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A BDDC algorithm for the mortar-type rotated Q_1 FEM for elliptic problems with discontinuous coefficients

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Abstract

In this paper, we propose a BDDC preconditioner for the mortar-type rotated Q_1 finite element method for second order elliptic partial differential equations with piecewise but discontinuous coefficients. We construct an auxiliary discrete space and build our algorithm on an equivalent auxiliary problem, and we present the BDDC preconditioner based on this constructed discrete space. Meanwhile, in the framework of the standard additive Schwarz methods, we describe this method by a complete variational form. We show that our method has a quasi-optimal convergence behavior, *i.e.*, the condition number of the preconditioned problem is independent of the jumps of the coefficients, and depends only logarithmically on the ratio between the subdomain size and the mesh size. Numerical experiments are presented to confirm our theoretical analysis.

MSC: 65N55; 65N30

Keywords: domain decomposition; BDDC algorithm; mortar; rotated Q_1 element; preconditioner

1 Introduction

The method of balancing domain decomposition by constraints (BDDC) was first introduced by Dohrmann in [1]. Mandel and Dohrmann restated the method in an abstract manner, and provided its convergence theory in [2]. The BDDC method is closely related to the dual-primal FETI (FETI-DP) method [3], which is one of dual iterative substructuring methods. Each BDDC and FETI-DP method is defined in terms of a set of primal continuity. The primal continuity is enforced across the interface between the subdomains and provides a coarse space component of the preconditioner. In [4], Mandel, Dohrmann, and Tezaur analyzed the relation between the two methods and established the corresponding theory.

In the last decades, the two methods have been widely analyzed and successfully been extended to many different types of partial differential equations. In [3], the two algorithms for elliptic problems were rederived and a brief proof of the main result was given. A BDDC algorithm for mortar finite element was developed in [5], meanwhile, the author also extended the FETI-DP algorithm to elasticity problems and Stokes problems in [6, 7], respectively. These algorithms are based on locally conforming finite element methods, and the coarse space components of the algorithms are related to the cross-points (*i.e.*, corners), which are often noteworthy points in domain decomposition methods (DDMs).

Since the cross-points are related to more than two subregions, thus it is not convenient to design the domain decomposition algorithm.

The BDDC method derives from the Neumann-Neumann domain decomposition method (see [8]). The difference is that the BDDC method applies an additive rather than a multiplicative coarse grid correction, and substructure spaces have some constraints which result in non-singular subproblems. Thus we need not modify the bilinear forms on subdomains, and we can solve each subproblem and coarse problem in parallel.

The rotated Q_1 element is an important nonconforming element. It was introduced by Rannacher and Turek in [9] for stokes equations originally, and it is the simplest example of a divergence-stable nonconforming element on quadrilaterals. Since its degree of freedom is integral average on element edge which is not related to the corners, and each degree of freedom on subdomain interfaces is only included in two neighboring subdomains, so it is easy to design the BDDC algorithm.

The mortar technique was introduced in [10]. This method is nonconforming domain decomposition methods with nonoverlapping subdomains. The meshes on different subdomains need not align across subdomain interfaces, and the matching of discretizations on adjacent subdomains is only enforced weakly. This offers the advantages of freely choosing highly varying mesh sizes on different subdomains and is very promising to approximate the problems with abruptly changing diffusion coefficients or local anisotropic.

In this paper, we study the BDDC algorithm for the mortar-type rotated Q_1 element for the second order elliptic problem with discontinuous coefficients, where the discontinuities lie only along the subdomain interfaces. Following the technique in [11], we construct an auxiliary discrete space and build our BDDC algorithm on an equivalent auxiliary problem. This approach overcomes the difficulty caused by the mortar condition and simplifies the implementation of the BDDC preconditioning iteration. Furthermore, since the rotated Q_1 element is not related to the subdomain's vertices, we can complete our theoretical analysis conveniently. It is proved that the condition number of the preconditioned operator is independent of the jumps of the coefficients and only depends logarithmically on the ratio between the subdomain size and mesh size. Numerical experiments are presented to confirm our theoretical analysis.

The rest of this paper is organized as follows: in Section 2, we introduce the model problem and the auxiliary problem. Section 3 gives the BDDC algorithm and proposes the BDDC preconditioner. Several technical tools are presented and analyzed in Section 4. In Section 5, we give the proof of the main result. Last section provides numerical experiments. For convenience, the symbols $\preceq$, $\succeq$ and $\asymp$ are used, and $x_1 \preceq y_1$, $x_2 \succeq y_2$, and $x_3 \asymp y_3$ mean that $x_1 \leq C_1 y_1$, $x_2 \geq C_2 y_2$, and $c_3 x_3 \leq y_3 \leq C_3 y_3$ for some constants C_1, C_2, C_3, and c_3 that are independent of discontinuous coefficients and mesh size.

2 Preliminaries

Let $\Omega \subset \mathcal{R}^2$ be a bounded, simply connect rectangular or L-shaped domain, we divide Ω into several nonoverlapping regular rectangular subdomains Ω_i $(i = 1, \ldots, N)$, *i.e.*, $\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i$. Consider the following model problem: Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = f(v), \quad \forall v \in H_0^1(\Omega), \tag{2.1}$$

where

$$a(u,v)=\sum_{i=1}^{N}\int_{\Omega_i}\rho_i(x)\nabla u\cdot\nabla v\,dx,\qquad f(v)=\sum_{i=1}^{N}\int_{\Omega_i}fv\,dx,$$

$f\in L^2(\Omega)$, the coefficients $\rho_i(x)$ $(i=1,\dots,N)$ are piecewise positive constants over Ω_i $(i=1,\dots,N)$.

For simplicity, we only consider the geometrically conforming case, *i.e.*, the intersection between the closure of two different subdomains is empty, or a vertex, or an edge. The subdomains $\{\Omega_i\}_{i=1}^N$ together form a coarse partition $\mathcal{T}_H(\Omega)$, we denote the diameter of each Ω_i by H_i. Let $\mathcal{T}_h(\Omega_i)$ be a quasi-uniform partition with the mesh size $O(h_i)$, made up of shape regular rectangles in Ω_i. The resulted partition can be nonmatched across adjacent subdomain interfaces. We denote the sets of edges of the triangulation $\mathcal{T}_h(\Omega_i)$ in Ω_i and $\partial\Omega_i$ by $\Omega_{i,h}^e$, $\partial\Omega_{i,h}^e$ respectively, and let $\Omega_{i,h}$, $\partial\Omega_{i,h}$ be the sets of vertices of the triangulation $\mathcal{T}_h(\Omega_i)$ that are in $\bar{\Omega}_i$, $\partial\bar{\Omega}_i$ respectively.

For each triangulation $\mathcal{T}_h(\Omega_i)$, the rotated Q_1 element space is defined by

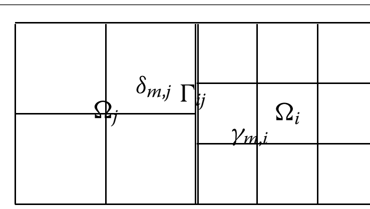
$$\begin{aligned}X_h(\Omega_i)=\Bigg\{&v\in L^2(\Omega_i):v|_E=a_E^1+a_E^2x+a_E^3y+a_E^4\big(x^2-y^2\big),a_E^i\in\mathcal{R};\\&\int_e v\,ds=0,\forall e\in\partial E\cap\partial\Omega,E\in\mathcal{T}_h(\Omega_i),\text{for }E_1,E_2\in\mathcal{T}_h(\Omega_i),\\&\text{if }\partial E_1\cap\partial E_2=e,\text{then }\int_e v|_{\partial E_1}\,ds=\int_e v|_{\partial E_2}\,ds\Bigg\}.\end{aligned}$$

Let the global discrete space $X_h(\Omega)=\prod_{i=1}^N X_h(\Omega_i)$. We equip the space $X_h(\Omega_i)$ with the following seminorm:

$$|v|^2_{H_h^1(\Omega_i)}=\sum_{E\in\mathcal{T}_h(\Omega_i)}|v|^2_{H^1(E)}.$$

We denote Γ_{ij} the common open edge of Ω_i and Ω_j, and let $\Gamma=\bigcup_{ij}\Gamma_{ij}$. Each Γ_{ij} can be regarded as two sides corresponding to the two subdomains Ω_i and Ω_j. We define one of the sides of Γ_{ij} as mortar denoted by $\gamma_{m,i}$ and the other one as nonmortar denoted by $\delta_{m,j}$, here m represents the indexing of Γ_{ij} (see Figure 1). We assume that: (1) the mortar for $\gamma_{m,i}=\delta_{m,j}=\Gamma_{ij}$ is chosen by the condition $\rho_j\preceq\rho_i$; (2) there is at least one subdomain which has two mortar sides associated with each cross point; (3) $h_i\preceq h_j$, *i.e.*, h_i/h_j is bounded. The first condition used in choosing mortar sides is essential (see the numerical tests in [12]). The last condition is technical but not essential for the convergence analysis. Along each Γ_{ij}, there are two independent and different 1-D meshes which are denoted by $\mathcal{T}_h^i(\gamma_{m,i})$ and $\mathcal{T}_h^j(\delta_{m,j})$. For each nonmortar side $\delta_{m,j}=\Gamma_{ij}$, we denote by $M^{h_j}(\delta_{m,j})\subset L^2(\Gamma_{ij})$ an auxiliary

Figure 1 Nonmatching grid.



test space whose functions are piecewise constant on $\mathcal{T}_h^j(\delta_{m,j})$. We denote by Q_m the L^2-orthogonal projection from the $L^2(\Gamma_{ij})$ space to the $M^{h_j}(\delta_{m,j})$ space.

Now we define the mortar-type rotated Q_1 space as follows:

$$\mathcal{V}_h = \left\{ v = \prod_{i=1}^{N} v_i \in X_h(\Omega) : Q_m(v_i|_{\gamma_{m,i}}) = Q_m(v_j|_{\delta_{m,j}}), \forall \gamma_{m,i} = \delta_{m,j} \subset \Gamma \right\}, \tag{2.2}$$

here $v_i|_{\gamma_{m,i}}$ is the restriction of $v_i \in X_h(\Omega_i)$ to the mortar side $\gamma_{m,i}$, and $v_j|_{\delta_{m,j}}$ is the restriction of $v_j \in X_h(\Omega_j)$ to the nonmortar side $\delta_{m,j}$. The condition in (2.2) for each interface is called *mortar condition*. The mortar-type rotated Q_1 element approximation of problem (2.1) is: find $u_h \in \mathcal{V}_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in \mathcal{V}_h, \tag{2.3}$$

where

$$a_h(u_h, v_h) = \sum_{i=1}^{N} a_{h,i}(u_h, v_h), \qquad a_{h,i}(u_h, v_h) = \sum_{E \in \mathcal{T}_h(\Omega_i)} \int_E \rho_i \nabla u_h \nabla v_h \, dx.$$

It can easily be shown that $a_h(\cdot,\cdot)$ is positive definite on $\mathcal{V}_h$, which yields the existence and uniqueness of the discrete solution. The error estimate between the discrete and the continuous solution is discussed in [13].

Since the mortar condition depends on both the degrees of freedom on the interfaces and the ones near the interfaces, it is difficult to construct a preconditioner directly for (2.3). To overcome this difficulty, we introduce a new discrete space and an auxiliary problem which is equivalent to problem (2.3).

For each $v \in \mathcal{V}_h$, we define an element $\tilde{v} = \prod_{i=1}^N \tilde{v}_i \in X_h(\Omega)$ that satisfies the following conditions:

- for any $e \in (\bigcup_{i=1}^N \Omega_{i,h}^e) \cup (\bigcup_m \mathcal{T}_h^i(\gamma_{m,i}))$,

$$\frac{1}{|e|} \int_e \tilde{v} \, ds = \frac{1}{|e|} \int_e v \, ds; \tag{2.4}$$

- for any $\psi \in M^{h_j}(\delta_{m,j})$,

$$\int_{\delta_{m,j}} \tilde{v}\psi \, ds = \int_{\delta_{m,j}} \bar{v}\psi \, ds, \tag{2.5}$$

  where $\bar{v} \in L^2(\gamma_{m,i})$ is a piecewise constant function on elements of $\mathcal{T}_h^i(\gamma_{m,i})$ such that $\bar{v}|_e = \frac{1}{|e|}\int_e v_i|_{\gamma_{m,i}} \, ds$ for any $e \in \mathcal{T}_h^i(\gamma_{m,i})$. Note that the average value of $\tilde{v}$ on $e \in \mathcal{T}_h^j(\delta_{m,j})$ can be calculated by (2.5).

By the above definition, all $\tilde{v}$ associated with v form a space $\tilde{\mathcal{V}}_h \subset X_h(\Omega)$ as

$$\tilde{\mathcal{V}}_h = \left\{ \tilde{v} = \prod_{i=1}^{N} \tilde{v}_i \in X_h(\Omega) : v \in \mathcal{V}_h \right\}.$$

For the two related spaces $\mathcal{V}_h$ and $\tilde{\mathcal{V}}_h$, we have the following result.

Lemma 2.1 ([12]) *For any pair of $v \in \mathcal{V}_h$, $\tilde{v} \in \tilde{\mathcal{V}}_h$ defined above, the following is true*:

$$a_h(v,v) \asymp a_h(\tilde{v},\tilde{v}). \tag{2.6}$$

Now we introduce the auxiliary problem, that is, to find $\tilde{u} \in \tilde{\mathcal{V}}_h$ which satisfies

$$a_h(\tilde{u},\tilde{v}) = f(\tilde{v}), \quad \forall \tilde{v} \in \tilde{\mathcal{V}}_h. \tag{2.7}$$

Define an operator $\tilde{A}_h : \tilde{\mathcal{V}}_h \to \tilde{\mathcal{V}}_h$ by

$$(\tilde{A}_h\tilde{v},\tilde{w}) = a_h(\tilde{v},\tilde{w}), \quad \forall \tilde{v},\tilde{w} \in \tilde{\mathcal{V}}_h.$$

From the above lemma, we only need to construct a preconditioner for the operator $\tilde{A}_h$.

3 BDDC algorithm

In this section, we introduce our BDDC preconditioner for problem (2.7) and describe the BDDC algorithm.

We first define a discrete harmonic operator $\mathcal{H}_i$ associated with the rotated Q_1 element: for any $v \in X_h(\Omega_i)$, let $\mathcal{H}_i v \in X_h(\Omega_i)$ such that

$$\begin{cases} a_{h,i}(\mathcal{H}_i v, w) = 0, & \forall w \in X_h^0(\Omega_i), \\ \frac{1}{|e|}\int_e \mathcal{H}_i v\,ds = \frac{1}{|e|}\int_e v\,ds, & \forall e \in \partial\Omega_{i,h}^e, \end{cases}$$

here $X_h^0(\Omega_i) = \{v \in X_h(\Omega_i) : \int_e v\,ds = 0, \forall e \in \partial\Omega_{i,h}^e\}$. Let $X_h(\partial\Omega_i) = \mathcal{H}_i(X_h(\Omega_i))$. We define $\mathcal{H}$ as a corresponding piecewise harmonic operator on the auxiliary space $\tilde{\mathcal{V}}_h$ by $\mathcal{H}|_{\Omega_i} = \mathcal{H}_i$.

In order to introduce our domain decomposition method, we decompose the auxiliary discrete space $\tilde{\mathcal{V}}_h$ as follows:

$$\tilde{\mathcal{V}}_h = X_h^P(\Omega) \oplus \tilde{\mathcal{V}}_h(\Gamma) \quad \text{and} \quad X_h^P(\Omega) = \prod_{i=1}^N X_h^0(\Omega_i), \tag{3.1}$$

where the space $\tilde{\mathcal{V}}_h(\Gamma)$ is a piecewise harmonic function space defined as

$$\tilde{\mathcal{V}}_h(\Gamma) = \mathcal{H}(\tilde{\mathcal{V}}_h) = \big\{v \in \tilde{\mathcal{V}}_h : v|_{\Omega_i} = \mathcal{H}_i(v|_{\Omega_i}), i = 1,2,\ldots,N\big\}.$$

We define a space $\tilde{X}_h(\Gamma) = \{v \in \prod_{i=1}^N X_h(\partial\Omega_i) : \int_{\gamma_{m,i}} v|_{\Omega_i}\,ds = \int_{\delta_{m,j}} v|_{\Omega_j}\,ds, \forall \gamma_{m,i} = \delta_{m,j} \subset \Gamma\}$. The space $\tilde{X}_h(\Gamma)$ is between $\tilde{\mathcal{V}}_h(\Gamma)$ and $\prod_{i=1}^N X_h(\partial\Omega_i)$, and our BDDC preconditioner is mainly constructed on this space.

As we know, the technical aspect in DDMs is that the preconditioner includes a coarse problem which can enhance the convergence. In view of the characteristic of the space $\tilde{X}_h(\Gamma)$, we select the standard coarse space $\mathcal{V}_H(\Omega)$ which is the rotated Q_1 finite element space associated with the coarse partition $\mathcal{T}_H(\Omega)$, and it satisfies primal constraints on subdomain interfaces.

The substructure space $\tilde{\mathcal{V}}_\Delta(\Gamma_i)$ with constraints is defined by

$$\tilde{\mathcal{V}}_\Delta(\Gamma_i) = \left\{v \in X_h(\partial\Omega_i) : \int_{\Gamma_{ij}} v\,ds = 0, \forall \Gamma_{ij} \subset \partial\Omega_i\right\}.$$

Denote $\tilde{\mathcal{V}}_\Delta(\Gamma) = \prod_{i=1}^N \tilde{\mathcal{V}}_\Delta(\Gamma_i)$. The coarse space and the product space $\tilde{\mathcal{V}}_\Delta(\Gamma)$ play an important role in the description and analysis of our iterative method.

To present our BDDC preconditioner, we introduce several space transfer operators. Define an interpolation operator $I_H : \tilde{\mathcal{V}}_h \to \mathcal{V}_H(\Omega)$ by

$$\frac{\int_{\Gamma_{ij}} I_H v\,ds}{|\Gamma_{ij}|} = \frac{\int_{\Gamma_{ij}} v\,ds}{|\Gamma_{ij}|}, \quad \forall \Gamma_{ij} \subset \Gamma.$$

The intergrid transfer operator $I_h : \mathcal{V}_H(\Omega) \to \tilde{X}_h(\Gamma)$ is defined by

$$\frac{\int_e I_h v\,ds}{|e|} = \frac{\int_e v\,ds}{|e|}, \quad \forall e \in \partial\Omega_{i,h}^e\ (i = 1, \dots, N).$$

Define an extension operator $R_i^T : X_h(\partial\Omega_i) \to \tilde{\mathcal{V}}_h(\Gamma)$ as

- for any $e \in \bigcup_{\gamma_{m,i}\subset\partial\Omega_i} \mathcal{T}_h^i(\gamma_{m,i})$, $\frac{1}{|e|}\int_e R_i^T v|_{\gamma_{m,i}}\,ds = \frac{1}{|e|}\int_e v|_{\gamma_{m,i}}\,ds$;
- for any $e \in \bigcup_{\gamma_{r,j}\not\subset\partial\Omega_i} \mathcal{T}_h^j(\gamma_{r,j})$, $\frac{1}{|e|}\int_e R_i^T v|_{\gamma_{r,j}}\,ds = 0$;
- for any $e \in \bigcup_n \mathcal{T}_h^j(\delta_{n,j})$, $\frac{1}{|e|}\int_e R_i^T v|_{\delta_{n,j}}\,ds$ satisfies (2.5).

Its transpose $R_i : \tilde{\mathcal{V}}_h(\Gamma) \to X_h(\partial\Omega_i)$ is defined by

$$(R_i w, v) = (w, R_i^T v), \quad \forall w \in \tilde{\mathcal{V}}_h(\Gamma), v \in X_h(\partial\Omega_i).$$

Denote $R_i^T|_{\tilde{\mathcal{V}}_\Delta(\Gamma_i)} : \tilde{\mathcal{V}}_\Delta(\Gamma_i) \to \tilde{\mathcal{V}}_h(\Gamma)$ by $R_{\Delta,i}^T$, the corresponding transpose $R_{\Delta,i} : \tilde{\mathcal{V}}_h(\Gamma) \to \tilde{\mathcal{V}}_\Delta(\Gamma_i)$ is defined by

$$(R_{\Delta,i} w, v) = (w, R_{\Delta,i}^T v), \quad \forall w \in \tilde{\mathcal{V}}_h(\Gamma), v \in \tilde{\mathcal{V}}_\Delta(\Gamma_i).$$

We also need to define another prolongation operator $E_i : X_h(\Omega_i) \to \tilde{\mathcal{V}}_h$ as follows:

- if $e \in \Omega_{i,h}^e$, then $\frac{1}{|e|}\int_e E_i v\,ds = \frac{1}{|e|}\int_e v\,ds$;
- if $e \in \mathcal{T}_h^i(\gamma_{m,i})$, $\gamma_{m,i} \subset \partial\Omega_i$, then $\frac{1}{|e|}\int_e E_i v\,ds = \frac{1}{|e|}\int_e v\,ds$;
- if $e \in \mathcal{T}_h^k(\gamma_{s,k})$, $\forall \gamma_{s,k}$, $k \neq i$, then $\frac{1}{|e|}\int_e E_i v\,ds = 0$;
- if $e \in \bigcup_s \mathcal{T}_h^j(\delta_{s,j})$, it follows from (2.5) that $\frac{1}{|e|}\int_e E_i v\,ds$ can be obtained by the edge average values on associated mortar sides;
- else, $\frac{1}{|e|}\int_e E_i v\,ds = 0$.

In what follows, we describe our BDDC preconditioning algorithm, we apply the basic framework of additive Schwarz method (or parallel subspace correction method [14]). From the decomposition (3.1), we only need to choose appropriate subspace solvers.

First of all, the coarse subspace solver $B_H : \mathcal{V}_H(\Omega) \to \mathcal{V}_H(\Omega)$ is defined by

$$(B_H u_H, v_H) = a_h(u_H, v_H), \quad \forall u_H, v_H \in \mathcal{V}_H(\Omega).$$

On each subdomain, similar operators $B_i : \tilde{\mathcal{V}}_\Delta(\Gamma_i) \to \tilde{\mathcal{V}}_\Delta(\Gamma_i)$ and $B_{P,i} : X_h^0(\Omega_i) \to X_h^0(\Omega_i)$ are defined, respectively, by

$$(B_i u, v) = a_{h,i}(u, v), \quad \forall u, v \in \tilde{\mathcal{V}}_\Delta(\Gamma_i),$$

$$(B_{P,i} u, v) = a_{h,i}(u, v), \quad \forall u, v \in X_h^0(\Omega_i).$$

Remark 3.1 The bilinear form on the coarse space can be different from that on substructure space, here we only use the exact solvers. On each subdomain, we avoid the possible singularity of local subproblem and we need not modify the bilinear forms.

Now we define our BDDC preconditioner as

$$B_{\text{bddc}} = R_0^T B_H^{-1} R_0 + \sum_{i=1}^N R_{\Delta,i}^T B_i^{-1} R_{\Delta,i} + \sum_{i=1}^N B_{p,i}^{-1},$$

where $R_0^T = \sum_{i=1}^N R_i^T I_h$, R_0 is the corresponding transpose defined by

$$(R_0 w, v) = \left(w, R_0^T v\right), \quad \forall w \in \tilde{\mathcal{V}}_h(\Gamma), v \in \mathcal{V}_H(\Omega).$$

Let P_0 be an operator from $\tilde{\mathcal{V}}_h(\Gamma)$ to $\mathcal{V}_H(\Omega)$ defined by

$$a_h(P_0 u, v) = a_h\left(u, R_0^T v\right), \quad \forall u \in \tilde{\mathcal{V}}_h(\Gamma), v \in \mathcal{V}_H(\Omega),$$

P_i and $P_{p,i}$ be the operators from $\tilde{\mathcal{V}}_h(\Gamma)$ to $\tilde{\mathcal{V}}_\Delta(\Gamma_i)$ and $X_h^0(\Omega_i)$ defined, respectively, by

$$a_{h,i}(P_i u, v) = a_h\left(u, R_{\Delta,i}^T v\right), \quad \forall u \in \tilde{\mathcal{V}}_h, v \in \tilde{\mathcal{V}}_\Delta(\Gamma_i),$$
$$a_h(P_{p,i} u, v) = a_h(u, v), \quad \forall u \in \tilde{\mathcal{V}}_h, v \in X_h^0(\Omega_i).$$

Then the BDDC preconditioned operator $P_{\text{bddc}} = B_{\text{bddc}} \tilde{A}_h$ can be written as

$$P_{\text{bddc}} = R_0^T P_0 + \sum_{i=1}^N R_{\Delta,i}^T P_i + \sum_{i=1}^N P_{p,i}.$$

We have the following main result.

Theorem 3.1 *The BDDC preconditioned operator P_{bddc} satisfies*

$$a_h(u, u) \preceq a_h(P_{\text{bddc}} u, u) \preceq \left(1 + \log \frac{H}{h}\right)^2 a_h(u, u), \quad \forall u \in \tilde{\mathcal{V}}_h,$$

where $H/h = \max_i(H_i/h_i)$.

4 Technical tools

In this section we state and prove a few technical lemmas necessary for the proof of Theorem 3.1. Our theoretical analysis is based on the substructuring theory of conforming elements.

We assume $V^h(\Omega_i)$ be the bilinear conforming element space associated with the partition $\mathcal{T}_h(\Omega_i)$. We split the interface $\partial\Omega_i$ into four open edges $\mathcal{E}$, and define a restriction operator $I_{\mathcal{E}}^0 : V^h(\partial\Omega_i) \to V^h(\partial\Omega_i)$ ($V^h(\partial\Omega_i) = V^h(\Omega_i)|_{\partial\Omega_i}$) as: for any $v \in V^h(\partial\Omega_i)$

$$I_{\mathcal{E}}^0 v = \begin{cases} v, & \text{on } \mathcal{E}, \\ 0, & \text{on } \partial\Omega_i \setminus \mathcal{E}. \end{cases}$$

For the operator $I_{\mathcal{E}}^0$, we have the following result.

Lemma 4.1 ([15]) *For an edge $\mathcal{E}$ of $\partial\Omega_i$, then for any $v \in V^h(\partial\Omega_i)$, we have*

$$\left\| I_{\mathcal{E}}^0 v \right\|_{H^{1/2}(\partial\Omega_i)} \preceq \left(1 + \log \frac{H_i}{h_i}\right) \|v\|_{H^{1/2}(\partial\Omega_i)}.$$

Remark 4.1 The above lemma is related to vertex-edge-face arguments in substructuring methods, in view of the characteristic for the rotated Q_1 element, here the results only concern the inequalities for faces.

Let $V^{h/2}(\Omega_i)$ be the conforming element space of bilinear continuous functions on the partition $\mathcal{T}_{h/2}(\Omega_i)$ which is constructed by joining the midpoints of the edges of elements of $\mathcal{T}_h(\Omega_i)$. We now introduce a local equivalence map $\mathcal{M}_i : X_h(\Omega_i) \to V^{h/2}(\Omega_i)$ as follows (*cf.* [13]).

Definition 4.2 Given $v \in X_h(\Omega_i)$, we define $\mathcal{M}_i v \in V^{h/2}(\Omega_i)$ by the values of $\mathcal{M}_i v$ at the vertices of the partition $\mathcal{T}_{h/2}(\Omega_i)$.

- If P is a central point of E, $E \in \mathcal{T}_h(\Omega_i)$, then

$$(\mathcal{M}_i v)(P) = \frac{1}{4} \sum_{e_i \in \partial E} \frac{1}{|e_i|} \int_{e_i} v\, ds.$$

- If P is a midpoint of one edge $e \in \partial E$, $E \in \mathcal{T}_h(\Omega_i)$, then

$$(\mathcal{M}_i v)(P) = \frac{1}{|e|} \int_e v\, ds.$$

- If $P \in \Omega_{i,h} \setminus \partial\Omega_{i,h}$, then

$$(\mathcal{M}_i v)(P) = \frac{1}{4} \sum_{e_i} \frac{1}{|e_i|} \int_{e_i} v\, ds,$$

  where the sum is taken over all edges e_i with the common vertex P, $e_i \in \partial E_i$, $E_i \in \mathcal{T}_h(\Omega_i)$.

- If $P \in \partial\Omega_{i,h}$, then

$$(\mathcal{M}_i v)(P) = \frac{|e_l|}{|e_l| + |e_r|} \left(\frac{1}{|e_l|} \int_{e_l} v\, ds \right) + \frac{|e_r|}{|e_l| + |e_r|} \left(\frac{1}{|e_r|} \int_{e_r} v\, ds \right),$$

  where $e_l \in \partial E_1 \cap \partial\Omega_i$ and $e_r \in \partial E_2 \cap \partial\Omega_i$ are the left and right neighbor edges of P, $E_1, E_2 \in \mathcal{T}_h(\Omega_i)$. If P is a vertex of Ω_i, then $E_1 = E_2$.

Define the pseudo-inverse map $\mathcal{M}_i^+ : V^{h/2}(\Omega_i) \to X_h(\Omega_i)$ by

$$\frac{1}{|e|} \int_e \mathcal{M}_i^+ v\, ds = v(P), \quad \forall v \in V^{h/2}(\Omega_i),$$

where $e \in \partial E$, $E \in \mathcal{T}_h(\Omega_i)$, P is the midpoint of e. Obviously, we have

$$\mathcal{M}_i^+ \mathcal{M}_i v = v, \quad \forall v \in X_h(\Omega_i).$$

For the operators $\mathcal{M}_i$ and $\mathcal{M}_i^+$, we have the following results (see [13]):

$$\begin{aligned} |\mathcal{M}_i v|_{H^1(\Omega_i)} &\asymp |v|_{H_h^1(\Omega_i)}, \quad \forall v \in X_h(\Omega_i); \\ \left|\mathcal{M}_i^+ v\right|_{H_h^1(\Omega_i)} &\preceq |v|_{H_h^1(\Omega_i)}, \quad \forall v \in V^{h/2}(\Omega_i). \end{aligned} \tag{4.1}$$

Lemma 4.3 *For any $u_i \in \tilde{\mathcal{V}}_\Delta(\Gamma_i)$, we can split u_i into $u_i = \sum_{\Gamma_{ij}\subset\partial\Omega_i} u_{ij}$, and we have*

$$|u_{ij}|_{H_h^1(\Omega_i)} \preceq \bigl(1 + \log(H_i/h_i)\bigr)|u_i|_{H_h^1(\Omega_i)}, \tag{4.2}$$

where $u_{ij} \in \tilde{\mathcal{V}}_\Delta(\Gamma_i)$, and for any $e \in \Gamma_{ij}^e$, $\int_e u_{ij}\,ds/|e| = \int_e u_i\,ds/|e|$; for any $e \in \partial\Omega_{i,h}^e \setminus \Gamma_{ij}$, $\int_e u_{ij}\,ds/|e| = 0$.

Proof By (4.1), Lemma 4.1, the inverse trace theorem, the trace theorem, and the Poincaré inequality, we obtain

$$\begin{aligned} |u_{ij}|_{H_h^1(\Omega_i)} &\leq \left|\left|\mathcal{M}_i^+ \mathcal{H}_i I_{\mathcal{E}}^0(\mathcal{M}_i u_{ij})\right|_{\partial\Omega_i}\right|_{H_h^1(\Omega_i)} \\ &\preceq \left|\left|\mathcal{H}_i I_{\mathcal{E}}^0(\mathcal{M}_i u_{ij})\right|_{\partial\Omega_i}\right|_{H_h^1(\Omega_i)} \\ &\preceq \left|\left|I_{\mathcal{E}}^0(\mathcal{M}_i u_{ij})\right|_{\partial\Omega_i}\right|_{H^{1/2}(\partial\Omega_i)} \\ &\preceq \bigl(1 + \log(H_i/h_i)\bigr)\|\mathcal{M}_i u_i\|_{H^{1/2}(\partial\Omega_i)} \\ &\leq \bigl(1 + \log(H_i/h_i)\bigr)\|\mathcal{M}_i u_i\|_{H^1(\Omega_i)} \\ &\preceq \bigl(1 + \log(H_i/h_i)\bigr)|\mathcal{M}_i u_i|_{H^1(\Omega_i)} \\ &\preceq \bigl(1 + \log(H_i/h_i)\bigr)|u_i|_{H_h^1(\Omega_i)}, \end{aligned}$$

where $\mathcal{H}_i$ is a piecewise bilinear conforming element harmonic operator, and we have used the minimal energy property of discrete harmonic functions. □

5 Proof of Theorem 3.1

In the proof of Theorem 3.1 we use the abstract framework of ASM methods (see [16]), we need to prove three assumptions. Assumption II follows from the standard coloring argument, we only need to prove Assumption I and Assumption III.

First we show the following stability of the decomposition.

Lemma 5.1 (Assumption I) *For any $u \in \tilde{\mathcal{V}}_h$, we have the following decomposition*:

$$u = R_0^T u_H + \sum_{i=1}^N R_{\Delta,i}^T u_i + \sum_{i=1}^N u_{p,i}, \quad u_H \in \mathcal{V}_H(\Omega), u_i \in \tilde{\mathcal{V}}_\Delta(\Gamma_i), u_{p,i} \in X_h^0(\Omega_i), \tag{5.1}$$

which satisfies

$$a_h(u_H, u_H) + \sum_{i=1}^N a_{h,i}(u_i, u_i) + \sum_{i=1}^N a_{h,i}(u_{p,i}, u_{p,i}) \preceq a_h(u, u). \tag{5.2}$$

Proof First we show the decomposition (5.1). For any function $u \in \tilde{\mathcal{V}}_h$, let $u_{p,i} = P_{p,i}u$ and $u_H = I_H u$, obviously $u - \sum_{i=1}^N u_{p,i} - I_h u_H$ is a piecewise discrete harmonic function. So we denote $u_\Delta = u - \sum_{i=1}^N u_{p,i} - I_h u_H$, $u_i = u_\Delta|_{\Omega_i}$. From the definition of I_H and I_h, we have

$$\int_{\Gamma_{ij}} u_i\,ds = \int_{\Gamma_{ij}} u_\Delta\,ds = \int_{\Gamma_{ij}} (u - I_h u_H)\,ds = \int_{\Gamma_{ij}} (u - u_H)\,ds = 0,$$

and by the definition of $R_{\Delta,i}^T$, we get

$$\begin{aligned} R_0^T u_H + \sum_{i=1}^N R_{\Delta,i}^T u_i + \sum_{i=1}^N u_{p,i} &= \sum_{i=1}^N R_i^T I_h u_H + \sum_{i=1}^N R_i^T \left(u - \sum_{i=1}^N u_{p,i} - I_h u_H \right) + \sum_{i=1}^N u_{p,i} \\ &= \sum_{i=1}^N R_i^T \left(u - \sum_{i=1}^N u_{p,i} \right) + \sum_{i=1}^N u_{p,i} \\ &= u - \sum_{i=1}^N u_{p,i} + \sum_{i=1}^N u_{p,i} \\ &= u, \end{aligned}$$

where we have used the fact $\sum_{i=1}^N R_i^T u = u$, $\forall u \in \tilde{\mathcal{V}}_h(\Gamma)$. Hence $u_i \in \tilde{\mathcal{V}}_\Delta(\Gamma_i)$ and the equality (5.1) holds.

Now we prove the stability of decomposition (5.2). Let $\bar{u}_{\Gamma_{ij}} = \int_{\Gamma_{ij}} u\,ds/|\Gamma_{ij}|$. Using Lemma 3.5 in [12], Poincaré-Friedrichs' inequality and scaling argument, we derive

$$\begin{aligned} \sum_{\Gamma_{ij},\Gamma_{ik} \subset \partial\Omega_i} |\bar{u}_{\Gamma_{ij}} - \bar{u}_{\Gamma_{ik}}|^2 &= \sum_{\Gamma_{ij},\Gamma_{ik} \subset \partial\Omega_i} \left(\frac{1}{|\Gamma_{ij}|} \int_{\Gamma_{ij}} (u - \bar{u}_{\Gamma_{ik}}) \right)^2 \\ &\preceq \sum_{\Gamma_{ik} \subset \partial\Omega_i} \left(\frac{1}{H_i^2} \|u - \bar{u}_{\Gamma_{ik}}\|_{L^2(\Omega_i)}^2 + |u|_{H_h^1(\Omega_i)}^2 \right) \\ &\preceq |u|_{H_h^1(\Omega_i)}^2. \end{aligned} \tag{5.3}$$

From (5.3) and the discrete equivalent norm, we have

$$a_h(u_H, u_H) = \sum_{i=1}^N a_{h,i}(u_H, u_H) \asymp \sum_{i=1}^N \rho_i \sum_{\Gamma_{ij},\Gamma_{ik} \subset \partial\Omega_i} |\bar{u}_{\Gamma_{ij}} - \bar{u}_{\Gamma_{ik}}|^2 \preceq a_h(u, u). \tag{5.4}$$

Since $P_{p,i}$ is an orthogonal projection with respect to $a_{h,i}(\cdot,\cdot)$, we obtain

$$\sum_{i=1}^N a_{h,i}(u_{p,i}, u_{p,i}) = \sum_{i=1}^N a_{h,i}(P_{p,i}u, P_{p,i}u) \le a_h(u, u). \tag{5.5}$$

Meanwhile, from the fact that the harmonic function has minimal energy norm and (5.4)-(5.5), we deduce

$$\begin{aligned} \sum_{i=1}^N a_{h,i}(u_i, u_i) &= a_h(u_\Delta, u_\Delta) \\ &= a_h\left(u - \sum_{i=1}^N u_{p,i} - I_h u_H, u - \sum_{i=1}^N u_{p,i} - I_h u_H \right) \end{aligned}$$

$$\preceq a_h(u,u) + \sum_{i=1}^{N} a_{h,i}(u_{p,i},u_{p,i}) + a_h(I_h u_H, I_h u_H) \tag{5.6}$$

$$\preceq a_h(u,u). \tag{5.7}$$

So (5.4)-(5.6) lead to (5.2). □

Next we state the local stability as follows.

Lemma 5.2 (Assumption III) *For any $u \in \tilde{\mathcal{V}}_\Delta(\Gamma_i)$, we have*

$$a_h\big(R_{\Delta,i}^T u, R_{\Delta,i}^T u\big) \preceq \left(1 + \log\frac{H}{h}\right)^2 a_{h,i}(u,u). \tag{5.8}$$

For any $u_H \in \mathcal{V}_H(\Omega)$, we have

$$a_h\big(R_0^T u_H, R_0^T u_H\big) \preceq \left(1 + \log\frac{H}{h}\right)^2 a_h(u_H,u_H). \tag{5.9}$$

Proof To prove (5.8) we first introduce a function $\theta_m = \prod_{i=1}^N \theta_{m,i} \in \tilde{\mathcal{V}}_h$ associated with a mortar side $\gamma_{m,i} \subset \Gamma$, which satisfies the following:

- for any $e \in \mathcal{T}_h^i(\gamma_{m,i})$, $\frac{1}{|e|}\int_e \theta_{m,i}|_{\gamma_{m,i}}\,ds = 1$;
- for any $e \in \bigcup_{r\neq m} \mathcal{T}_h^i(\gamma_{r,i})$, $\frac{1}{|e|}\int_e \theta_{m,i}|_{\gamma_{r,i}}\,ds = 0$;
- for any $e \in \bigcup_n \mathcal{T}_h^j(\delta_{n,j})$, $\frac{1}{|e|}\int_e \theta_{m,j}|_{\delta_{n,j}}\,ds$ satisfies (2.5).

Then we can decompose $R_{\Delta,i}^T u \in \tilde{\mathcal{V}}_h(\Gamma)$ as follows:

$$R_{\Delta,i}^T u = R_i^T u = \mathcal{H}\Bigg(\sum_{\gamma_{m,i}\subset\partial\Omega_i} \mathcal{I}_h\big(\theta_{m,i}(E_i u)\big)\Bigg) = \sum_{\gamma_{m,i}\subset\partial\Omega_i} \mathcal{H}\big(\mathcal{I}_h\big(\theta_{m,i}(E_i u)\big)\big), \tag{5.10}$$

here we have used the fact that the degrees of freedom on the interface Γ of the function u are as same as that of $\sum_{\gamma_{m,i}\subset\partial\Omega_i} \mathcal{I}_h(\theta_{m,i}(E_i u))$, and the operator $\mathcal{I}_h$ is defined by the average values on the edge elements, *i.e.*,

$$\frac{1}{|e|}\int_e \mathcal{I}_h\big(\theta_{m,i}(E_i u)\big) = \frac{1}{|e|}\int_e \theta_{m,i}\,ds \cdot \frac{1}{|e|}\int_e E_i u\,ds, \quad \forall e \in \partial\Omega_{h,i}^e.$$

Note that the support of $\mathcal{H}(\mathcal{I}_h(\theta_{m,i}(E_i u)))$ is on $\bar{\Omega}_i \cup \bar{\Omega}_j$, and using Lemma 3.4 in [12] we have

$$a_h\big(\mathcal{H}\mathcal{I}_h\big(\theta_{m,i}(E_i u)\big), \mathcal{H}\mathcal{I}_h\big(\theta_{m,i}(E_i u)\big)\big) \lesssim \rho_i \big|\mathcal{H}\mathcal{I}_h\big(\theta_{m,i}(E_i u)\big)\big|_{H_h^1(\Omega_i)}^2. \tag{5.11}$$

Since the degrees of freedom on the interface $\partial\Omega_i$ of $\mathcal{H}_i(\mathcal{I}_h(\theta_{m,i}(E_i u)))$ are only nonzero on the edge $\gamma_{m,i}$, using Lemma 4.3, we deduce

$$\begin{aligned} \big|\mathcal{H}\mathcal{I}_h\big(\theta_{m,i}(E_i u)\big)\big|_{H_h^1(\Omega_i)}^2 &= \big|\mathcal{H}_i\big(\mathcal{I}_h\big(\theta_{m,i}(E_i u)\big)\big)\big|_{H_h^1(\Omega_i)}^2 \\ &\preceq \big(1 + \log(H_i/h_i)\big)^2 |\mathcal{H}_i u|_{H_h^1(\Omega_i)}^2 \\ &\preceq \big(1 + \log(H_i/h_i)\big)^2 |u|_{H_h^1(\Omega_i)}^2. \end{aligned} \tag{5.12}$$

From (5.10)-(5.12), we complete the proof of (5.8).

Using similar techniques to those in (5.8), and summing over all subdomains, we can complete the proof of (5.9). □

6 Numerical results

In this section, we show numerical results of our method using the model problem

$$\begin{cases} -\operatorname{div}(\rho \nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = [0,1]^2$. The domain is composed of $M \times M$ sub-squares, their mesh sizes are H, and the sub-squares are divided into smaller ones with mesh sizes h_m in mortar subdomains; and h_n in nonmortar subdomains. The coefficient ρ is either 1 or 10^k ($k = 2, 4, 6$).

We use the preconditioned conjugate gradient (PCG) method with zero initial guess for the discrete system of equations. The stopping criterion for the PCG method is when the 2-norm of the residual is reduced by the factor of 10^{-6} of the initial guess. An estimate for the condition number of the corresponding system is computed by using the Lanczos algorithm.

In Table 1, we show the number of iterations and the condition numbers with different ratio H/h_n. In Figure 2, we plot the condition number as the function $(1 + \log(H/h))^2$ for

Table 1 The number of iterations and condition numbers for $h_m/h_n = 2/3$

$M \times M$	$H/h_n = 4$			$H/h_n = 16$		
	$k = 2$	$k = 4$	$k = 6$	$k = 2$	$k = 4$	$k = 6$
4×4	10 (3.30)	10 (3.30)	10 (3.30)	12 (5.15)	12 (5.15)	12 (5.15)
8×8	11 (3.30)	11 (3.34)	11 (3.32)	12 (5.30)	13 (5.34)	13 (5.34)
16×16	11 (3.24)	12 (3.21)	13 (3.21)	13 (5.32)	13 (5.39)	14 (5.39)
32×32	11 (3.24)	12 (3.24)	12 (3.24)	15 (5.37)	15 (5.41)	15 (5.41)

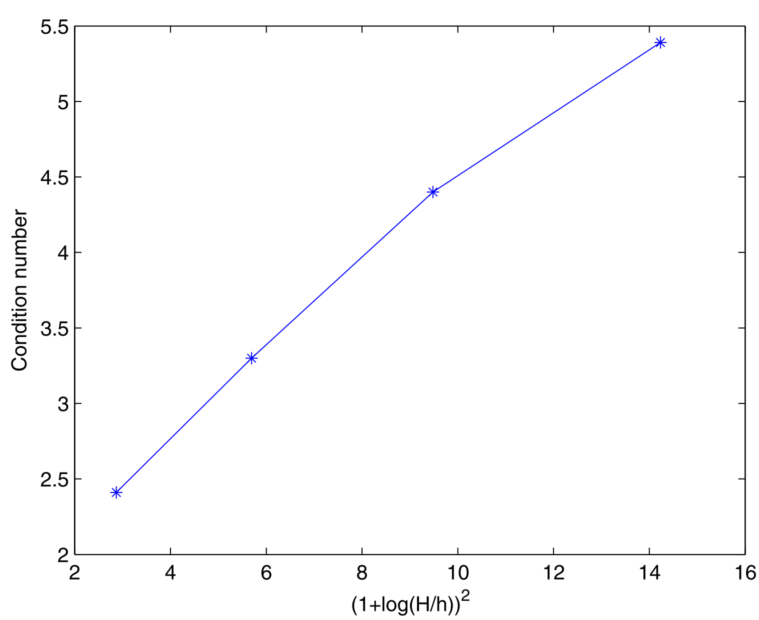


Figure 2 Plot of the condition numbers as the function of $(1 + \log(H/h))^2$.

16 domains. From the results in Table 1 and Figure 2, we see that the convergence of our method is quasi-optimal since the number of iterations is independent of the jumps of the coefficients, and almost independent of the mesh size.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All results belong to YJ and JC. All authors read and approved the final manuscript.

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