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Finite element analysis of a nonlinear parabolic equation modeling epitaxial thin-film growth

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Abstract

In this paper, we consider a nonlinear model describing crystal surface growth. For the equation, the finite element method is presented and a nice error estimate is derived in the L^2 norm by means of a finite element biharmonic projection approximation.

1 Introduction

The finite element method is essentially a discretization method for the approximate solution of partial differential equations. It has the natural advantage of keeping the physical properties of the primitive problems. There are many papers that have already been published to study the finite element method for a fourth-order nonlinear parabolic equation (see [1–6]).

In this paper, we consider the finite element analysis for the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} + \gamma u_{xxxx} - (|u_x|^2 u_x - u_x)_x = 0, & (x,t) \in (0,\pi) \times (0,T), \\ u_x(x,t) = u_{xxx}(x,t) = 0, & x = 0, \pi, \\ u(x,0) = u_0(x), & \text{in } (0,\pi), \end{cases} \tag{1}$$

where γ is a positive constant.

Problem (1) arises in epitaxial growth of nanoscale thin films [7, 8], where $u(x,t)$ denotes the height from the surface of the film in epitaxial growth. The term u_{xxxx} denotes the capillarity-driven surface diffusion, u_{xx} denotes diffusion due to evaporation-condensation and $|u_x|^2 u_x$ corresponds to the upward hopping of atoms. During the past years, many authors have paid much attention to problem (1), for example [7, 9–12].

Here, we give the existence and uniqueness of a global solution for problem (1) (see[7]).

Theorem 1.1 *Suppose that* $H_E^2(0,\pi) = \{u \in H^2(0,\pi) : u_x(0) = u_x(\pi) = 0\}$, *and* $u_0 \in H_E^2(0,\pi) \cap W^{1,4}(0,\pi)$, *then there exists a unique global solution* $u(x,t)$ *for problem* (1), *such that*

$$u(x,t) \in C^0\big([0,T]; L^2(0,\pi)\big) \cap L^\infty\big([0,T]; H_E^2(0,\pi)\big) \cap L^\infty\big([0,T]; W^{1,4}(0,\pi)\big).$$

The outline of this paper is as follows. In the next section, we establish a semi-discrete approximation and derive its error bound. In Section 3, the full-discrete approximation

for problem (1) is studied. In the last section, some numerical experiments which confirm our results are presented.

Throughout this paper, we denote the L^2, L^p, L^∞, H^k norms in $(0,1)$ simply by $\|\cdot\|, \|\cdot\|_{L^p}$, $|\cdot|_\infty$, and $\|\cdot\|_k$. Define the inner product of L^2 space as $(\cdot,\cdot)$, we have the space

$$L^p(0,T;X) = \left\{ u(t) : \|u\|_{L^p(X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}} < \infty \right\}, \quad 1 \le p \le \infty.$$

On the other hand, the letters C, C' denote generic constants independent of the finite element division size and not necessarily the same at different occurrences.

2 Semi-discrete approximation

Let $I_h : 0 = x_0 < x_1 < \cdots < x_N = \pi$ be a finite element division for the interval $I = [0,\pi]$, $h_i = x_i - x_{i-1}$, $h = \max h_i$. Let $S_h^{(k)}$ be the piecewise polynomial spline space with the degree $k \ge 3$, and

$$S_h^{(k)} \subset H_E^2(I) = \left\{ u \in H^2(I) : u_x(0) = u_x(\pi) = 0 \right\}.$$

The weak formulation of problem (1) reads

$$\begin{cases} (u_t, v) + \gamma(D^2 u, D^2 v) + (|Du|^2 Du - Du, Dv) = 0, & \forall v \in H_E^2(I), \\ u(x,0) = u_0(x), \end{cases} \tag{2}$$

where $u_t = \frac{\partial u}{\partial t}$, $Du = \frac{\partial u}{\partial x}$. Based on (2), we define the semi-discrete finite element approximation to problem (1). Find $u_h(t) : (0,T] \to S_h^{(k)}$ such that

$$\begin{cases} (u_{h,t}, v_h) + \gamma(D^2 u_h, D^2 v_h) + (|Du_h|^2 Du_h - Du_h, Dv_h) = 0, & \forall v_h \in S_h^{(k)}, \\ (u_h(0) - u(0), v_h) = 0, & \forall v_h \in S_h^{(k)}. \end{cases} \tag{3}$$

It is clear that the conservation of mass for (3) holds as it does for the classical solution. Setting $v_h = 1 \in S_h^{(k)}$ in (3), we get

$$\int_0^\pi u_h(t,x) \, dx = \int_0^\pi u_h(0,x) \, dx, \quad 0 \le t \le T.$$

Theorem 2.1 *Let $u_h(0) \in H_E^2(I)$, then there exists a unique approximation solution $u_h(t) \in S_h^{(k)}$ for problem* (3), *such that*

$$\left\| u_h(t) \right\|_2 \le C, \quad 0 \le t \le T, \tag{4}$$

where C is a positive constant depending only on γ and $\|u_h(0)\|_2$, independent of h.

Proof The equation of problem (3) is an ordinary differential equation and according to ODE theory, there exists a unique local solution to problem (3) in the interval $[0, t_n]$. If we have (4), then according to the extension theorem, we can also obtain the existence of unique global solution. So, we only need to prove (4).

Setting $v_h = u_h$ in (3), we derive

$$\frac{1}{2}\frac{d}{dt}\left\|u_h(t)\right\|^2 + \gamma\left\|D^2 u_h\right\|^2 + \left(|Du_h|^2 Du_h, Du_h\right)$$

$$= \|Du_h\|^2 = -\left(u_h, D^2 u_h\right) \leq \frac{\gamma}{2}\left\|D^2 u_h\right\|^2 + \frac{1}{2\gamma}\|u_h\|^2.$$

Therefore

$$\frac{d}{dt}\left\|u_h(t)\right\|^2 + \gamma\left\|D^2 u_h\right\|^2 \leq \frac{1}{\gamma}\|u_h\|^2. \tag{5}$$

Letting $\beta = \frac{1}{\gamma}$, we have

$$\frac{d}{dt}\left(e^{-\beta t}\left\|u_h(t)\right\|^2\right) \leq 0. \tag{6}$$

Integrating (6) with respect to the time t, we get

$$\left\|u_h(t)\right\|^2 \leq e^{\beta t}\left\|u_h(0)\right\|^2, \quad 0 \leq t \leq T, \beta = \frac{1}{\gamma}. \tag{7}$$

Setting $v_h = u_{h,t}$ in (3), we deduce that

$$\|u_{h,t}\|^2 + \gamma\left(D^2 u_h, D^2 u_{h,t}\right) + \left(|Du_h|^2 Du_h - Du_h, Du_{h,t}\right) = 0. \tag{8}$$

Let

$$F_h(t) = \frac{\gamma}{2}\left\|D^2 u_h\right\|^2 + \frac{1}{4}\|Du_h\|_{L^4}^4 - \frac{1}{2}\|Du_h\|^2.$$

Differentiating $F_h(t)$ with respect to t, using (8), we get

$$\frac{d}{dt}F_h(t) = \gamma\int_\Omega D^2 u_h D^2 u_{h,t}\, dx + \int_\Omega |Du_h|^2 Du_h Du_{h,t}\, dx - \int_\Omega Du_h Du_{h,t}\, dx$$

$$= -\|u_{h,t}\|^2 \leq 0.$$

Therefore $F_h(t) \leq F_h(0)$, that is

$$\frac{\gamma}{2}\left\|D^2 u_h\right\|^2 + \frac{1}{4}\|Du_h\|_{L^4}^4 + \frac{1}{2}\left\|Du_h(0)\right\|^2$$

$$\leq \frac{\gamma}{2}\left\|D^2 u_h(0)\right\|^2 + \frac{1}{4}\left\|Du_h(0)\right\|_{L^4}^4 + \frac{1}{2}\|Du_h\|^2$$

$$\leq \frac{\gamma}{2}\left\|D^2 u_h(0)\right\|^2 + \frac{1}{4}\left\|Du_h(0)\right\|_{L^4}^4 + \frac{\gamma}{4}\left\|D^2 u_h\right\|^2 + \frac{1}{4\gamma}\|u_h\|^2$$

$$\leq \frac{\gamma}{2}\left\|D^2 u_h(0)\right\|^2 + \frac{1}{4}\left\|Du_h(0)\right\|_{L^4}^4 + \frac{\gamma}{4}\left\|D^2 u_h\right\|^2 + \frac{1}{4\gamma}e^{\beta T}\left\|u_h(0)\right\|^2. \tag{9}$$

Hence

$$\left\|D^2 u_h\right\|^2 \leq C. \tag{10}$$

We also have

$$\|Du_h\|^2 = -(u_h, D^2 u_h) \le \frac{1}{2}\|u_h\|^2 + \frac{1}{2}\|D^2 u_h\|^2. \tag{11}$$

By (7), (10), and (11), we complete the proof of Theorem 2.1. □

Remark 2.1 By the above argument, we can obtain a better result. Let $C_1' = \min\{\frac{\gamma}{4}, \frac{1}{4}\}$, $C_2' = \max\{\frac{\gamma}{2}, \frac{1}{4}, \frac{1}{4\gamma}e^{\beta T}\}$. It then follows from (9) that

$$C_1'\big(\|D^2 u_h\|^2 + \|Du_h\|_{L^4}^4\big) \le C_2'\big(\|D^2 u_h(0)\|^2 + \|Du_h(0)\|_{L^4}^4 + \|u_h(0)\|^2\big).$$

Furthermore, we have

$$\|u_h(t)\|_2^2 + \|Du_h(t)\|_{L^4}^4 \le C_1\big(\|u_h(0)\|_2^2 + \|Du_h(0)\|_{L^4}^4\big),$$

where C_1 is a positive constant dependent only on γ, independent of $u_h(0)$ and h.

In order to consider the error estimate, we first introduce a finite element approximation projection for a steady-state problem. Let $u, v \in H_E^2(I)$, define $a(u, v) \equiv \gamma(D^2 u, D^2 v) + (u, v)$, define the biharmonic projection $R_h : u \to R_h u \in S_h^{(k)}$ such that

$$a(u - R_h u, v_h) \equiv \gamma\big(D^2(u - R_h u), D^2 v_h\big) + (u - R_h u, v_h) = 0, \quad \forall v_h \in S_h^{(k)}. \tag{12}$$

It then follows (12) that

$$\|u\|_2^2 \le c_0 a(u, u), \quad \forall u \in H_E^2(I), \tag{13}$$

where c_0 is a positive constant depends only on γ and μ. Hence, $a(u, v)$ is a symmetrical positive determined bilinear form, and there exists a unique solution $u_h \in S_h^{(k)}$ for problem (12).

Based on the standard finite element method for a biharmonic equation (see [13]), we have

$$\|u - R_h u\| + h\|u - R_h u\|_1 + h^2\|u - R_h u\|_2 \le Ch^{r+1}\|u\|_{r+1}, \quad 2 \le r \le k. \tag{14}$$

Now, we consider the error estimate for the semi-discrete finite element solution. Let u be the solution of (2), and u_h be the solution of (3). Denote $\eta(t) = u - R_h u$ and $\theta(t) = R_h u - u_h$, then

$$u - u_h = u - R_h u + R_h u - u_h = \eta(t) + \theta(t). \tag{15}$$

Combining (2) and (4) gives

$$(u_t - u_{h,t}, v_h) + \gamma\big(D^2 u - D^2 u_h, D^2 v_h\big) + \big(|Du|^2 Du - |Du_h|^2 Du_h, Dv_h\big)$$

$$- (Du - Du_h, Dv_h) = 0,$$

that is

$$(\eta_t + \theta_t, v_h) + \gamma\left(D^2\eta + D^2\theta, D^2v_h\right) + \left(|Du|^2Du - |Du_h|^2Du_h, Dv_h\right)$$
$$- (D\eta + D\theta, Dv_h) = 0. \tag{16}$$

It then follows from (12) and (16) that

$$(\theta_t, v_h) + \gamma\left(D^2\theta, D^2v_h\right)$$
$$= (\eta - \eta_t, v_h) - \left(|Du|^2Du - |Du_h|^2Du_h, Dv_h\right) + (D\theta, Dv_h). \tag{17}$$

Lemma 2.1 *Let u be the solution of* (2), *u_h be the solution of* (3), *$u \in L^\infty(0, T; H^2(I))$. Then there exists a constant $C = C(u, u_h(0), \gamma)$ such that*

$$-\left(|Du|^2Du - |Du_h|^2Du_h, D\theta\right) \le C\left(\|\theta\|^2 + \|\eta\|^2\right) + \frac{\gamma}{4}\left\|D^2\theta\right\|^2.$$

Proof First of all, we give some estimates which will be used in this proof. It follows from Theorem 2.1, (12), and (13) that

$$\|u - u_h\|_1 \le \|u\|_1 + C \le C, \tag{18}$$

and

$$\|\eta\|_2 = \|u - R_hu\|_2 \le \|u\|_2 + C\|u\|_2 \le C. \tag{19}$$

We notice that

$$|Du|^2Du - |Du_h|^2Du_h = \left[(Du)^2 + DuDu_h + (Du_h)^2\right](Du - Du_h).$$

By Sobolev's embedding theorem, we have $H^2(I) \hookrightarrow W^{1,\infty}(I)$. Hence

$$|Du|_\infty \le C', \qquad |Du_h|_\infty \le C', \qquad |D\theta|_\infty \le C'\|\theta\|_2.$$

Thus, using the method of integration by parts, we get

$$-\left(|Du|^2Du - |Du_h|^2Du_h, D\theta\right)$$
$$= -\left(\left[(Du)^2 + DuDu_h + (Du_h)^2\right](Du - Du_h), D\theta\right)$$
$$= \left(\left[(Du)^2 + DuDu_h + (Du_h)^2\right](\theta + \eta), D^2\theta\right)$$
$$\quad + \left(\theta + \eta, D\theta\left[2DuD^2u + DuD^2u_h + D^2uDu_h + 2Du_hD^2u_h\right]\right)$$
$$\le \|\theta + \eta\|\left\|D^2\theta\right\| + |D\theta|_\infty$$
$$\quad \times \|\theta + \eta\|\left(2|Du|_\infty\left\|D^2u\right\| + |Du|_\infty\left\|D^2u_h\right\| + |Du_h|_\infty\left\|D^2u\right\| + 2|Du_h|_\infty\left\|D^2u_h\right\|\right)$$
$$\le \|\theta + \eta\|\left\|D^2\theta\right\| + C'\|\theta\|_2\|\theta + \eta\|$$
$$\le C\left(\|\theta\|^2 + \|\eta\|^2\right) + \varepsilon\left\|D^2\theta\right\|^2.$$

Then the proof of Lemma 2.1 is completed. □

Remark 2.2 We use the integration by parts for the term $-(|Du|^2Du - |Du_h|^2Du_h, D\theta)$ in the proof of Lemma 2.1. Then a better convergency is obtained.

Theorem 2.2 *Let u be the solution of* (2), *u_h be the solution of* (3), *$u(0) \in H^{k+1}(I)$, $u_t \in L^2(0,T;H^{k+1}(I))$, and the initial value satisfies*

$$\|u(0) - u_h(0)\| \le Ch^{k+1}\|u(0)\|_{k+1}. \tag{20}$$

Then we have the following error estimate:

$$\|u(t) - u_h(t)\| \le C'h^{k+1}\left(\|u(0)\|_{k+1}^2 + \int_0^T \|u_t(\tau)\|_{k+1}^2 \, d\tau\right)^{\frac{1}{2}}, \quad 0 \le t \le T,$$

where the constant $C' = C'(u, u_h(0), \gamma)$.

Proof By (14)-(15) and Lemma 2.1, we only need to estimate $\theta(t)$. Setting $v_h = \theta$ in (17), using Cauchy's inequality, we immediately conclude that

$$\frac{1}{2}\frac{d}{dt}\|\theta\|^2 + \gamma\|D^2\theta\|^2$$

$$\le \left(\|\eta\| + \|\eta_t\|\right)\|\theta\| - \left(|Du|^2Du - |Du_h|^2Du_h, D\theta\right) + \|D^2\theta\|\|\theta\|$$

$$\le \frac{1}{2}\left(\|\eta\| + \|\eta_t\|\right)^2 + C\left(\|\theta\|^2 + \|\eta\|^2\right) + \frac{\gamma}{4}\|D^2\theta\|^2 + \frac{1}{\gamma}\|\theta\|^2 + \frac{\gamma}{4}\|D^2\theta\|^2$$

$$\le \left(C + \frac{1}{2}\right)\left(\|\eta\| + \|\eta_t\|\right)^2 + \frac{\gamma}{2}\|D^2\theta\|^2 + \left(C + \frac{1}{\gamma}\right)\|\theta\|^2.$$

It then follows from the above inequality that

$$\frac{d}{dt}\|\theta\|^2 + \gamma\|D^2\theta\|^2 \le (2C+1)\left(\|\eta\| + \|\eta_t\|\right)^2 + \left(2C + \frac{2}{\gamma}\right)\|\theta\|^2.$$

By Gronwall's inequality, we deduce that

$$\|\theta\|^2 \le C'\left(\|\theta(0)\|^2 + \int_0^t \left(\|\eta(\tau)\|^2 + \|\eta_t(\tau)\|^2\right) d\tau\right).$$

Combing (20) and (14) (noticing that $(R_h u)_t = R_h u_t$), and using the triangle inequality, we complete the proof of Theorem 2.2. □

Remark 2.3 In Theorem 2.2, we give the L^2-norm error estimate for the semi-discrete approximation. In fact, we want to obtain some better result for the error estimates. Our best guess on the H^2-norm error estimate is

$$\|u(t) - u_h(t)\|_2 \le C'h^{k-1}\left(\|u(0)\|_{k+1}^2 + \int_0^T \|u_t(\tau)\|_{k+1}^2 \, d\tau\right)^{\frac{1}{2}}, \quad 0 \le t \le T,$$

where the constant $C' = C'(u, u_h(0), \gamma)$. We will prove it in the next step.

3 Full-discrete approximation

For any given positive integer M, let $\Delta t = T/M$ denote the size of the time discretization. Denote $U^n = U(x, t_n)$ for $t_n = n\Delta t$, $n = 0, 1, \ldots, M$. Introduce the forward Euler difference formula,

$$u_t(t_n) = \frac{u^n - u^{n-1}}{\Delta t} + \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} (\tau - t_n) u_{tt}(\tau)\, d\tau = \delta_t u^n + \varepsilon^n, \tag{21}$$

where $u^n = u(t_n)$.

Now, we define the full-discrete finite element form to approximate problem (2): Find $U^n \in S_h^{(k)}$ ($n = 1, 2, \ldots, M$) such that

$$\begin{cases} (\delta_t U^n, v_h) + \gamma(D^2 U^n, D^2 v_h) - (DU^n, Dv_h) + ((DU^{n-1})^2 DU^n, Dv_h) = 0, \\ \quad \forall v_h \in S_h^{(k)}, \\ (U^0 - u(0), v_h) = 0, \quad \forall v_h \in S_h^{(k)}. \end{cases} \tag{22}$$

For the above form, if U^{n-1} is known and Δt sufficiently small, by solving a positive definite system of linear equations which is equal to (22), we can obtain U^n. Let

$$u^n - U^n = u^n - R_h u^n + R_h u^n - U^n = \eta^n + \theta^n, \quad \theta^n \in S_h^{(k)}.$$

Using (2) and (12), $R_h u^n$ satisfies

$$\begin{aligned} (\delta_t R_h u^n, v_h) &+ \gamma(D^2 R_h u^n, D^2 v_h) \\ &= (\eta^n - \varepsilon^n - \delta_t \eta^n, v_h) + (DR_h u^n, Dv_h) - (|Du^n|^2 Du^n, Dv_h). \end{aligned} \tag{23}$$

Adding (22) and (23), $\forall v_h \in S_h^{(k)}$, we have

$$\begin{aligned} (\delta_t \theta^n, v_h) &+ \gamma(D^2 \theta^n, D^2 v_h) \\ &= (\eta^n - \varepsilon^n - \delta_t \eta^n, v_h) + (D\theta^n, Dv_h) - (|Du^n|^2 Du^n - |DU^{n-1}|^2 DU^n, Dv_h). \end{aligned} \tag{24}$$

Theorem 3.1 *Let u be the solution of* (2), *U^n be the solution of* (22), *$u(0) \in H^{k+1}(I)$, $u_t \in L^2(0, T; H^{k+1}(I))$, $u_{tt} \in L^2(0, T; L^2(I))$, $\Delta t/h^2 \leq c$, and $U^0 \in S_h^{(k)}$ satisfies*

$$\left\| u(0) - U^0 \right\|_i \leq C h^{k+1-i} \left\| u(0) \right\|_2,$$

where $i = 0, 1$. Then if h is sufficiently small, there exists a constant $C = C(u, \gamma)$ which is independent of h, Δt, and n, such that

$$\left\| u^n - U^n \right\| \leq C(\Delta t + h^k), \quad n = 0, 1, \ldots, M - 1. \tag{25}$$

Proof First of all, we give a posterior hypothesis: There exists a h_0; when $0 < h \leq h_0$, we have

$$\left\| Du^m - DU^m \right\|_\infty \leq C, \quad m = 1, 2, \ldots, n - 1. \tag{26}$$

We will prove the correctness of (26) in the end of this proof.

Setting $v_h = \theta^n$ in (24), we derive

$$\left(\delta_t \theta^n, \theta^n\right) + \gamma \left\|D^2\theta^n\right\|^2$$

$$\leq \left\|\eta^n - \varepsilon^n - \delta_t \eta^n\right\| \left\|\theta^n\right\| + \left\|D\theta^n\right\|^2 - \left\||Du^n|^2 Du^n - |DU^{n-1}|^2 DU^n\right\| \left\|D\theta^n\right\|$$

$$\leq \frac{1}{2}\left\|\eta^n - \varepsilon^n - \delta_t \eta^n\right\|^2 + \frac{1}{2}\left\|\theta^n\right\|^2 + 2\left\|D\theta^n\right\|^2$$

$$\quad + \frac{1}{4}\left\||Du^n|^2 Du^n - |DU^{n-1}|^2 DU^n\right\|^2$$

$$\leq \frac{1}{2}\left\|\eta^n - \varepsilon^n - \delta_t \eta^n\right\|^2 + \left(\frac{1}{2} + \frac{2}{\gamma}\right)\left\|\theta^n\right\|^2 + \frac{\gamma}{2}\left\|D^2\theta^n\right\|^2$$

$$\quad + \frac{1}{4}\left\||Du^n|^2 Du^n - |DU^{n-1}|^2 DU^n\right\|^2.$$

Hence

$$\left\|\theta^n\right\|^2 + \gamma \left\|D^2\theta^n\right\|^2$$

$$\leq \left\|\theta^{n-1}\right\|^2 + 2\Delta t\left\{\frac{1}{2}\left\|\eta^n - \varepsilon^n - \delta_t \eta^n\right\|^2 + \left(\frac{1}{2} + \frac{2}{\gamma}\right)\left\|\theta^n\right\|^2\right.$$

$$\left. + \frac{1}{4}\left\||Du^n|^2 Du^n - |DU^{n-1}|^2 DU^n\right\|^2\right\}. \tag{27}$$

Using (14) and (21), we get

$$\left\|\eta^n\right\| \leq Ch^{k+1}\left\|u(t_n)\right\|_{k+1},$$

$$\left\|\varepsilon^n\right\| \leq \int_{t_{n-1}}^{t_n} \left\|u_{tt}(\tau)\right\| d\tau \leq (\Delta t)^{\frac{1}{2}}\left(\int_{t_{n-1}}^{t_n} \left\|u_{tt}(\tau)\right\|^2 d\tau\right)^{\frac{1}{2}},$$

$$\left\|\delta_t \eta^n\right\| = \left\|\frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} \eta_t(\tau) d\tau\right\| \leq C\frac{1}{\Delta t}h^{k+1}\int_{t_{n-1}}^{t_n} \left\|u_t(\tau)\right\|_{k+1} d\tau$$

$$\leq C\frac{1}{(\Delta t)^{\frac{1}{2}}}h^{k+1}\left(\int_{t_{n-1}}^{t_n} \left\|u_t(\tau)\right\|_{k+1}^2 d\tau\right)^{\frac{1}{2}}.$$

In addition, we have

$$\left\||Du^n|^2 Du^n - |DU^{n-1}|^2 DU^n\right\|$$

$$= \left\||Du^n|^2 Du^n - |Du^{n-1}|^2 Du^n + |Du^{n-1}|^2 Du^n - |DU^{n-1}|^2 Du^n\right.$$

$$\quad + \left. |DU^{n-1}|^2 Du^n - |DU^{n-1}|^2 DU^n\right\|$$

$$\leq \left\||Du^n|^2 Du^n - |Du^{n-1}|^2 Du^n\right\| + \left\||Du^{n-1}|^2 Du^n - |DU^{n-1}|^2 Du^n\right\|$$

$$\quad + \left\||DU^{n-1}|^2 Du^n - |DU^{n-1}|^2 DU^n\right\|$$

$$\leq \left|Du^n\right|_\infty \left|Du^n + Du^{n-1}\right|_\infty \left\|Du^n - Du^{n-1}\right\|$$

$$\quad + \left|Du^n\right|_\infty^2 \left|Du^{n-1} + DU^{n-1}\right|_\infty \left\|Du^{n-1} - DU^{n-1}\right\|$$

$$\quad + \left|DU^{n-1}\right|_\infty^2 \left\|Du^n - DU^n\right\|. \tag{28}$$

By Theorem 1.1 and Sobolev's embedding theorem, we have

$$
\begin{aligned}
&\left|Du^n\right|_\infty \le C\left\|u^n\right\|_2 \le C, \\
&\left|Du^n + Du^{n-1}\right|_\infty \le C\left(\left|Du^n\right|_\infty + \left|Du^{n-1}\right|_\infty\right) \le C\left(\left\|u^n\right\|_2^2 + \left\|u^{n-1}\right\|_2^2\right) \le C,
\end{aligned}
\tag{29}
$$

and

$$
\begin{aligned}
&\left|DU^{n-1}\right|_\infty \le \left|Du^{n-1} - DU^{n-1}\right|_\infty + \left|Du^{n-1}\right|_\infty \le C + \left|Du^{n-1}\right|_\infty, \\
&\left|Du^{n-1} + DU^{n-1}\right|_\infty \le \left|Du^{n-1} - DU^{n-1}\right|_\infty + 2\left|Du^{n-1}\right|_\infty \le C + \left|Du^{n-1}\right|_\infty.
\end{aligned}
\tag{30}
$$

We have used the posterior hypothesis in (30). Adding (28)-(30) gives

$$
\begin{aligned}
&\left\|\left|Du^n\right|^2 Du^n - \left|DU^{n-1}\right|^2 DU^n\right\| \\
&\quad \le C\left(\left\|Du^n - Du^{n-1}\right\| + \left\|Du^{n-1} - DU^{n-1}\right\| + \left\|Du^n - DU^n\right\|\right) \\
&\quad \le C\left(\left\|Du^n - Du^{n-1}\right\| + \left\|D\theta^{n-1} + D\eta^{n-1}\right\| + \left\|D\theta^n + D\eta^n\right\|\right) \\
&\quad \le C\left(\int_{t_{n-1}}^{t_n} \left\|Du_t(\tau)\right\| d\tau + \left\|D\eta^{n-1}\right\| + \left\|D\theta^{n-1}\right\| + \left\|D\eta^n\right\| + \left\|D\theta^n\right\|\right) \\
&\quad \le C\left[(\Delta t)^{\frac{1}{2}}\left(\int_{t_{n-1}}^{t_n} \left\|Du_t(\tau)\right\|^2 d\tau\right)^{\frac{1}{2}} + h^k\left(\left\|u(t_n)\right\|_{k+1} + \left\|u(t_{n-1})\right\|_{k+1}\right)\right. \\
&\qquad\left. + \left\|D\theta^{n-1}\right\| + \left\|D\theta^n\right\|\right].
\end{aligned}
$$

Taking the above estimates into (27), we derive

$$
\begin{aligned}
&\left\|\theta^n\right\|^2 - \left\|\theta^{n-1}\right\|^2 + \gamma\left\|D^2\theta^n\right\|^2 \\
&\quad \le C\Delta t\left(\left\|D\theta^n\right\|^2 + \left\|D\theta^{n-1}\right\|^2 + h^{2k}\left\|u(t_n)\right\|_{k+1}^2\right) \\
&\qquad + C\left[h^{2k} + (\Delta t)^2\right]\left(\int_{t_{n-1}}^{t_n}\left(\left\|u_t(\tau)\right\|_{k+1}^2 + \left\|u_{tt}\right\|^2\right) d\tau\right).
\end{aligned}
$$

Taking the sum of n, noticing that $\|D\theta^0\| \le Ch^k\|u(0)\|_1$, $n\Delta t = t_n \le T$, we obtain

$$
\begin{aligned}
&\left\|\theta^n\right\|^2 + \gamma\sum_{i=1}^n\left\|D^2\theta^i\right\|^2 \\
&\quad \le C\Delta t\left[\left\|D\theta^0\right\|^2 + \sum_{i=1}^n\left\|D\theta^i\right\|^2\right] \\
&\qquad + C\left[(\Delta t)^2 + h^{2k}\right]\left(\int_0^{t_n}\left(\left\|u_t(\tau)\right\|_{k+1}^2 + \left\|u_{tt}\right\|^2\right) d\tau\right) \\
&\quad \le C\Delta t\left[\left\|D\theta^0\right\|^2 + \frac{1}{2}\sum_{i=1}^n\left(\left\|D^2\theta^i\right\|^2 + \left\|\theta^i\right\|^2\right)\right] \\
&\qquad + C\left[(\Delta t)^2 + h^{2k}\right]\left(\int_0^{t_n}\left(\left\|u_t(\tau)\right\|_{k+1}^2 + \left\|u_{tt}\right\|^2\right) d\tau\right).
\end{aligned}
$$

Let Δt be sufficiently small, which satisfies $\frac{C}{2\gamma}\Delta t \le 1$ and $C\Delta t < \frac{1}{2}$, and we deduce

$$\|\theta^n\|^2 \le C\Delta t \sum_{i=1}^{n-1} \|\theta^i\|^2 + C\big[(\Delta t)^2 + h^{2k}\big].$$

Using the discrete Gronwall inequality, (12), and the triangle inequality, we obtain (25).

Now, in order to complete the proof of Theorem 3.1, we only need to prove the posterior hypothesis (26). Use the inductive method. When $m = 0$, based on the initial approximation assumption and the finite element inverse inequality, letting $h \le h_0$ and h_0 be sufficiently small, we obtain (26). If we assume that (26) is correct for $m = l - 1$, based on the above proof, we can easily see that the estimate (25) is correct for $n = l$, where C is a constant independent of n, Δt, and h (noticing that $t_n \le T$). Using the finite element inverse inequality, the interpolation approximation properties, and (25), we have

$$
\begin{aligned}
\big\|Du^l - DU^l\big\|_\infty &\le C\big\|u^l - U^l\big\|_2 \le C\big(\big\|u^l - u_I^l\big\|_2 + \big\|u_I^l - U^l\big\|_2\big) \\
&\le C\big(\big\|u^l - u_I^l\big\|_2 + h^{-2}\big\|u_I^l - U^l\big\|\big) \\
&\le C\big[\big\|u^l - u_I^l\big\|_2 + h^{-2}\big(\big\|u_I^l - u^l\big\| + \big\|u^l - U^l\big\|\big)\big] \\
&\le C\big(h^{k-2} + h^{-2}\Delta t\big) \le C,
\end{aligned}
$$

where $u_i \in S_h^{(3)}$ is the Hermite type interpolation approximation of the function u. Hence, (26) is correct for $m = l$. Then, using the inductive method, the correctness of (26) is proved, and the proof of Theorem 3.1 is completed. □

Remark 3.1 In Theorem 2.2, we give the L^2-norm error estimate for the full-discrete approximation. In fact, we want to obtain some better result on the error estimates. Our best guess on the H^2-norm error estimate is

$$\big\|u^n - U^n\big\|_2 \le C\big(\Delta t + h^{k-2}\big), \quad n = 0, 1, \dots, M - 1.$$

We will prove it in the next step.

4 Numerical approximation

In this section, using the full-discrete form (22), we approximate the solution of problem (1). Let $u_0 = \cos x$, $\gamma = 1$, $T = 1$, $h = \frac{\pi}{400}$, $\Delta t = \frac{1}{100}$. We get the solution which evolves from $t = 0$ to $t = 1$ (*cf.* Figure 1).

In addition, we consider the change of error when the time $t = 0.5$. Since there is no exact solution to problem (1) to the best of our knowledge, we make a comparison between the solution of (22) on coarse meshes and the fine mesh.

Choose $\Delta t = 0.1000, 0.0500, 0.0250, 0.0167, 0.0125$, respectively, to solve (22). Set $u_N^{\min}(x, 0.5)$ as the solution for $\Delta t_{\min} = 0.01$. Denote

$$\text{err}(0.5, \Delta t) = \left(\int_0^1 \big[u_N^k(x, 0.5) - u_N^{\min}(x, 0.5)\big]^2 dx\right)^{\frac{1}{2}}, \quad k = 1, 2, 3, 4. \tag{31}$$
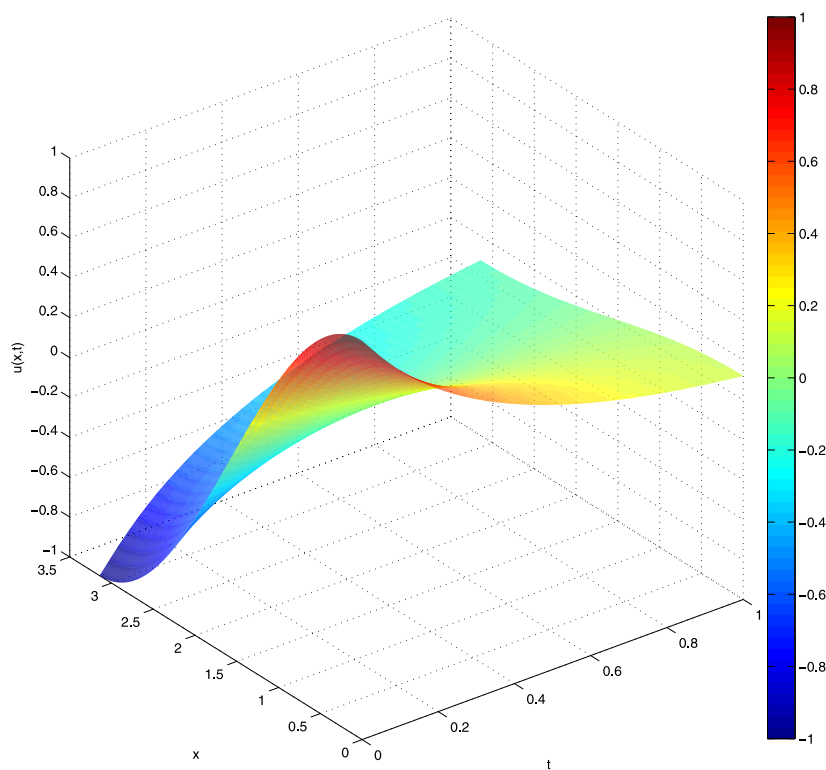
Then the error is shown in Table 1.

Figure 1 The approximation solution of the full-discrete form.

Table 1 The error for difference time step at $t = 0.5$

Δt	err$(0.5, \Delta t)$	$\frac{\text{err}(0.5,\Delta t)}{\Delta t}$	$\frac{\text{err}(0.5,\Delta t)}{(\Delta t)^2}$
0.1000	0.0253	0.2530	2.5300
0.0500	0.0117	0.2340	4.6800
0.0250	0.0045	0.1800	7.2000
0.0167	0.0020	0.1198	7.1736
0.0125	$7.5745e^{-4}$	0.0606	4.848

In Table 1, it is easy to see that the third column $\frac{\text{err}(0.5,\Delta t)}{\Delta t}$ is monotone decreasing along with the time step's waning and the fourth column $\frac{\text{err}(0.5,\Delta t)}{(\Delta t)^2}$ is not monotone decreasing along with the time step's waning. Then the order of convergence for time is of $O(\Delta t)$ and $O((\Delta t)^2)$. It is easy to see that the result of the numerical analysis on time is better than the theoretical result. The reason may be the existence of a nonlinear term or the limit of the theoretical proof tool.

Now, we consider the error for the difference h at $t = 0.5$. We choose $h = 0.1517, 0.0785$, 0.0628, 0.0524, respectively, to solve (22). Set $u_N^{\min}(x, 0.5)$ as the solution for $h = \frac{\pi}{400}$, $\Delta t = \frac{1}{100}$, and use the discrete L^2 norm to obtain the error err$(0.5, h)$ for $t = 0.5$. Then the error is shown in Table 2.

In Table 2, it is easy to see that the fourth column $\frac{\text{err}(0.5,h)}{h^2}$ is monotone decreasing along with the space step's waning. The fifth column $\frac{\text{err}(0.5,h)}{h^3}$ is not monotone increasing along with the space step's waning, and it tends to a positive constant when the space subdivision

Table 2 The error for difference h at $t = 0.5$

h	err($0.5, h$)	$\frac{\text{err}(0.5,h)}{h}$	$\frac{\text{err}(0.5,h)}{h^2}$	$\frac{\text{err}(0.5,h)}{h^3}$
0.1571	$2.6459e^{-7}$	$1.6842e^{-6}$	$1.0721e^{-5}$	$6.8241e^{-5}$
0.1047	$3.6393e^{-8}$	$3.4759e^{-7}$	$3.3199e^{-6}$	$3.1709e^{-5}$
0.0785	$1.8007e^{-8}$	$2.2939e^{-7}$	$2.9222e^{-6}$	$3.7225e^{-5}$
0.0628	$4.9313e^{-9}$	$7.8524e^{-8}$	$1.2504e^{-6}$	$1.9911e^{-5}$
0.0524	$9.5773e^{-10}$	$1.8277e^{-8}$	$3.4880e^{-7}$	$6.6566e^{-6}$

Table 3 The error for the differences h and Δt at $t = 0.5$

h	Δt	err($0.5, h, \Delta t$)	$\frac{\text{err}(0.5,h,\Delta t)}{h^3 + \Delta t}$
0.3927	0.1000	0.0525	0.3270
0.2614	0.0500	0.0266	0.3286
0.1963	0.0250	0.0125	0.3839
0.1571	0.0167	0.0062	0.3018
0.1257	0.0125	0.0026	0.1795

is small enough. Hence, we can find a positive constant C, such that

$$\frac{\text{err}(0.5, h)}{h^3} \leq \mathcal{C},$$

which means the order of the error estimates is $O(h)^3$.

On the other hand, we consider the error for difference h and Δt at $t = 0.5$. We choose $(h, \Delta t) = (\frac{\pi}{8}, \frac{1}{10}), (\frac{\pi}{10}, \frac{1}{20}), (\frac{\pi}{16}, \frac{1}{40}), (\frac{\pi}{20}, \frac{1}{60}), (\frac{\pi}{25}, \frac{1}{80})$, respectively, to solve (22), set $u_N^{\min}(x, 0.5)$ as the solution for $h = \frac{\pi}{400}$, $\Delta t = \frac{1}{100}$, use the discrete L^2 norm to obtain the error err($0.5, h, \Delta t$) for $t = 0.5$, which is shown in Table 3.

In Table 3, it is easy to see that the fourth column $\frac{\text{err}(0.5,h,\Delta t)}{h^3 + \Delta t}$ tends to a positive constant. Hence, we can find a positive constant C, such that

$$\frac{\text{err}(0.5, h, \Delta t)}{h^3 + \Delta t} \leq \mathcal{C},$$

which means that the order of error estimates is of $O(h^3 + \Delta t)$.

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