t) = \|x_n\|, x_n'(t) = 0, t \in \left[\frac{1}{n}, 1-\frac{1}{n}\right]\right\}$$

and

$$t_n' = \inf\left\{t : x_n(t) = \|x_n\|, x_n'(t) = 0, t \in \left[\frac{1}{n}, 1-\frac{1}{n}\right]\right\},$$

where $\|x_n\| = \max_{\frac{1}{n}\le t\le 1-\frac{1}{n}} x_n(t) \le R$. Then

$$t_n, t_n' \in \left[\frac{1}{n}, 1-\frac{1}{n}\right], \quad x_n(t_n) = x_n\bigl(t_n'\bigr) = \|x_n\|, \qquad x_n'(t_n) = x_n'\bigl(t_n'\bigr) = 0.$$

Using $x_n(t)$, 1, t_n in place of $y(t)$, λ and t^* in Lemma 4.2, from equation (4.9); we have

$$\int_{b_n}^{\|x_n\|}\frac{dx}{F(x)} \le \left(1+\frac{\bar{G}(R)}{F(R)}\right)\int_{t_n}^{1-\frac{1}{n}}\int_0^t q(s)\,ds\,dt$$

and using $x_n(t)$, 1, t_n' in place of $y(t)$, λ and t_* in Lemma 4.2, from equation (4.10), we obtain easily

$$\int_{x_n(\frac{1}{n})+b_n}^{\|x_n\|}\frac{dx}{F(x)} \le \left(1+\frac{\bar{G}(R)}{F(R)}\right)\int_{\frac{1}{n}}^{t_n'}\int_0^t q(s)\,ds\,dt.$$

It follows from the above inequalities that $a = \inf\{t_n'\} > 0$ and $b = \sup\{t_n\} < 1$.

(1) Fixing $z \in (b,1)$, we get $b_n < x_n(z) < \|x_n\| \le R$. From equation (4.9) of the proof in Lemma 4.2, one easily has

$$\int_{b_n}^{x_n(z)}\frac{dx}{F(x)} \le \left(1+\frac{\bar{G}(R)}{F(R)}\right)\int_z^{1-\frac{1}{n}}\int_0^t q(s)\,ds\,dt, \quad z \in (b,1).$$

Letting $n \to +\infty$ in the above inequality and noticing $b_n \to 0$, we have

$$\int_0^{x(z)}\frac{dx}{F(x)} \le \left(1+\frac{\bar{G}(R)}{F(R)}\right)\int_z^1\int_0^t q(s)\,ds\,dt, \quad z \in (b,1). \tag{4.17}$$

It follows from equation (4.17) that $x(1) = \lim_{z\to 1^-} x(z) = 0$.

(2) Fixing $z \in (0,a)$, we get $x_n(\frac{1}{n}) + b_n < x_n(z) < \|x_n\| \le R$. From equation (4.10) in the proof of Lemma 4.2, we easily get

$$\int_{x_n(\frac{1}{n})+b_n}^{x_n(z)}\frac{dx}{F(x)} \le \left(1+\frac{\bar{G}(R)}{F(R)}\right)\int_{\frac{1}{n}}^{z}\int_0^t q(s)\,ds\,dt, \quad z \in (0,a). \tag{4.18}$$

Since $\lim_{n\to+\infty} x_n(t) = x(t)$ and $\|x_n\| \leq R$, the Lebesgue Dominated Convergent theorem guarantees that

$$\lim_{n\to+\infty}\int_{\frac{1}{n}}^{1-\frac{1}{n}} x_n(t)\,d\alpha_1(t) = \int_0^1 x(t)\,d\alpha_1(t), \qquad \lim_{n\to+\infty}\int_{\frac{1}{n}}^{1-\frac{1}{n}} x_n(t)\,d\alpha_2(t) = \int_0^1 x(t)\,d\alpha_2(t).$$

Since H is continuous, we have

$$\lim_{n\to+\infty} x_n\left(\frac{1}{n}\right) = \lim_{n\to+\infty}\left(1-\frac{2}{n}\right)H\bigl(\phi_n(x_n)\bigr) = H\bigl(\phi(x)\bigr). \tag{4.19}$$

Letting $n \to +\infty$ in equation (4.18) and noticing $b_n \to 0$ and equation (4.19), we have

$$\int_{H(\phi(x))}^{x(z)} \frac{dx}{F(x)} \leq \left(1+\frac{\bar{G}(R)}{F(R)}\right)\int_0^z\int_0^t q(s)\,ds\,dt, \quad z\in(0,a). \tag{4.20}$$

It follows from equation (4.20) that $x(0) = \lim_{z\to 0+} x(z) = H(\phi(x))$. This complete the proof. □

Example 4.1 Consider

$$y''(t) + \frac{1}{8}\left(\frac{1}{217}y^2(t) + \frac{1}{100}\left(\frac{1}{y^2(t)} - \frac{y^3(t)}{t^{10}} - \frac{3}{t^4}\right)\right) = 0, \quad 0 < t < 1, \tag{4.21}$$

with boundary conditions

$$y(0) = \frac{1}{100}\left|\int_0^1 y(s)\,d\alpha_1(s) + \int_0^1 y(s)\,d\alpha_2(s)\right|^3, \qquad y(1) = 0, \tag{4.22}$$

where

$$d\alpha_1(s) = -\frac{1}{10}\cos 4\pi s\,ds, \qquad d\alpha_2(s) = \frac{1}{9}\bigl(e^s - 2\bigr)\,ds.$$

Then the BVP (4.21)-(4.22) has at least one positive solution.

Let $q(t) = \frac{1}{8}, f(t,y) = \frac{1}{217}y^2 + \frac{1}{100}(\frac{1}{y^2} - \frac{y^3}{t^{10}} - \frac{3}{t^4})$, $G(y) = \frac{1}{217}y^2$, $F(y) = \frac{1}{100y^2}$, $b(t) = \frac{1}{2}t^2$, $a(t) = \frac{7}{8t^4}$. Let $R = 2$ and $H(y) = \frac{1}{100}|y|^3$. We have

$$\int_0^2 \frac{1}{F(y)}\,dy\left(1+\frac{G(2)}{F(2)}\right)^{-1} > \frac{200}{9} > \frac{1}{16} = \int_0^1 (1-s)q(s)\,ds,$$

$$\max_{y\in[0,c_0 r]} H(r) = \frac{1}{100}(c_0 r)^3 < r, \quad \forall r\in(0,2],$$

where $c_0 = \int_0^1 |d\alpha_1(s)| + \int_0^1 |d\alpha_2(s)| < 1$ and

$$f(t,y) \geq a(t), \quad \forall 0 < y \leq b(t), t\in(0,1).$$

Then (H_1)-(H_5) hold. Now Theorem 4.1 guarantees that the BVP (4.21)-(4.22) has at least one positive solution.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author finished the paper himself.

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