Lomtatidze and Šremr *Boundary Value Problems* 2014, 2014:52
http://www.boundaryvalueproblems.com/content/2014/1/52

Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

Carathéodory solutions to a hyperbolic differential inequality with a non-positive coefficient and delayed arguments

Alexander Lomtatidze and Jiří Šremr*

Dedicated to Professor Ivan Kiguradze.

*Correspondence: sremr@ipm.cz
Institute of Mathematics, Academy of Sciences of the Czech Republic, branch in Brno, Žižkova 22, Brno, 616 62, Czech Republic

Abstract

New effective conditions are found for the validity of a theorem on differential inequalities corresponding to the Darboux problem for linear hyperbolic differential equations with argument deviations.

MSC: 35L10; 35L15

Keywords: Carathéodory solution; hyperbolic differential equation with arguments deviations; Darboux problem; differential inequality

1 Introduction

On the rectangle $\mathcal{D} = [a,b] \times [c,d]$ we consider the Darboux problem

$$\frac{\partial^2 u(t,x)}{\partial t\, \partial x} = p(t,x)u\big(\tau(t,x),\mu(t,x)\big) + q(t,x), \tag{1.1}$$

$$u(t,c) = \varphi(t) \quad \text{for } t \in [a,b], \qquad u(a,x) = \psi(x) \quad \text{for } x \in [c,d], \tag{1.2}$$

where $p,q\colon \mathcal{D} \to \mathbb{R}$ are Lebesgue integrable functions, $\tau\colon \mathcal{D} \to [a,b]$ and $\mu\colon \mathcal{D} \to [c,d]$ are measurable functions, and $\varphi\colon [a,b] \to \mathbb{R}$, $\psi\colon [c,d] \to \mathbb{R}$ are absolutely continuous functions such that $\varphi(a) = \psi(c)$. By a *solution* to problem (1.1), (1.2) we mean a function $u\colon \mathcal{D} \to \mathbb{R}$ absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory[a] which satisfies equality (1.1) almost everywhere in $\mathcal{D}$ and verifies the initial conditions (1.2).

It is well known that theorems on differential inequalities (maximum principles in other terminology) play an important role in the theory of both ordinary and partial differential equations. For example, theorems on hyperbolic differential inequalities dealing with classical as well as Carathéodory solutions are studied in [1–8]. By using these statements, in particular, the method of lower and upper functions and monotone iterative techniques can be developed to derive solvability results for hyperbolic equations subjected to various initial conditions (Darboux, Cauchy, Goursat, *etc.*) as is done, *e.g.*, in [4, 8–11].

In this paper we continue the study of theorems on linear hyperbolic differential inequalities initiated in [6], where a more general functional-differential equation with a linear operator $\ell\colon C(\mathcal{D};\mathbb{R}) \to L(\mathcal{D};\mathbb{R})$ on the right-hand side is investigated and (1.1) is considered as a particular case of it.

We have introduced the following definition in [6].

Definition 1.1 Let $p\colon \mathcal{D} \to \mathbb{R}$ be a Lebesgue integrable function and $\tau\colon \mathcal{D} \to [a,b]$, $\mu\colon \mathcal{D} \to [c,d]$ be measurable functions. We say that the principle on differential inequalities (maximum principle) holds for (1.1) and we write $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$ if for any function $u\colon \mathcal{D} \to \mathbb{R}$ absolutely continuous on $\mathcal{D}$ in the sense of Carathéodory[b] satisfying the inequalities

$$\begin{aligned}
&\frac{\partial^2 u(t,x)}{\partial t\,\partial x} \geq p(t,x)u\bigl(\tau(t,x),\mu(t,x)\bigr) \quad \text{for a.e. } (t,x) \in \mathcal{D},\\
&\frac{\partial u(t,c)}{\partial t} \geq 0 \quad \text{for a.e. } t \in [a,b], \qquad \frac{\partial u(a,x)}{\partial x} \geq 0 \quad \text{for a.e. } x \in [c,d],\\
&u(a,c) \geq 0,
\end{aligned}$$

the relation

$$u(t,x) \geq 0 \quad \text{for } (t,x) \in \mathcal{D} \tag{1.3}$$

holds.

It is also mentioned in [6] that under the assumption $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$, problem (1.1), (1.2) has a unique (Carathéodory) solution and this solution satisfies (1.3) provided

$$\begin{aligned}
&q(t,x) \geq 0 \quad \text{for a.e. } (t,x) \in \mathcal{D}, \qquad \varphi(a) = \psi(c) \geq 0,\\
&\varphi'(t) \geq 0 \quad \text{for a.e. } t \in [a,b], \qquad \psi'(x) \geq 0 \quad \text{for a.e. } x \in [c,d].
\end{aligned}$$

Moreover, some efficient conditions are given in [6] for the validity of the inclusion $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$ in the case, where

$$p(t,x) \geq 0 \quad \text{for a.e. } (t,x) \in \mathcal{D}. \tag{1.4}$$

From those results it follows that, in the case (1.4), the hyperbolic equation (1.1) is similar in a certain sense to first-order ordinary differential equations, which is already noted in the book of Walter [2]. It is worth mentioning here that Definition 1.1 is in compliance with the formulation of a theorem on differential inequalities given in [5, Theorem 1], where the case (1.4) is also considered.

On the other hand, if

$$p(t,x) \leq 0 \quad \text{for a.e. } (t,x) \in \mathcal{D}, \tag{1.5}$$

then the explanation of Walter that hyperbolic equations are 'similar' to first-order ordinary differential equations does not hold because even for (1.1) with

$$\tau(t,x) := t, \qquad \mu(t,x) := x \quad \text{for } (t,x) \in \mathcal{D}, \tag{1.6}$$

oscillatory solutions may occur. In this case, the properties of hyperbolic equation (1.1) are 'closer' to properties of ordinary differential equations of the second order. In [6], we got a general sufficient condition for the validity of the inclusion $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$ in the case (1.5) under the assumption that (1.1) is delayed in both arguments, *i.e.*, if the inequalities

$$\big|p(t,x)\big|\big(\tau(t,x)-t\big) \le 0, \qquad \big|p(t,x)\big|\big(\mu(t,x)-x\big) \le 0 \quad \text{for a.e. } (t,x) \in \mathcal{D} \tag{1.7}$$

hold (see Lemma 3.3 below). Using that general result, we have also proved in [6] that if p, μ, and τ satisfy conditions (1.5) and (1.7), then $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$ provided

$$\iint_{\mathcal{D}} \big|p(t,x)\big| \,\mathrm{d}t\,\mathrm{d}x \le 1. \tag{1.8}$$

Note that assumption (1.7) is not restrictive in the case (1.5) because it is necessary, as is shown in [12]. Moreover, inequality (1.8) cannot, in general, be improved (see [6, Example 6.2]). However, it does not mean that inequality (1.8) is necessary and cannot be weakened in particular cases. In this paper, we give efficient criteria for the validity of the inclusion $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$ in the case when (1.5) holds optimally for equations which are 'close' to the equation without argument deviations,

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = p(t,x)u(t,x) + q(t,x). \tag{1.9}$$

It is well known that, without any additional assumptions, the Darboux problem (1.9), (1.2) has a unique (Carathéodory) solution u (see [13, Existensatz, $\alpha(y) = 0$] and [14, Remarks (b), (c)]) and this solution admits the integral representation

$$\begin{aligned} u(t,x) = {}& Z(t,x,a,c)\varphi(a) \\ &+ \int_a^t Z(t,x,s,c)\varphi'(s)\,\mathrm{d}s + \int_c^x Z(t,x,a,\eta)\psi'(\eta)\,\mathrm{d}\eta \\ &+ \int_a^t \int_c^x Z(t,x,s,\eta)q(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for } (t,x) \in \mathcal{D} \end{aligned} \tag{1.10}$$

(see [15, Theorem 8.1] and [16, Section 3.4] for continuous p, q), where Z are the Riemann functions of the homogeneous equation

$$\frac{\partial^2 u(t,x)}{\partial t\,\partial x} = p(t,x)u(t,x). \tag{1.11}$$

Recall that, for any $(t_0,x_0) \in \mathcal{D}$, the Riemann function $Z(t_0,x_0,\cdot,\cdot)$ is defined as a solution to (1.11) satisfying the initial conditions

$$u(t,x_0) = 1 \quad \text{for } t \in [a,b], \qquad u(t_0,x) = 1 \quad \text{for } x \in [c,d].$$

Therefore, it follows from equality (1.10) and Definition 1.1 that a theorem on differential inequalities holds for (1.9) if and only if

$$Z(t,x,s,\eta) \ge 0 \quad \text{for } a \le s \le t \le b, c \le \eta \le x \le d. \tag{1.12}$$

Unfortunately, the Riemann functions can be explicitly written only in some simple cases. In particular, for (1.9) with a constant non-positive coefficient p the following proposition holds (see, *e.g.*, [16, Section 3.4] or [15, Example 8.1]).

Proposition 1.1 *Let $k \leq 0$, the functions τ, μ be defined by relations* (1.6), *and*

$$p(t,x) := k \quad \textit{for } (t,x) \in \mathcal{D}. \tag{1.13}$$

Then $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$ if and only if

$$|k| \leq \frac{j_0^2}{4(b-a)(d-c)}, \tag{1.14}$$

where j_0 denotes the first positive zero of the Bessel function J_0.

In Section 2, we consider the case (1.5) and we present new effective conditions for the validity of the inclusion $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$ that are proved later in Section 3 by comparing (1.1) with a linear hyperbolic equation without argument deviations.

2 Main results

For any $\nu > -1$, let J_ν denote the Bessel function of the first kind and order ν and let j_ν be the first positive zero of the function J_ν. Moreover, we put

$$E_\nu(s) := \begin{cases} s^{-\nu} J_\nu(s) & \text{for } s > 0, \\ 2^{-\nu} \frac{1}{\Gamma(1+\nu)} & \text{for } s = 0, \end{cases} \tag{2.1}$$

where Γ is the standard gamma function.

Theorem 2.1 *Let $p\colon \mathcal{D} \to \mathbb{R}$ be a Lebesgue integrable function and $\tau: \mathcal{D} \to [a,b]$, $\mu\colon \mathcal{D} \to [c,d]$ be measurable functions satisfying conditions* (1.5) *and* (1.7). *Moreover, let there exist numbers $\lambda \in\,]0,1]$, $\alpha \in [0,1[$, and $\beta \in [0,\alpha]$ such that the inequalities*

$$\bigl[(t-a)(x-c)\bigr]^{1-\lambda} \bigl|p(t,x)\bigr| \leq \frac{\lambda^2}{4} \frac{j_{-\alpha}^2}{[(b-a)(d-c)]^\lambda}, \tag{2.2}$$

$$\begin{aligned} &\bigl[(t-a)(x-c)\bigr]^{1-\lambda} \bigl(E_{-\alpha}\bigl(z\bigl(\tau(t,x),x\bigr)\bigr) - E_{-\alpha}\bigl(z(t,x)\bigr)\bigr)\bigl|p(t,x)\bigr| \\ &\quad \leq \frac{\lambda^2 \beta}{2} \frac{j_{-\alpha}^2}{[(b-a)(d-c)]^\lambda} E_{1-\alpha}\bigl(z(t,x)\bigr), \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\bigl[(t-a)(x-c)\bigr]^{1-\lambda} \bigl(E_{-\alpha}\bigl(z\bigl(t,\mu(t,x)\bigr)\bigr) - E_{-\alpha}\bigl(z(t,x)\bigr)\bigr)\bigl|p(t,x)\bigr| \\ &\quad \leq \frac{\lambda^2 (\alpha-\beta)}{2} \frac{j_{-\alpha}^2}{[(b-a)(d-c)]^\lambda} E_{1-\alpha}\bigl(z(t,x)\bigr) \end{aligned} \tag{2.4}$$

are fulfilled a.e. on $\mathcal{D}$, where

$$z(t,x) := j_{-\alpha} \left[\frac{(t-a)(x-c)}{(b-a)(d-c)}\right]^{\frac{\lambda}{2}} \quad \textit{for } (t,x) \in \mathcal{D}. \tag{2.5}$$

Then the theorem on differential inequalities holds for (1.1), *i.e.*, $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$.

Remark 2.1 As we have mentioned above, assumption (1.7) in Theorem 2.1 is necessary for the validity of the inclusion $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$ in the case where inequality (1.5) holds (see [12]).

Remark 2.2 Theorem 2.1 cannot be improved in the sense that assumption (2.2) cannot be, in general, replaced by the assumption

$$\big[(t-a)(x-c)\big]^{1-\lambda}\big|p(t,x)\big| \le \frac{(1+\varepsilon)\lambda^2}{4}\frac{j_{-\alpha}^2}{[(b-a)(d-c)]^{\lambda}},$$

no matter how small $\varepsilon > 0$ is. Indeed, if p, τ, and μ are defined by (1.6) and (1.13) with $k \le 0$, then assumptions (2.3) and (2.4) of Theorem 2.1 hold with $\alpha = \beta = 0$, $\lambda = 1$, and inequality (2.2) takes the form (1.14), which is, in this case, necessary for the validity of the inclusion $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$ as is stated in Proposition 1.1.

Remark 2.3 Observe that if $\tau(t,x) = t$ for a.e. $(t,x) \in \mathcal{D}$ then the left-hand side of inequality (2.3) is equal to zero. Therefore, assumption (2.3) of Theorem 2.1 says how 'close' $\tau(t,x)$ must be to t, and this 'closeness' is understood through the composition of the functions $E_{-\alpha}$ and z. Similarly, 'closeness' of $\mu(t,x)$ to x is required in assumption (2.4).

The meaning of assumptions (2.3) and (2.4) of Theorem 2.1 is more transparent in the following two corollaries.

Corollary 2.1 *Let* $p\colon \mathcal{D} \to \mathbb{R}$ *be a Lebesgue integrable function and* $\tau\colon \mathcal{D} \to [a,b]$, $\mu\colon \mathcal{D} \to [c,d]$ *be measurable functions satisfying conditions* (1.5) *and* (1.7). *Moreover, let there exist numbers* $\lambda \in]0,1]$, $\alpha \in [0,1[$, *and* $\beta \in [0,\alpha]$ *such that inequalities* (2.2),

$$(t-a)^{1-\frac{\lambda}{2}}(x-c)\big((t-a)^{\frac{\lambda}{2}} - \big(\tau(t,x)-a\big)^{\frac{\lambda}{2}}\big)\big|p(t,x)\big| \le \frac{\lambda^2\beta}{2}j_{-\alpha}^{*}, \tag{2.6}$$

$$(t-a)(x-c)^{1-\frac{\lambda}{2}}\big((x-c)^{\frac{\lambda}{2}} - \big(\mu(t,x)-c\big)^{\frac{\lambda}{2}}\big)\big|p(t,x)\big| \le \frac{\lambda^2(\alpha-\beta)}{2}j_{-\alpha}^{*} \tag{2.7}$$

are fulfilled a.e. on $\mathcal{D}$, *where*

$$j_{-\alpha}^{*} := \frac{E_{1-\alpha}(j_{-\alpha})}{E_{1-\alpha}(0)}. \tag{2.8}$$

Then $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$.

Remark 2.4 It follows from the proof of Corollary 2.1 that the number $j_{-\alpha}^{*}$ on the right-hand side of inequalities (2.6) and (2.7) can be replaced by

$$\operatorname{ess\,inf}\left\{\frac{E_{1-\alpha}(z(t,x))}{E_{1-\alpha}(z(\tau(t,x),x))} : (t,x) \in \mathcal{D}\right\}$$

and

$$\operatorname{ess\,inf}\left\{\frac{E_{1-\alpha}(z(t,x))}{E_{1-\alpha}(z(t,\mu(t,x)))} : (t,x) \in \mathcal{D}\right\},$$

respectively, where the function z is defined by (2.5).

Corollary 2.2 *Let $p\colon \mathcal{D} \to \mathbb{R}$ be a Lebesgue integrable function and $\tau\colon \mathcal{D} \to [a,b]$, $\mu\colon \mathcal{D} \to [c,d]$ be measurable functions satisfying conditions* (1.5) *and* (1.7). *Moreover, let there exist numbers $\alpha \in [0,1[$ and $\beta \in [0,\alpha]$ such that the inequalities*

$$\left|p(t,x)\right| \leq \frac{j_{-\alpha}^2}{4(b-a)(d-c)}, \tag{2.9}$$

$$(x-c)\big(t-\tau(t,x)\big)\left|p(t,x)\right| \leq \beta j_{-\alpha}^*, \tag{2.10}$$

$$(t-a)\big(x-\mu(t,x)\big)\left|p(t,x)\right| \leq (\alpha-\beta) j_{-\alpha}^* \tag{2.11}$$

are fulfilled a.e. on $\mathcal{D}$, where the number $j_{-\alpha}^*$ is defined by formula (2.8). *Then $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$.*

Remark 2.5 Corollary 2.2 improves Corollary 2.1 with $\lambda = 1$. Indeed, for a.e. $(t,x) \in \mathcal{D}$ such that $p(t,x) \neq 0$ we have

$$\begin{aligned}\sqrt{t-a}\big(\sqrt{t-a}-\sqrt{\tau(t,x)-a}\big) &= \frac{\sqrt{t-a}}{\sqrt{t-a}+\sqrt{\tau(t,x)-a}}\big(t-\tau(t,x)\big)\\ &\geq \frac{1}{2}\big(t-\tau(t,x)\big)\end{aligned}$$

and thus inequality (2.6) with $\lambda = 1$ yields the validity of inequality (2.10). Analogously, inequality (2.11) follows from inequality (2.7) with $\lambda = 1$.

3 Proofs

The following notation is used throughout this section.

- The first-order partial derivatives of a function $v\colon \mathcal{D} \to \mathbb{R}$ at a point $(t,x) \in \mathcal{D}$ are denoted by

$$v'_{[1]}(t,x) \qquad \left(\text{or} \quad \frac{\partial v(t,x)}{\partial t}\right), \qquad v'_{[2]}(t,x) \qquad \left(\text{or} \quad \frac{\partial v(t,x)}{\partial x}\right).$$

- The second-order mixed partial derivative of a function $v\colon \mathcal{D} \to \mathbb{R}$ at a point $(t,x) \in \mathcal{D}$ is denoted by

$$v''_{[1,2]}(t,x) \qquad \left(\text{or} \quad \frac{\partial^2 v(t,x)}{\partial t\, \partial x}\right).$$

To prove the main results stated in the previous section we need the next three lemmas.

Lemma 3.1 *Let $\nu > -1$. Then the function E_ν defined by* (2.1) *has the following properties*:

(i)
$$E_\nu(s) = 2^{-\nu}\sum_{m=0}^{+\infty}\frac{(-1)^m(\frac{s}{2})^{2m}}{m!\Gamma(\nu+m+1)} \quad \textit{for } s \geq 0.$$

(ii) $E_\nu(0) > 0$ *and j_ν is the first positive zero of the function E_ν.*

(iii) $E'_\nu(s) = -sE_{\nu+1}(s)$ *for $s \geq 0$.*

(iv) $j_\nu < j_{\nu+1}$.

(v) *The function* $E_\nu^{(i)}\colon [0,+\infty[\,\to \mathbb{R}$ *is continuous for* $i = 0,1,2$.
(vi) $s^2E_\nu''(s) + (1 + 2\nu)sE_\nu'(s) + s^2E_\nu(s) = 0$ *for* $s \geq 0$.

Proof (i), (ii): It follows from (2.1), the definition of the function J_ν (see, *e.g.*, [17, Chapter III, Section 3.12]), and the fact that $\Gamma(x) > 0$ for every $x > 0$.

(iii): Since the series in assertion (i) converges uniformly on every closed subinterval of $[0,+\infty[$, we can take its derivative term-by-term and thus assertion (iii) follows immediately from (i).

(iv): See [17, Chapter XV, Section 15.21].

(v): It follows from assertions (i) and (iii).

(vi): The function J_ν is a solution to the Bessel equation and thus we have

$$J_\nu''(s) + \frac{1}{s}J_\nu'(s) + \left(1 - \frac{\nu^2}{s^2}\right)J_\nu(s) = 0 \quad \text{for } s > 0$$

(see, *e.g.*, [17, Chapter III, Section 3.12]). Consequently, by direct calculation we can check that the function E_ν satisfies the desired equality for every $s > 0$. It remains to mention that for $s = 0$, the validity of the desired equality is obvious. □

Lemma 3.2 ([18, Theorem 2.1]) *The following three statements are equivalent*:

(1) *The function* $v\colon \mathcal{D} \to \mathbb{R}$ *is absolutely continuous on* $\mathcal{D}$ *in the sense of Carathéodory.*[c]
(2) $v \in C^*(\mathcal{D};\mathbb{R})$, *i.e., the function* $v\colon \mathcal{D} \to \mathbb{R}$ *admits the representation*

$$v(t,x) = e + \int_a^t k(s)\,\mathrm{d}s + \int_c^x l(\eta)\,\mathrm{d}\eta + \int_a^t \int_c^x f(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \textit{for } (t,x) \in \mathcal{D},$$

where $e \in \mathbb{R}$ *and* $k\colon [a,b] \to \mathbb{R}$, $l\colon [c,d] \to \mathbb{R}$, *and* $f\colon \mathcal{D} \to \mathbb{R}$ *are Lebesgue integrable functions*.
(3) *The function* $v\colon \mathcal{D} \to \mathbb{R}$ *satisfies the conditions*:
   (a) *the function* $v(\cdot,x)\colon [a,b] \to \mathbb{R}$ *is absolutely continuous for every* $x \in [c,d]$ *and the function* $v(a,\cdot)\colon [c,d] \to \mathbb{R}$ *is absolutely continuous*;
   (b) *the function* $v'_{[1]}(t,\cdot)\colon [c,d] \to \mathbb{R}$ *is absolutely continuous for almost all* $t \in [a,b]$;
   (c) *the function* $v''_{[1,2]}\colon \mathcal{D} \to \mathbb{R}$ *is Lebesgue integrable*.

Lemma 3.3 *Let* $p\colon \mathcal{D} \to \mathbb{R}$ *be a Lebesgue integrable function and* $\tau\colon \mathcal{D} \to [a,b]$, $\mu\colon \mathcal{D} \to [c,d]$ *be measurable functions satisfying conditions* (1.5) *and* (1.7). *Assume that there exists a function*[d] $\gamma \in C^*(\mathcal{D};\mathbb{R})$ *such that*

$$\gamma''_{[1,2]}(t,x) \leq p(t,x)\gamma\big(\tau(t,x),\mu(t,x)\big) \quad \textit{for a.e. } (t,x) \in \mathcal{D}, \tag{3.1}$$

$$\gamma'_{[1]}(t,c) \leq 0 \quad \textit{for a.e. } t \in [a,b], \qquad \gamma'_{[2]}(a,x) \leq 0 \quad \textit{for a.e. } x \in [c,d] \tag{3.2}$$

and

$$\gamma(t,x) > 0 \quad \textit{for } (t,x) \in [a,b[\,\times[c,d[. \tag{3.3}$$

Then $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$.

Proof It follows from [6, Theorem 3.5] with the operator ℓ defined by the relation

$$\ell(v)(t,x) := p(t,x)v\big(\tau(t,x),\mu(t,x)\big)$$

for a.e. $(t,x) \in \mathcal{D}$ and all continuous functions $v\colon \mathcal{D} \to \mathbb{R}$. □

Proof of Theorem 2.1 Let

$$\gamma(t,x) := E_{-\alpha}\big(z(t,x)\big) \quad \text{for } (t,x) \in \mathcal{D}, \tag{3.4}$$

where the functions $E_{-\alpha}$ and z are defined by (2.1) and (2.5), respectively. It is clear that

$$0 \le z(t,x) < j_{-\alpha} \quad \text{for } (t,x) \in \mathcal{D} \setminus \big\{(b,d)\big\},$$

and thus, in view of Lemma 3.1(ii), the function γ satisfies inequalities (3.2) and (3.3). Since the functions $z(\cdot,x)$ and $z(t,\cdot)$ are absolutely continuous for every $x \in [c,d]$ and $t \in [a,b]$, by virtue of Lemma 3.1(v), we conclude that the functions $\gamma(\cdot,x)$ and $\gamma(t,\cdot)$ are absolutely continuous for every $x \in [c,d]$ and $t \in [a,b]$, respectively. Moreover, we have

$$\gamma'_{[1]}(t,x) = E'_{-\alpha}\big(z(t,x)\big)\frac{\lambda}{2}\frac{z(t,x)}{t-a} \quad \text{for } (t,x) \in \,]a,b] \times [c,d] \tag{3.5}$$

and

$$\gamma'_{[2]}(t,x) = E'_{-\alpha}\big(z(t,x)\big)\frac{\lambda}{2}\frac{z(t,x)}{x-c} \quad \text{for } (t,x) \in [a,b] \times \,]c,d]. \tag{3.6}$$

Now, in view of Lemma 3.1(v), it follows from (3.5) that the function $\gamma'_{[1]}(t,\cdot)$ is absolutely continuous for every $t \in \,]a,b]$ and

$$\gamma''_{[1,2]}(t,x) = \frac{\lambda^2}{4(t-a)(x-c)}\big[z^2(t,x)E''_{-\alpha}\big(z(t,x)\big) + z(t,x)E'_{-\alpha}\big(z(t,x)\big)\big]$$

for every $(t,x) \in \,]a,b] \times \,]c,d]$. Therefore, by using equalities (2.5), (3.4)-(3.6), and Lemma 3.1(vi), we get

$$\begin{aligned}\gamma''_{[1,2]}(t,x) = {}& -\frac{\lambda^2}{4}\frac{j^2_{-\alpha}}{[(b-a)(d-c)]^{\lambda}}\frac{1}{[(t-a)(x-c)]^{1-\lambda}}\gamma(t,x) \\ & + \frac{\lambda\beta}{x-c}\gamma'_{[1]}(t,x) + \frac{\lambda(\alpha-\beta)}{t-a}\gamma'_{[2]}(t,x) \quad \text{for } (t,x) \in \,]a,b] \times \,]c,d],\end{aligned} \tag{3.7}$$

which shows, in particular, that the function $\gamma''_{[1,2]}$ is Lebesgue integrable on $\mathcal{D}$. Consequently, Lemma 3.2 guarantees that $\gamma \in C^*(\mathcal{D};\mathbb{R})$. Moreover, by using Lemma 3.1(iii), (iv), we get

$$\begin{aligned}&\gamma'_{[1]}(t,x) \le 0 \quad \text{for } (t,x) \in \,]a,b] \times [c,d], \\ &\gamma'_{[2]}(t,x) \le 0 \quad \text{for } (t,x) \in [a,b] \times \,]c,d],\end{aligned} \tag{3.8}$$

and thus (3.7) implies that

$$\gamma''_{[1,2]}(t,x) \le 0 \quad \text{for } (t,x) \in \,]a,b] \times \,]c,d]. \tag{3.9}$$

Now, by virtue of Lemma 3.3, to prove the theorem it remains to show that the function γ satisfies differential inequality (3.1). For this purpose we put

$$\delta_1(t,x) := \frac{(t-a)^{1-\lambda\alpha}}{z^{1-\alpha}(t,x)J_{1-\alpha}(z(t,x))} \quad \text{for } (t,x) \in]a,b] \times]c,d], \tag{3.10}$$

$$\delta_2(t,x) := \frac{(x-c)^{1-\lambda\alpha}}{z^{1-\alpha}(t,x)J_{1-\alpha}(z(t,x))} \quad \text{for } (t,x) \in]a,b] \times]c,d] \tag{3.11}$$

and

$$\Omega := \frac{\lambda}{2} \frac{j_{-\alpha}}{[(b-a)(d-c)]^{\frac{\lambda}{2}}}. \tag{3.12}$$

Observe that, in view of equalities (3.5), (3.10), and Lemma 3.1(iii), we have

$$\delta_1(t,x)\gamma'_{[1]}(t,x) = -\frac{\lambda}{2} \frac{z^{1+\alpha}(t,x)E_{1-\alpha}(z(t,x))}{(t-a)^{\lambda\alpha}J_{1-\alpha}(z(t,x))} \quad \text{for } (t,x) \in]a,b] \times]c,d]$$

and thus, by using (2.1), (2.5), and (3.12), we get

$$\delta_1(t,x)\gamma'_{[1]}(t,x) = -\frac{\lambda}{2}\left(\frac{2}{\lambda}\Omega\right)^{2\alpha}(x-c)^{\lambda\alpha} \quad \text{for } (t,x) \in]a,b] \times]c,d].$$

Consequently,

$$\text{the function } \delta_1(\cdot,x)\gamma'_{[1]}(\cdot,x):\ ]a,b] \to \mathbb{R} \text{ is constant for every } x \in]c,d]. \tag{3.13}$$

We can show in a similar manner that

$$\text{the function } \delta_2(t,\cdot)\gamma'_{[2]}(t,\cdot):\ ]c,d] \to \mathbb{R} \text{ is constant for every } t \in]a,b]. \tag{3.14}$$

On the other hand, in view of (2.1), (2.5) and (3.12), and Lemma 3.1(iii), it follows from equality (3.10) that

$$\begin{aligned}\frac{1}{\delta_1(t,x)} &= \frac{2}{\lambda}\frac{(t-a)^{\lambda\alpha}}{z^{2\alpha}(t,x)}\frac{\lambda}{2}\frac{z(t,x)}{t-a}z(t,x)E_{1-\alpha}\big(z(t,x)\big) \\ &= -\left(\frac{2}{\lambda}\right)^{1-2\alpha}\frac{E'_{-\alpha}(z(t,x))z'_{[1]}(t,x)}{\Omega^{2\alpha}(x-c)^{\lambda\alpha}} \quad \text{for } (t,x) \in]a,b] \times]c,d],\end{aligned}$$

whence we get

$$\begin{aligned}\int_{\tau(t,x)}^{t}\frac{\mathrm{d}s}{\delta_1(s,x)} &= -\left(\frac{2}{\lambda}\right)^{1-2\alpha}\frac{1}{\Omega^{2\alpha}(x-c)^{\lambda\alpha}}\int_{\tau(t,x)}^{t}E'_{-\alpha}\big(z(s,x)\big)z'_{[1]}(s,x)\,\mathrm{d}s \\ &= \left(\frac{2}{\lambda}\right)^{1-2\alpha}\frac{E_{-\alpha}(z(\tau(t,x),x)) - E_{-\alpha}(z(t,x))}{\Omega^{2\alpha}(x-c)^{\lambda\alpha}}\end{aligned} \tag{3.15}$$

for a.e. $(t,x) \in \mathcal{D}$ because $z(\cdot,x)$ is an increasing absolutely continuous function for every $x \in [c,d]$. Similarly, we can show that

$$\int_{\mu(t,x)}^{x}\frac{\mathrm{d}\eta}{\delta_2(t,\eta)} = \left(\frac{2}{\lambda}\right)^{1-2\alpha}\frac{E_{-\alpha}(z(t,\mu(t,x))) - E_{-\alpha}(z(t,x))}{\Omega^{2\alpha}(t-a)^{\lambda\alpha}} \tag{3.16}$$

for a.e. $(t,x) \in \mathcal{D}$. We have proved that $\gamma \in C^*(\mathcal{D};\mathbb{R})$ and therefore, by using Lemma 3.2, we get

$$\begin{aligned}-\gamma\bigl(\tau(t,x),\mu(t,x)\bigr) = -\gamma(t,x) &+ \int_{\tau(t,x)}^{t} \gamma'_{[1]}(s,x)\,\mathrm{d}s + \int_{\mu(t,x)}^{x} \gamma'_{[2]}(t,\eta)\,\mathrm{d}\eta \\ &- \int_{\tau(t,x)}^{t}\int_{\mu(t,x)}^{x} \gamma''_{[1,2]}(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s \quad \text{for a.e. } (t,x) \in \mathcal{D}.\end{aligned}$$

Multiplying both sides of the latter equality by $|p(t,x)|$ and using inequalities (1.5), (1.7), (3.9), and properties (3.13), (3.14), for a.e. $(t,x) \in \mathcal{D}$ we obtain

$$\begin{aligned}p(t,x)\gamma\bigl(\tau(t,x),\mu(t,x)\bigr) \geq -\bigl|p(t,x)\bigr|\gamma(t,x) &+ \bigl|p(t,x)\bigr|\delta_1(t,x)\gamma'_{[1]}(t,x)\int_{\tau(t,x)}^{t}\frac{\mathrm{d}s}{\delta_1(s,x)} \\ &+ \bigl|p(t,x)\bigr|\delta_2(t,x)\gamma'_{[2]}(t,x)\int_{\mu(t,x)}^{x}\frac{\mathrm{d}\eta}{\delta_2(t,\eta)}.\end{aligned} \tag{3.17}$$

Now, combining (2.3), (3.10), (3.15) and (2.4), (3.11), (3.16), we get

$$\bigl|p(t,x)\bigr|\delta_1(t,x)\int_{\tau(t,x)}^{t}\frac{\mathrm{d}s}{\delta_1(s,x)} \leq \frac{\lambda\beta}{x-c} \quad \text{for a.e. } (t,x) \in \mathcal{D} \tag{3.18}$$

and

$$\bigl|p(t,x)\bigr|\delta_2(t,x)\int_{\mu(t,x)}^{x}\frac{\mathrm{d}\eta}{\delta_2(t,\eta)} \leq \frac{\lambda(\alpha-\beta)}{t-a} \quad \text{for a.e. } (t,x) \in \mathcal{D}, \tag{3.19}$$

respectively. Finally, by virtue of inequalities (2.2), (3.3) and (3.8), (3.18), (3.19), it follows from (3.7) and (3.17) that the function γ satisfies also assumption (3.1) of Lemma 3.3 and thus $(p,\tau,\mu) \in \mathcal{S}_{\mathrm{ac}}(\mathcal{D})$. □

Proof of Corollary 2.1 According to Lemma 3.1(iii) and (v), for any $s_1, s_2 \in [0, j_{-\alpha}]$, $s_1 \leq s_2$, there exists $\xi \in [s_1, s_2]$ such that

$$E_{-\alpha}(s_2) - E_{-\alpha}(s_1) = (s_2 - s_1)E'_{-\alpha}(\xi) = -(s_2 - s_1)\xi E_{1-\alpha}(\xi)$$

and thus we get

$$E_{-\alpha}(s_1) - E_{-\alpha}(s_2) \leq (s_2 - s_1)s_2 E_{1-\alpha}(0) \quad \text{for } 0 \leq s_1 \leq s_2 \leq j_{-\alpha} \tag{3.20}$$

and

$$E_{1-\alpha}(j_{-\alpha}) \leq E_{1-\alpha}(s) \quad \text{for } 0 \leq s \leq j_{-\alpha} \tag{3.21}$$

because the function $E_{1-\alpha}$ is decreasing on $[0, j_{-\alpha}]$ as follows from Lemma 3.1(iii), (iv). Moreover, in view of assumption (1.7), for a.e. $(t,x) \in \mathcal{D}$ such that $p(t,x) \not\equiv 0$ we have

$$z\bigl(\tau(t,x),x\bigr) \leq z(t,x),$$

where the function z is defined by (2.5). Consequently, it follows from (2.6), (3.20), and (3.21) that

$$\begin{aligned}
&\big[(t-a)(x-c)\big]^{1-\lambda}\big(E_{-\alpha}\big(z\big(\tau(t,x),x\big)\big)-E_{-\alpha}\big(z(t,x)\big)\big)\big|p(t,x)\big| \\
&\quad\le \big[(t-a)(x-c)\big]^{1-\lambda}\big(z(t,x)-z\big(\tau(t,x),x\big)\big)z(t,x)E_{1-\alpha}(0)\big|p(t,x)\big| \\
&\quad\le \frac{\lambda^2\beta}{2}\frac{j_{-\alpha}^2}{[(b-a)(d-c)]^\lambda}E_{1-\alpha}(j_{-\alpha}) \\
&\quad\le \frac{\lambda^2\beta}{2}\frac{j_{-\alpha}^2}{[(b-a)(d-c)]^\lambda}E_{1-\alpha}\big(z(t,x)\big) \quad \text{for a.e. } (t,x)\in\mathcal{D},
\end{aligned}$$

i.e., inequality (2.3) holds for a.e. $(t,x)\in\mathcal{D}$. We can show in a similar manner that inequality (2.4) holds for a.e. $(t,x)\in\mathcal{D}$, where the function z is defined by (2.5). Therefore, the assertion of the corollary follows from Theorem 2.1. □

Proof of Corollary 2.2 Let $\lambda = 1$ and the function z be defined by (2.5). For any $x\in[c,d]$ we put

$$f_x(t) := E_{-\alpha}\big(z(t,x)\big) \quad \text{for } t\in[a,b].$$

Since the function $z(\cdot,x)$ is absolutely continuous for every $x\in[c,d]$, by using Lemma 3.1(v), we conclude that the function f_x is absolutely continuous for every $x\in[c,d]$, as well. Moreover, by virtue of Lemma 3.1(iii), we get

$$\begin{aligned}
f_x'(t) &= E_{-\alpha}'\big(z(t,x)\big)z_{[1]}'(t,x) \\
&= -E_{1-\alpha}\big(z(t,x)\big)\frac{z^2(t,x)}{2(t-a)} \quad \text{for } (t,x)\in\,]a,b]\times[c,d].
\end{aligned} \tag{3.22}$$

It follows from Lemma 3.1(iii), (iv) that the function $E_{1-\alpha}$ is decreasing on $[0,j_{-\alpha}]$ and thus we have

$$E_{1-\alpha}(j_{-\alpha}) \le E_{1-\alpha}\big(z(t,x)\big) \le E_{1-\alpha}(0) \quad \text{for } (t,x)\in\mathcal{D}. \tag{3.23}$$

Now (3.22) and (3.23) yield

$$f_x'(t) \ge -\frac{j_{-\alpha}^2}{2(b-a)(d-c)}E_{1-\alpha}(0)(x-c) \quad \text{for } (t,x)\in\,]a,b]\times[c,d].$$

Consequently, for any $x\in[c,d]$ and $t_1,t_2\in[a,b]$, $t_1\le t_2$, we get

$$\begin{aligned}
E_{-\alpha}\big(z(t_2,x)\big)-E_{-\alpha}\big(z(t_1,x)\big) &= f_x(t_2)-f_x(t_1) = \int_{t_1}^{t_2} f_x'(s)\,\mathrm{d}s \\
&\ge -\frac{j_{-\alpha}^2}{2(b-a)(d-c)}E_{1-\alpha}(0)(x-c)(t_2-t_1).
\end{aligned} \tag{3.24}$$

On the other hand, observe that, in view of assumption (1.7), for a.e. $(t,x)\in\mathcal{D}$ such that $p(t,x)\not\equiv 0$ we have

$$a\le\tau(t,x)\le t\le b.$$

Therefore, by virtue of assumption (2.10) and condition (3.23), (3.24) shows that

$$\begin{aligned}
&\big(E_{-\alpha}\big(z\big(\tau(t,x),x\big)\big) - E_{-\alpha}\big(z(t,x)\big)\big)\big|p(t,x)\big| \\
&\quad\le \frac{j_{-\alpha}^2}{2(b-a)(d-c)} E_{1-\alpha}(0)(x-c)\big(t-\tau(t,x)\big)\big|p(t,x)\big| \\
&\quad\le \frac{\beta}{2}\frac{j_{-\alpha}^2}{(b-a)(d-c)} E_{1-\alpha}(j_{-\alpha}) \\
&\quad\le \frac{\beta}{2}\frac{j_{-\alpha}^2}{(b-a)(d-c)} E_{1-\alpha}\big(z(t,x)\big) \quad \text{for a.e. } (t,x)\in\mathcal{D},
\end{aligned}$$

i.e., inequality (2.3) with $\lambda = 1$ holds for a.e. $(t,x)\in\mathcal{D}$. We can show in a similar manner that inequality (2.4) with $\lambda = 1$ holds for a.e. $(t,x)\in\mathcal{D}$, as well. Consequently, all assumptions of Theorem 2.1 with $\lambda = 1$ are satisfied and thus $(p,\tau,\mu)\in\mathcal{S}_{\mathrm{ac}}(\mathcal{D})$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AL and JŠ obtained the results in a joint research. Both authors read and approved the final manuscript.

Acknowledgements
The research was supported by RVO: 67985840.

Endnotes
[a] This notion is introduced in [19] (see also [18] and Lemma 3.2).
[b] See Lemma 3.2.
[c] This notion is introduced in [19] (see also [18]).
[d] See Lemma 3.2.

Received: 28 January 2014 Accepted: 20 February 2014 Published: 12 Mar 2014

References
1. Lakshmikantham, V, Leela, S: Differential and Integral Inequalities. Academic Press, New York (1969)
2. Walter, W: Differential and Integral Inequalities. Springer, Berlin (1970)
3. Karpowicz, A: Carathéodory solutions of hyperbolic functional differential inequalities with first order derivatives. Ann. Pol. Math. **94**(1), 53-78 (2008)
4. Lakshmikantham, V, Pandit, SG: The method of upper, lower solutions and hyperbolic partial differential equations. J. Math. Anal. Appl. **105**, 466-477 (1985)
5. Karpowicz, A: The Darboux problem for hyperbolic functional differential equations and inequalities in the sense of Carathéodory. Math. Inequal. Appl. **13**(2), 405-425 (2010)
6. Lomtatidze, A, Mukhigulashvili, S, Šremr, J: Nonnegative solutions of the characteristic initial value problem for linear partial functional-differential equations of hyperbolic type. Math. Comput. Model. **47**(11-12), 1292-1313 (2008)
7. Domoshnitsky, A, Lomtatidze, A, Maghakyan, A, Šremr, J: Linear hyperbolic functional-differential equations with essentially bounded right-hand side. Abstr. Appl. Anal. **2011**, Article ID 242965 (2011). doi:10.1155/2011/242965
8. Kiguradze, T: Some boundary value problems for systems of linear partial differential equations of hyperbolic type. Mem. Differ. Equ. Math. Phys. **1**, 1-144 (1994)
9. Człapiński, T: Iterative methods for the Darboux problem for partial functional-differential equations. J. Inequal. Appl. **4**(2), 141-161 (1999)
10. Kiguradze, T: On a boundary value problem for quasilinear hyperbolic systems. Dokl. Akad. Nauk **328**(2), 135-138 (1993) (in Russian)
11. Kiguradze, T: On bounded in a strip solutions of quasilinear partial differential equations of hyperbolic type. Appl. Anal. **58**(3-4), 199-214 (1995)
12. Šremr, J: Some remarks on linear partial functional-differential inequalities of hyperbolic type. Ukr. Math. J. **60**(2), 327-337 (2008)
13. Deimling, K: Das Picard-Problem für $u_{xy} = f(x,y,u,u_x,u_y)$ unter Carathéodory-Voraussetzungen. Math. Z. **114**(4), 303-312 (1970) (in German)
14. Deimling, K: Absolutely continuous solutions of Cauchy problem for $u_{xy} = f(x,y,u,u_x,u_y)$. Ann. Mat. Pura Appl. **89**, 381-391 (1971)
15. Šremr, J: On the Darboux problem for linear hyperbolic functional-differential equations. Preprint IM-2013-8, Institute of Mathematics, AS CR (2013)
16. Tricomi, FG: Lezioni sulle equazioni a derivate parziali. Corso di analisi superiore. Editrice Gheroni, Torino (1954) (in Italian)

17. Watson, GN: A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge (1944)
18. Šremr, J: Absolutely continuous functions of two variables in the sense of Carathéodory. Electron. J. Differ. Equ. **2010**, 154 (2010)
19. Carathéodory, C: Vorlesungen über reelle Funktionen. Teubner, Leipzig (1918) (in German)

10.1186/1687-2770-2014-52
Cite this article as: Lomtatidze and Šremr: Carathéodory solutions to a hyperbolic differential inequality with a non-positive coefficient and delayed arguments. *Boundary Value Problems* 2014, 2014:52