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Asymptotic profile of solutions to the semilinear beam equation

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Abstract

In this paper, we investigate the initial value problem for the semilinear beam equation. Under a small condition on the initial value, we prove the global existence and optimal decay estimate of solutions. Moreover, we show that as time tends to infinity, the solution is asymptotic to a diffusion wave, which is given explicitly in terms of the solution of parabolic equation.

MSC: 35L30; 35L75

Keywords: beam equation; decay estimate; asymptotic profile; diffusion wave

1 Introduction

We investigate the initial value problem for the following semilinear beam equation:

$$u_{tt} - \alpha u_{xx} + u_t + u_{xxxx} = f(u_x)_x \tag{1.1}$$

with the initial value

$$t = 0: \quad u = u_0(x), \qquad u_t = u_1(x). \tag{1.2}$$

Here $u = u(x, t)$ is the unknown function of $x \in \mathbb{R}$ and $t > 0$, $b > 0$, and $\alpha > 0$ are constants. The nonlinear term $f(v)$ is a given smooth function of $v \in \mathbb{R}$. More precisely,

$$f(v) = f(0) + f'(0)v + g(v),$$

where $g(v) = O(v^2)$ for $v \to 0$.

This initial value problem was studied by [1, 2] when f satisfies

$$\begin{cases} |f(v)| \leq C|v|^p, \qquad |f'(v)| \leq C|v|^{p-1}, \\ |f(v_1) - f(v_2)| \leq C(|v_1|^{p-1} + |v_2|^{p-1})|v_1 - v_2|, \\ |f'(v_1) - f'(v_2)| \leq C(|v_1|^{p-2} + |v_2|^{p-2})|v_1 - v_2| \end{cases} \tag{1.3}$$

with $p \geq 2$, so that $f'(0) = 0$. Here C is independent of v, v_1, and v_2. [1] proved that there exists a global solution $u \in C([0, \infty); H^2 \cap L^1)$ to the problem (1.1), (1.2) under smallness condition on the initial data $u_0 \in W^{2,1} \cap H^2$ and $u_1 \in L^2 \cap L^1$. In particular, they showed the decay estimates:

$$\|u\|_{L^q} \leq C(1 + t)^{-\frac{1}{2}(1 - \frac{1}{q})}, \quad 1 \leq q \leq \infty \tag{1.4}$$

and

$$\left\|\partial_x^2 u\right\|_{L^q} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{q})-1}, \quad 2 \le q \le \infty. \tag{1.5}$$

In addition to the above assumptions, suppose that the initial data $u_0, u_1 \in L_1^1$, Takeda and Yoshikawa [2] established the following asymptotic profile of global solution:

$$\left\|u(t) - MG_\alpha(t) - (\tilde{M}+N)\partial_x G_\alpha(t)\right\|_{L^q} \le C(1+t)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{1}{2}}, \quad 1 \le q \le \infty, \tag{1.6}$$

where

$$M = \int_{-\infty}^{\infty} (u_0 + u_1)(x)\, dx, \qquad G_\alpha(x,t) = F^{-1}\big[e^{-\alpha\xi^2 t}\big], \qquad \tilde{M} = -\int_{-\infty}^{\infty} x(u_0 + u_1)(x)\, dx$$

and

$$N = \int_0^{\infty} \int_{-\infty}^{\infty} f\big(\partial_x u(x,t)\big)\, dx\, dt.$$

The main purpose of our present paper is two-fold: first, we try to recover all the results about global existence and decay estimate of solution in Takeda and Yoshikawa [1] under some assumptions on the initial data $(u_0(x), u_1(x))$ and the nonlinear function f, which is much weaker than those needed in Takeda and Yoshikawa's arguments. More precisely, the condition $u_0(x) \in W^{2,1} \cap H^2$, $u_1(x) \in L^2 \cap L^1$ and (1.3) have been relaxed to $u_0(x) \in H^{s+2} \cap L^1$, $u_1(x) \in H^s \cap L^1$ ($s \ge 0$) and the nonlinear function f satisfies $f'(0) > -\alpha$ in this paper. Second, we show that the solution is asymptotic to a diffusion wave, given explicitly in terms of the solution of parabolic equation that is different from the one in [2]. For the details, we refer to Theorem 4.1. Moreover, under some additional assumptions on the initial data, we also prove that the convergence rates of our new asymptotic profile are better than that obtained by [2]. For details, we refer to Theorem 4.2.

The study of the global existence and asymptotic behavior of solutions to hyperbolic-type equations has a long history. We refer to [3, 4] for hyperbolic equations, [5–7] for the damped wave equation and [8–16] for various aspects of dissipation of the plate equation.

The paper is organized as follows. In Section 2, we study the decay property of the solution to the linear problem. Then, in Section 3, we prove the global existence and decay estimate of the solutions. Finally, we prove that the solution is asymptotic to a diffusion wave, which is given explicitly in terms of the solution of the parabolic equation in Section 4.

Notations We give some notations which are used in this paper. Let $\mathcal{F}[u]$ denote the Fourier transform of u defined by

$$\hat{u}(\xi) = \mathcal{F}[u] = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx,$$

and we denote its inverse transform by $\mathcal{F}^{-1}$.

For $1 \le p \le \infty$, $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. The usual Sobolev space of s is defined by $W^{s,p} = (I - \partial_x^2)^{-\frac{s}{2}} L^p$ with the norm $\|f\|_{W^{s,p}} = \|(I-\partial_x^2)^{\frac{s}{2}} f\|_{L^p}$.

Finally, in this paper, we denote every positive constant by the same symbol C or c which will not lead to any confusion. $[\cdot]$ is the Gauss symbol.

2 Linear problem

2.1 Solution formula

The aim of this subsection is to derive the solution formula for the problem (1.1), (1.2). We first investigate the linearized equation of (1.1):

$$u_{tt} - \beta u_{xx} + u_t + u_{xxxx} = 0, \tag{2.1}$$

with the initial data in (1.2), where $\beta = \alpha + f'(0) > 0$. We apply the Fourier transform to (2.1). This yields

$$\hat{u}_{tt} + \xi^2 \hat{u}_t + (\beta \xi^2 + \xi^4)\hat{u} = 0. \tag{2.2}$$

The corresponding initial values are given as

$$t = 0: \quad \hat{u} = \hat{u}_0(\xi), \qquad \hat{u}_t = \hat{u}_1(\xi). \tag{2.3}$$

The characteristic equation of (2.2) is

$$\lambda^2 + \lambda + \beta \xi^2 + \xi^4 = 0. \tag{2.4}$$

Let $\lambda = \lambda_\pm(\xi)$ be the corresponding eigenvalues of (2.4), we obtain

$$\lambda_\pm(\xi) = \frac{-1 \pm \sqrt{1 - 4\xi^2(\beta + \xi^2)}}{2}. \tag{2.5}$$

The solution to the problem (2.2), (2.3) in the Fourier space is then given explicitly in the form

$$\hat{u}(\xi, t) = \hat{G}(\xi, t)(\hat{u}_1(\xi) + \hat{u}_0(\xi)) + \hat{H}(\xi, t)\hat{u}_0(\xi), \tag{2.6}$$

where

$$\hat{G}(\xi, t) = \frac{1}{\lambda_+(\xi) - \lambda_-(\xi)} (e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}) \tag{2.7}$$

and

$$\hat{H}(\xi, t) = \frac{1}{\lambda_+(\xi) - \lambda_-(\xi)} ((\lambda_+(\xi) + 1)e^{\lambda_-(\xi)t} - (\lambda_-(\xi) + 1)e^{\lambda_+(\xi)t}). \tag{2.8}$$

We define $G(x, t)$ and $H(x, t)$ by

$$G(x, t) = \mathcal{F}^{-1}[\hat{G}(\xi, t)](x) \tag{2.9}$$

and

$$H(x, t) = \mathcal{F}^{-1}[\hat{H}(\xi, t)](x), \tag{2.10}$$

respectively, where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. Then, applying $\mathcal{F}^{-1}$ to (2.6), we obtain

$$u(t) = G(t) * (u_1 + u_0) + H(t) * u_0. \tag{2.11}$$

By the Duhamel principle, we obtain the solution formula to (1.1), (1.2),

$$u(t) = G(t) * (u_1 + u_0) + H(t) * u_0 + \int_0^t G(t - \tau) * g(u_x)_x(\tau)\, d\tau, \tag{2.12}$$

where $g(v)$ is a smooth function; it satisfies $g(v) = O(v^2)$.

2.2 Decay property

The aim of this subsection is to establish decay estimates of the solution operators $G(t)$ and $H(t)$ appearing in (2.9) and (2.10), respectively.

Lemma 2.1 *The solution of the problem* (2.2), (2.3) *satisfies*

$$\left|\hat{u}_t(\xi, t)\right|^2 + \left(1 + \xi^2\right)^2 \left|\hat{u}(\xi, t)\right|^2 \le Ce^{-c\rho(\xi)t}\left(\left|\hat{u}_1(\xi)\right|^2 + \left(1 + \xi^2\right)^2 \left|\hat{u}_0(\xi)\right|^2\right) \tag{2.13}$$

for $\xi \in \mathbb{R}$ and $t \ge 0$, where $\rho(\xi) = \frac{\xi^2}{1 + \xi^2}$.

Proof Multiplying (2.2) by $\bar{\hat{u}}_t$ and taking the real part yields

$$\frac{1}{2}\frac{d}{dt}\left\{|\hat{u}_t|^2 + \beta\xi^2|\hat{u}|^2 + \xi^4|\hat{u}|^2\right\} + |\hat{u}_t|^2 = 0. \tag{2.14}$$

Multiplying (2.2) by $\bar{\hat{u}}$ and taking the real part, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\{|\hat{u}|^2 + 2\operatorname{Re}(\hat{u}_t\bar{\hat{u}})\right\} + \beta\xi^2|\hat{u}|^2 + \xi^4|\hat{u}|^2 - |\hat{u}_t|^2 = 0. \tag{2.15}$$

Multiplying both sides of (2.14) by 2 and summing up the resulting equation and (2.15) yields

$$\frac{d}{dt}E + F = 0, \tag{2.16}$$

where

$$E = |\hat{u}_t|^2 + \left(\frac{1}{2} + \beta\xi^2 + \xi^4\right)|\hat{u}|^2 + \operatorname{Re}(\hat{u}_t\bar{\hat{u}})$$

and

$$F = |\hat{u}_t|^2 + \beta\xi^2|\hat{u}|^2 + \xi^4|\hat{u}|^2.$$

A simple computation implies that

$$CE_0 \le E \le CE_0, \tag{2.17}$$

where

$$E_0 = |\hat{u}_t|^2 + (1 + \xi^2)^2 |\hat{u}|^2.$$

Note that

$$F \geq c\rho(\xi)E_0.$$

It follows from (2.17) that

$$F \geq c\rho(\xi)E. \tag{2.18}$$

Using (2.16) and (2.18), we get

$$\frac{d}{dt}E + c\rho(\xi)E \leq 0.$$

Thus

$$E(\xi, t) \leq e^{-c\rho(\xi)t}E(\xi, 0),$$

which together with (2.17) proves the desired estimates (2.13). Then we have completed the proof of the lemma. □

Lemma 2.2 *Let $\hat{G}(\xi, t)$ and $\hat{H}(\xi, t)$ be the fundamental solution of* (2.1) *in the Fourier space, which are given in* (2.7) *and* (2.8), *respectively. Then we have the estimates*

$$\left|\hat{G}_t(\xi, t)\right|^2 + (1 + \xi^2)^2 \left|\hat{G}(\xi, t)\right|^2 \leq Ce^{-c\rho(\xi)t} \tag{2.19}$$

and

$$\left|\hat{H}_t(\xi, t)\right|^2 + (1 + \xi^2)^2 \left|\hat{H}(\xi, t)\right|^2 \leq C(1 + \xi^2)^2 e^{-c\rho(\xi)t} \tag{2.20}$$

for $\xi \in \mathbb{R}$ and $t \geq 0$, where $\rho(\xi) = \frac{\xi^2}{1+\xi^2}$.

Proof Firstly, we investigate the problem (2.1), (1.2) with $\hat{u}_0(\xi) = 0$, from (2.6), we obtain

$$\hat{u}(\xi, t) = \hat{G}(\xi, t)\hat{u}_1(\xi), \qquad \hat{u}_t(\xi, t) = \hat{G}_t(\xi, t)\hat{u}_1(\xi).$$

Substituting the equalities into (3.1) with $\hat{u}_0(\xi) = 0$, we get (2.19).

In what follows, we consider the problem (2.1), (1.2) with $\hat{u}_1(\xi) = 0$; it follows from (2.6) that

$$\hat{u}(\xi, t) = (\hat{G} + \hat{H})(\xi, t)\hat{u}_0(\xi), \qquad \hat{u}_t(\xi, t) = (\hat{G}_t + \hat{H}_t)(\xi, t)\hat{u}_0(\xi).$$

Substituting the equalities into (2.13) with $\hat{u}_1(\xi) = 0$, we get the desired estimate (2.20). The lemma is proved. □

Let

$$G_0(x,t) = F^{-1}\big[e^{-\beta\xi^2 t}\big](x) = \frac{1}{\sqrt{4\pi\beta t}}e^{e^{-\frac{x^2}{4\beta t}}}$$

be the fundamental solution to $u_t - \beta u_{xx} = 0$.

Lemma 2.3 *Let $\hat{G}(\xi,t)$ and $\hat{H}(\xi,t)$ be the fundamental solution of (2.1) in the Fourier space, which are given in (2.6) and (2.7), respectively. Then there is a small positive number R_0 such that if $|\xi| \le R_0$ and $t \ge 0$, we have the following estimates*:

$$\big|(\hat{G} - \hat{G}_0)(\xi,t)\big| + \big|\hat{H}(\xi,t)\big| \le C\xi^2 e^{-c\xi^2 t} + Ce^{-ct}, \tag{2.21}$$

$$\big|\hat{G}_t(\xi,t)\big| \le C\xi^2 e^{-c\xi^2 t} + Ce^{-ct} \tag{2.22}$$

and

$$\big|\partial_t(\hat{G} - \hat{G}_0)(\xi,t)\big| + \big|\hat{H}_t(\xi,t)\big| \le C\xi^4 e^{-c\xi^2 t} + Ce^{-ct}. \tag{2.23}$$

Proof For sufficiently small ξ, using the Taylor formula, we get

$$\lambda_+(\xi) = -\beta\xi^2 + O\big(\xi^4\big), \qquad \lambda_-(\xi) = -1 + O\big(\xi^2\big),$$

$$\frac{1}{\lambda_+(\xi) - \lambda_-(\xi)} = 1 + 2\beta\xi^2 + O\big(\xi^4\big). \tag{2.24}$$

We rewrite $\hat{G}(\xi,t)$ in (2.6) as

$$\hat{G}(\xi,t) = e^{-\beta\xi^2 t} + \left(\frac{1}{\lambda_+(\xi) - \lambda_-(\xi)} - 1\right)e^{\lambda_+(\xi)t}$$

$$+ \big(e^{(\lambda_+(\xi)+\beta\xi^2)t} - 1\big)e^{-\beta\xi^2 t} - \frac{e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)}. \tag{2.25}$$

For sufficiently small ξ, from (2.24) and (2.25), we immediately obtain

$$\big|(\hat{G} - \hat{G}_0)(\xi,t)\big| \le C\xi^2 e^{-c\xi^2 t} + Ce^{-ct}.$$

For sufficiently small ξ, from (2.7) and (2.24), we immediately get

$$\big|\hat{H}(\xi,t)\big| \le C\xi^2 e^{-c\xi^2 t} + Ce^{-ct}.$$

Thus we get (2.21). The other estimates are proved similarly and we omit the details. The proof of Lemma 2.3 is completed. □

Lemma 2.4 *Let $G(x,t)$ and $H(x,t)$ be the fundamental solutions of (2.1), which are given in (2.9) and (2.10), respectively. Let $1 \le p \le 2$, and let k, j, and l be nonnegative integers. Then we have*

$$\big\|\partial_x^k G(t) * \phi\big\|_{L^2} \le C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k-j}{2}}\big\|\partial_x^j \phi\big\|_{L^p} + Ce^{-ct}\big\|\partial_x^{k+l-2}\phi\big\|_{L^2}, \tag{2.26}$$

$$\left\| \partial_x^k H(t) * \phi \right\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+2-j}{2}} \left\| \partial_x^j \phi \right\|_{L^p} + Ce^{-ct} \left\| \partial_x^{k+l} \phi \right\|_{L^2} \tag{2.27}$$

for $0 \leq j \leq k$, where $k + l - 2 \geq 0$ in (2.26). Similarly, we have

$$\left\| \partial_x^k G_t(t) * \phi \right\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+2-j}{2}} \left\| \partial_x^j \phi \right\|_{L^p} + Ce^{-ct} \left\| \partial_x^{k+l} \phi \right\|_{L^2}, \tag{2.28}$$

$$\left\| \partial_x^k H_t(t) * \phi \right\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+4-j}{2}} \left\| \partial_x^j \phi \right\|_{L^p} + Ce^{-ct} \left\| \partial_x^{k+l+2} \phi \right\|_{L^2} \tag{2.29}$$

for $0 \leq j \leq k + 2$ in (2.28) and $0 \leq j \leq k + 4$ in (2.29).

Proof We only prove (2.26). By the Plancherel theorem and (2.19), the Hausdorff-Young inequality, we obtain

$$\left\| \partial_x^k G(t) * \phi \right\|_{L^2}^2 = \int_{|\xi| \leq 1} |\xi|^{2k} |\hat{G}(\xi, t)|^2 |\hat{\phi}(\xi)|^2 \, d\xi + \int_{|\xi| \geq 1} |\xi|^{2k} |\hat{G}(\xi, t)|^2 |\hat{\phi}(\xi)|^2 \, d\xi$$

$$\leq C \int_{|\xi| \leq 1} |\xi|^{2k} e^{-c\xi^2 t} |\hat{\phi}(\xi)|^2 \, d\xi + Ce^{-ct} \int_{|\xi| \geq 1} |\xi|^{2k} (1 + |\xi|^2)^{-2} |\hat{\phi}(\xi)|^2 \, d\xi$$

$$=: \mathcal{A}_1 + \mathcal{A}_2.$$

For the term $\mathcal{A}_1$, letting $\frac{1}{p'} + \frac{1}{p} = 1$, we have

$$\mathcal{A}_1 \leq C \int_{|\xi| \leq 1} |\xi|^{2k-2} e^{-c\xi^2 t} |\hat{\phi}(\xi)|^2 \, d\xi$$

$$\leq C \left\| |\xi|^j \hat{\phi} \right\|_{L^{p'}}^2 \left(\int_{|\xi| \leq 1} |\xi|^{2(k-j)p} e^{-cq\xi^2 t} \, d\xi \right)^{\frac{1}{p}}$$

$$\leq C(1+t)^{-(\frac{1}{p}-\frac{1}{2})-(k-j)} \left\| \partial_x^j \phi \right\|_{L^p}^2,$$

where we used the Hölder inequality with $\frac{2}{p'} + \frac{1}{q} = 1$ and the Hausdorff-Young inequality $\|\hat{v}\|_{L^{p'}} \leq C \|v\|_{L^p}$ for $v = \partial_x^j \phi$. On the other hand, we can estimate the term $\mathcal{A}_2$ simply as

$$\mathcal{A}_2 \leq Ce^{-ct} \int_{|\xi| \geq 1} |\xi|^{2k-4} |\hat{\phi}(\xi)|^2 \, d\xi$$

$$\leq Ce^{-ct} \int_{|\xi| \geq 1} |\xi|^{2(k+l-2)} |\hat{\phi}(\xi)|^2 \, d\xi$$

$$\leq Ce^{-ct} \left\| \partial_x^{k+l-2} \phi \right\|_{L^2}^2,$$

where $k + l - 2 \geq 0$.

Combining the above three inequalities yields (2.26). This completes the proof of Lemma 2.4. □

From Lemma 2.4, we immediately have the following corollary.

Corollary 2.1 *Let $G(x, t)$ and $H(x, t)$ be the fundamental solution of* (2.1), *which are given in* (2.8) *and* (2.9), *respectively. Let $1 \leq p \leq 2$, and let k, j, and l be nonnegative integers. Then we have*

$$\left\| \partial_x^k G(t) * g_x \right\|_{L^2} \leq C(1+t)^{-\frac{1}{2}(\frac{1}{p}-\frac{1}{2})-\frac{k+1-j}{2}} \left\| \partial_x^j g \right\|_{L^p} + Ce^{-ct} \left\| \partial_x^{k+l-1} g \right\|_{L^2} \tag{2.30}$$

for $0 \leq j \leq k + 1$ and $k + l - 1 \geq 0$. Also we have

$$\left\| \partial_x^k G_t(t) * g_x \right\|_{L^2} \leq C(1 + t)^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+3-j}{2}} \left\| \partial_x^j g \right\|_{L^p} + Ce^{-ct} \left\| \partial_x^{k+l+1} g \right\|_{L^2} \tag{2.31}$$

for $0 \leq j \leq k + 3$.

Lemma 2.5 *Let $G(x, t)$ be the fundamental solution of (2.1), given in (2.8) and let $G_0(x, t)$ be the fundamental solution of (2.1), given in (2.8). Let $1 \leq p \leq 2$, and let k, j, and l be nonnegative integers. Then we have*

$$\left\| \partial_x^k (G - G_0)(t) * \phi \right\|_{L^2} \leq C(1 + t)^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+2-j}{2}} \left\| \partial_x^j \phi \right\|_{L^p} + Ce^{-ct} \left\| \partial_x^{k+l} \phi \right\|_{L^2} \tag{2.32}$$

for $0 \leq j \leq k + 2$. Similarly,

$$\left\| \partial_x^k (G - G_0)_t(t) * \phi \right\|_{L^2} \leq C(1 + t)^{-\frac{1}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+4-j}{2}} \left\| \partial_x^j \phi \right\|_{L^p} + Ce^{-ct} \left\| \partial_x^{k+l+2} \phi \right\|_{L^2} \tag{2.33}$$

for $0 \leq j \leq k + 4$.

Proof The proof of Lemma 2.5 is similar to the proof of Lemma 2.4. By employing (2.21) and (2.23), we can prove Lemma 2.5. We omit the details. □

3 Global existence and asymptotic behavior of solutions to (1.1), (1.2)

The purpose of this section is to prove global existence and optimal decay estimate of solutions to the initial value problem (1.1), (1.2). We need the following lemma, which comes from [17] (see also [18]).

Lemma 3.1 *Assume that $f = f(v)$ is a smooth function. Suppose that $f(v) = O(|v|^{1+\theta})$ ($\theta \geq 1$ is an integer) when $|v| \leq v_0$. Then for integer $m \geq 0$, if $v \in W^{m,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\|v\|_{L^\infty} \leq v_0$, the following inequalities hold:*

$$\left\| \partial_x^m f(v) \right\|_{L^r} \leq C \|v\|_{L^p} \left\| \partial_x^m v \right\|_{L^q} \|v\|_{L^\infty}^{\theta-1}, \tag{3.1}$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $1 \leq p, q, r \leq +\infty$.

Theorem 3.1 *Let $s \geq 0$. Suppose that $f(v)$ is a smooth function and $f'(0) > -\alpha$. Assume that $u_0 \in H^{s+2} \cap L^1$, $u_1 \in H^s \cap L^1$. Put*

$$E_0 := \|u_0\|_{H^{s+2} \cap L^1} + \|u_1\|_{H^s \cap L^1}.$$

Then there exists a positive constant δ_1 such that if $E_0 \leq \delta_0$, and the initial value problem (1.1), (1.2) has a unique global solution $u(x, t)$ satisfying

$$u \in C^0\big([0, \infty); H^{s+2}(\mathbb{R})\big) \cap C^1\big([0, \infty); H^s(\mathbb{R})\big).$$

Moreover, the solution satisfies the decay estimate

$$\left\| \partial_x^k u(t) \right\|_{L^2} \leq CE_0(1 + t)^{-\frac{1}{4} - \frac{k}{2}} \tag{3.2}$$

and

$$\left\|\partial_x^k u_t(t)\right\|_{L^2} \le CE_0(1+t)^{-\frac{5}{4}-\frac{k}{2}}, \tag{3.3}$$

where $0 \le k \le s + 2$ in (3.2) and $0 \le k \le s$ in (3.3).

Proof The existence and uniqueness of small solutions can be proved by the contraction mapping principle. Here we only show the decay estimates (3.2) and (3.3) for the solution u of (2.12) satisfying $\|u(t)\|_{L^\infty} \le M_0$ with some M_0. To this end, we introduce the quantity

$$X(t) = \sum_{k=0}^{s+2} \sup_{0 \le \tau \le t} (1+\tau)^{\frac{1}{4}+\frac{k}{2}} \left\|\partial_x^k u(\tau)\right\|_{L^2}. \tag{3.4}$$

Here we note that

$$\left\|u_x(t)\right\|_{L^\infty} \le CX(t)(1+t)^{-1}, \tag{3.5}$$

provided that $s \ge 0$. This follows from the Gagliardo-Nirenberg inequality, $\|u_x\|_{L^\infty} \le C\|u_{xx}\|_{L^2}^{\frac{1}{2}}\|u_x\|_{L^2}^{\frac{1}{2}}$ and the definition of $X(t)$ in (3.4).

Applying ∂_x^k to (2.12) and taking the L^2 norm, we obtain

$$\left\|\partial_x^k u(t)\right\|_{L^2} \le \left\|\partial_x^k G(t) * (u_0 + u_1)\right\|_{L^2} + \left\|\partial_x^k H(t) * u_0\right\|_{L^2}$$

$$+ \int_0^t \left\|G(t-\tau) * \partial_x g(u_x(\tau))\right\|_{L^2} d\tau$$

$$=: I_1 + I_2 + J. \tag{3.6}$$

Firstly, we estimate I_1. We apply (2.26) with $p = 1$, $j = 0$ and $l = 0$ ($l = 2$ for $k = 0$, $l = 1$ for $k = 1$). This yields

$$I_1 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u_0 + u_1\|_{L^1} + Ce^{-ct}\left\|\partial_x^{(k-2)_+}(u_0 + u_1)\right\|_{L^2} \le CE_0(1+t)^{-\frac{1}{4}-\frac{k}{2}}, \tag{3.7}$$

where $(k-2)_+ = \max\{k-2, 0\}$. Similarly, applying (2.27) with $p = 1$, $j = 0$, and $l = 0$ to the term I_2, we have

$$I_2 \le C(1+t)^{-\frac{1}{4}-\frac{k}{2}}\|u_0\|_{L^1} + Ce^{-ct}\left\|\partial_x^k u_0\right\|_{L^2} \le CE_0(1+t)^{-\frac{1}{4}-\frac{k}{2}}. \tag{3.8}$$

We estimate the nonlinear term J. We divide J into two parts and write $J = J_1 + J_2$, where J_1 and J_2 are corresponding to the time intervals $[0, t/2]$ and $[t/2, t]$, respectively. For the term J_1, we apply (2.30) with $p = 1$, $j = 0$, and $l = 0$. This yields

$$J_1 \le C \int_0^{t/2} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}} \left\|g(u_x)(\tau)\right\|_{L^1} d\tau$$

$$+ C \int_0^{t/2} e^{-c(t-\tau)} \left\|\partial_x^{(k-1)_+} g(u_x)(\tau)\right\|_{L^2} d\tau$$

$$=: J_{11} + J_{12}. \tag{3.9}$$

Here we see that $\|g(u_x)\|_{L^1} \leq C\|u_x\|_{L^2}^2$ by Lemma 3.1. Thus we have $\|g(u_x)(\tau)\|_{L^1} \leq CX(t)^2(1+\tau)^{-\frac{3}{2}}$. Therefore we can estimate the term J_{11} as

$$J_{11} \leq CX(t)^2 \int_0^{t/2} (1+t-\tau)^{-\frac{3}{4}-\frac{k}{2}}(1+\tau)^{-\frac{3}{2}}\,d\tau$$

$$\leq CX(t)^2(1+t)^{-\frac{3}{4}-\frac{k}{2}} \int_0^{t/2} (1+\tau)^{-\frac{3}{2}}\,d\tau$$

$$\leq CX(t)^2(1+t)^{-\frac{3}{4}-\frac{k}{2}}.$$

On the other hand, we have $\|\partial_x^{(k-1)_+}g(u_x)\|_{L^2} \leq C\|u_x\|_{L^\infty}\|\partial_x^k u\|_{L^2}$ by Lemma 3.1. Therefore, using (3.5), we find that $\|\partial_x^{(k-1)_+}g(u_x)\|_{L^2} \leq CX(t)^2(1+\tau)^{-\frac{5}{4}-\frac{k}{2}}$. Consequently, we can estimate the term J_{12} as

$$J_{12} \leq CX(t)^2 \int_0^{t/2} e^{-c(t-\tau)}(1+\tau)^{-\frac{5}{4}-\frac{k}{2}}\,d\tau \leq CX(t)^2 e^{-ct}.$$

Finally, we estimate the term J_2 on the time interval $[t/2,t]$. Applying (2.30) with $p=2$, $j=k$, and $l=0$, and using $\|\partial_x^{(k-1)_+}g(u_x)(\tau)\|_{L^2} \leq CX(t)^2(1+\tau)^{-\frac{5}{4}-\frac{k}{2}}$, we obtain

$$J_2 \leq C \int_{t/2}^t (1+t-\tau)^{-1}\left\|\partial_x^{(k-1)_+}g(u_x)(\tau)\right\|_{L^2}\,d\tau + C \int_{t/2}^t e^{-c(t-\tau)}\left\|\partial_x^{(k-1)_+}g(u_x)(\tau)\right\|_{L^2}\,d\tau$$

$$\leq CX(t)^2 \int_{t/2}^t (1+t-\tau)^{-1}(1+\tau)^{-\frac{5}{4}-\frac{k}{2}}\,d\tau$$

$$\leq CX(t)^2(1+t)^{-\frac{5}{4}-\frac{k}{2}+\varepsilon}. \tag{3.10}$$

Thus we have shown that

$$J \leq CX(t)^2(1+t)^{-\frac{3}{4}-\frac{k}{2}}.$$

Substituting all these estimates into (3.6), we obtain

$$(1+t)^{\frac{1}{4}+\frac{k}{2}}\left\|\partial_x^k u(t)\right\|_{L^2} \leq CE_0 + CX(t)^2$$

for $0 \leq k \leq s+2$. Consequently, we have $X(t) \leq CE_0 + CX(t)^2$, from which we can deduce $X(t) \leq CE_0$, provided that E_0 is suitably small. This proves the decay estimate (3.2).

In what follows, we prove the decay estimate (3.3) for the time derivative u_t. For this purpose we differentiate (2.12) with respect to t to obtain

$$u_t(t) = G_t(t) * (u_0 + u_1) + H_t(t) * u_0 + \int_0^t G_t(t-\tau) * \partial_x g(u_x)(\tau)\,d\tau. \tag{3.11}$$

Applying ∂_x^k to (3.11) and taking the L^2 norm, we have

$$\left\|\partial_x^k u_t(t)\right\|_{L^2} \leq \left\|\partial_x^k G_t(t) * (u_0 + u_1)\right\|_{L^2} + \left\|\partial_x^k H_t(t) * u_0\right\|_{L^2}$$

$$+ \int_0^t \left\|\partial_x^k G_t(t-\tau) * \partial_x g(u_x)(\tau)\right\|_{L^2}\,d\tau$$

$$=: I_1' + I_2' + J', \tag{3.12}$$

where $0 \leq k \leq s$. For the term I_1', we apply (2.28) with $p = 1$, $j = 0$, and $l = 0$ to get

$$I_1' \leq C(1 + t)^{-\frac{5}{4} - \frac{k}{2}} \|u_0 + u_1\|_{L^1} + e^{-ct} \|\partial_x^k (u_1 + u_0)\|_{L^2} \leq CE_0 (1 + t)^{-\frac{5}{4} - \frac{k}{2}}.$$

Also, for the term I_2', applying (2.29) with $p = 1$, $j = 0$ and $l = 0$, we have

$$I_2' \leq C(1 + t)^{-\frac{9}{4} - \frac{k}{2}} \|u_0\|_{L^1} + e^{-ct} \|\partial_x^{k+2} u_0\|_{L^2} \leq CE_0 (1 + t)^{-\frac{9}{4} - \frac{k}{2}}.$$

To estimate the nonlinear term J', we rewrite $J' = J_1' + J_2'$, where J_1' and J_2' correspond to the time intervals $[0, t/2]$ and $[t/2, t]$, respectively. For the term J_1', we apply (2.31) with $p = 1$, $j = 0$, and $l = 0$. This yields

$$
\begin{aligned}
J_1' &\leq C \int_0^{t/2} (1 + t - \tau)^{-\frac{7}{4} - \frac{k}{2}} \|g(u_x)(\tau)\|_{L^1} \, d\tau \\
&\quad + C \int_0^{t/2} e^{-c(t-\tau)} \|\partial_x^{k+1} g(u_x)(\tau)\|_{L^2} \, d\tau \\
&=: J_{11}' + J_{12}'.
\end{aligned}
$$

Since $\|g(u_x)(\tau)\|_{L^1} \leq CX(t)^2 (1 + \tau)^{-\frac{3}{2}}$ as before, we can estimate the term J_{11}' as

$$
\begin{aligned}
J_{11}' &\leq CX(t)^2 \int_0^{t/2} (1 + t - \tau)^{-\frac{7}{4} - \frac{k}{2}} (1 + \tau)^{-\frac{3}{2}} \, d\tau \\
&\leq CX(t)^2 (1 + t)^{-\frac{7}{4} - \frac{k}{2}}.
\end{aligned}
$$

Also, the term J_{12}' is estimated similarly as before and we have $J_{12}' \leq CX(t)^2 e^{-ct}$. Finally, we estimate the term J_2' by applying (2.31) with $p = 2$, $j = k + 2$, and $l = 0$ and obtain

$$
\begin{aligned}
J_2' &\leq C \int_{t/2}^t (1 + t - \tau)^{-1} \|\partial_x^{k+1} g(u_x)(\tau)\|_{L^2} \, d\tau + C \int_{t/2}^t e^{-c(t-\tau)} \|\partial_x^{k+1} g(u_x)(\tau)\|_{L^2} \, d\tau \\
&\leq CX(t)^2 \int_{t/2}^t (1 + t - \tau)^{-1} (1 + \tau)^{-\frac{9}{4} - \frac{k}{2}} \, d\tau \\
&\leq CX(t)^2 (1 + t)^{-\frac{9}{4} - \frac{k}{2} + \varepsilon},
\end{aligned}
$$

where we used the estimate $\|\partial_x^{k+1} g(u_x)(\tau)\|_{L^2} \leq CX(t)^2 (1 + \tau)^{-\frac{9}{4} - \frac{k}{2}}$. Consequently we have shown that

$$J' \leq CX(t)^2 (1 + t)^{-\frac{9}{4} - \frac{k}{2}}.$$

Substituting all these estimates together with the previous estimate $X(t) \leq CE_0$ into (3.12), we arrive at the desired estimate (3.3) for $0 \leq k \leq s$. This completes the proof of Theorem 3.1. □

The above proof of Theorem 3.1 shows that the solution u to the integral equation (2.12) is asymptotic to the linear solution u_L given by the formula $u_L(t) := G(t) * (u_0 + u_1) + H(t) * u_0$ in (2.11) as $t \to \infty$. This result is stated as follows.

Corollary 3.1 *Assume the same conditions of Theorem* 3.1. *Then the solution u of the problem* (1.1), (1.2), *which is constructed in Theorem* 3.1, *can be approximated by the solution u_L to the linearized problem* (2.1), (1.2) *as $t \to \infty$. More precisely, we have the following asymptotic relations*:

$$\left\| \partial_x^k (u - u_L)(t) \right\|_{L^2} \le C E_0^2 (1 + t)^{-\frac{3}{4} - \frac{k}{2}},$$

$$\left\| \partial_x^k (u - u_L)_t(t) \right\|_{L^2} \le C E_0^2 (1 + t)^{-\frac{7}{4} - \frac{k}{2}}$$

for $0 \le k \le s + 2$ and $0 \le k \le s$, respectively, where $u_L(t) := G(t) * (u_0 + u_1) + H(t) * u_0$ is the linear solution.

4 Asymptotic profile

In this section, our aim is to establish an asymptotic profile to our global solution that is constructed in Theorem 4.1. In the previous section, we have shown that the solution u to the problem (1.1), (1.2) can be approximated by the linear solution u_L. In what follows, we shall derive a simpler asymptotic profile of the linear solution u_L.

Let v be the solution to the initial data problem

$$v_t - \beta v_{xx} = 0, \qquad t = 0: \quad v = (u_0 + u_1)(x). \tag{4.1}$$

Then

$$v = G_0(x, t) * (u_0 + u_1)(x) \tag{4.2}$$

gives a asymptotic profile of the linear solution u_L. In fact we have the following.

Lemma 4.1 *Let $s \ge 0$. Assume that $u_0, u_1 \in H^{s+2} \cap L^1$ and put $E_0 = \|(u_0, u_1)\|_{H^{s+2} \cap L^1}$. Let u_L be the linear solution and let v be defined by* (4.2). *Then we have*

$$\left\| \partial_x^k (u_L - v)(t) \right\|_{L^2} \le C E_0 (1 + t)^{-\frac{5}{4} - \frac{k}{2}} \tag{4.3}$$

for $0 \le k \le s + 2$.

Proof Note that $(u_L - v)(t) = (G - G_0)(t) * (u_0 + u_1) + H(t) * u_0$, so for the proof of (4.3), it suffices to show the following estimates:

$$\left\| \partial_x^k (G - G_0)(t) * \phi \right\|_{L^2} \le C(1 + t)^{-\frac{5}{4}} \|\phi\|_{L^1} + C e^{-ct} \left\| \partial_x^k \phi \right\|_{L^2},$$

$$\left\| \partial_x^k H(t) * \phi \right\|_{L^2} \le C(1 + t)^{-\frac{5}{4}} \|\phi\|_{L^1} + C e^{-ct} \left\| \partial_x^k \phi \right\|_{L^2},$$

where $0 \le k \le s + 2$. These estimates can be obtained by (2.27) and (2.32). Here we omit the details. □

When $M_0 = \int_{\mathbb{R}} (u_0 + u_1)(x) \, dx \ne 0$, we call

$$\bar{v}(x, t) = M_0 G_0(x, t + 1) \tag{4.4}$$

the diffusion wave with the amount M_0. Obviously, $\bar{v}(x, t)$ satisfies the following problem:

$$
\begin{cases}
\bar{v}_t - \beta \bar{v}_{xx} = 0, \\
t = 0: \quad \bar{v} = M_0 G_0(x, 1).
\end{cases}
\tag{4.5}
$$

Therefore, $v - \bar{v}$ satisfies

$$
\begin{cases}
(v - \bar{v})_t - \beta(v - \bar{v})_{xx} = 0, \\
t = 0: \quad v - \bar{v} = u_0 + u_1 - M_0 G_0(x, 1).
\end{cases}
\tag{4.6}
$$

It is not difficult to prove the following lemma.

Lemma 4.2 *Assume that $\int_{-\infty}^{x}(u_0(y) + u_1(y) - M_0 G_0(y, 1))\, dy \in L^1$, then*

$$
\left\| \partial_x^k (v - \bar{v})(t) \right\|_{L^2} \leq C(1 + t)^{-\frac{3}{4} - \frac{k}{2}}.
\tag{4.7}
$$

Combining Corollary 3.1, Lemma 4.1, and Lemma 4.2, we immediately have the following.

Theorem 4.1 *Under the same assumption as Theorem* 3.1, *and also assuming that $M_0 = \int_{\mathbb{R}}(u_0 + u_1)(x)\, dx \neq 0$ and $\int_{-\infty}^{x}(u_0(y) + u_1(y) - M_0 G_0(y, 1))\, dy \in L^1$, we let u be the global solution to the problem* (1.1), (1.2), *which is constructed in Theorem* 3.1 *and we let $\bar{v}$ be the diffusion wave defined by* (4.4). *Then we have*

$$
\left\| \partial_x^k (u - \bar{v})(t) \right\|_{L^2} \leq C(1 + t)^{-\frac{3}{4} - \frac{k}{2}}.
\tag{4.8}
$$

We have $M_0 = \int_{-\infty}^{\infty}(u_0 + u_1)(x)\, dx = 0$ and $M_1 = \int_{-\infty}^{\infty} U_0(x)\, dx \neq 0$, where $U_0(x) := \int_{-\infty}^{x}(u_0 + u_1)(y)\, dy$. We consider the initial value problem

$$
\begin{cases}
\rho_t - \beta \rho_{xx} = 0, \\
t = 0: \quad \rho = U_0(x)
\end{cases}
\tag{4.9}
$$

and

$$
\begin{cases}
\bar{\rho}_t - \beta \bar{\rho}_{xx} = 0, \\
t = 0: \quad \bar{\rho} = M_1 G_0(x, 1).
\end{cases}
\tag{4.10}
$$

When $\int_{-\infty}^{x}(U_0(y) - M_0 G_0(y, 1))\, dy \in L^1$, by applying Lemma 4.2, we have

$$
\left\| \partial_x^k (\rho - \bar{\rho})(t) \right\|_{L^2} \leq C(1 + t)^{-\frac{3}{4} - \frac{k}{2}}.
\tag{4.11}
$$

We call

$$
\tilde{v}(x, t) = M_1 G_{0x}(x, t + 1)
\tag{4.12}
$$

the diffusion wave with the amount M_1. Obviously, $v - \tilde{v}$ satisfies

$$
v - \tilde{v} = \partial_x \rho - \partial_x \bar{\rho}.
\tag{4.13}
$$

From (4.11) and (4.13), we immediately obtain the following lemma.

Lemma 4.3 *Assume that $\int_{-\infty}^{x}(U_0(y) - M_1 G_0(y,1))\,dy \in L^1$, then*

$$\left\|\partial_x^k(v - \tilde{v})(t)\right\|_{L^2} \le C(1 + t)^{-\frac{5}{4} - \frac{k}{2}}. \tag{4.14}$$

Corollary 3.1, Lemma 4.1, and Lemma 4.3 immediately give the following result.

Theorem 4.2 *Under the same assumption as Theorem* 3.1, *also assuming that $M_0 = \int_{\mathbb{R}}(u_0 + u_1)(x)\,dx = 0$, $M_1 = \int_{\mathbb{R}} U_0(x)\,dx \neq 0$, and $\int_{-\infty}^{x}(U_0(y) - M_1 G_0(y,1))\,dy \in L^1$, where $U_0(x) := \int_{-\infty}^{x}(u_0 + u_1)(y)\,dy$, we let u be the global solution to the problem* (1.1), (1.2), *which is constructed in Theorem* 3.1 *and we let $\tilde{v}$ be the diffusion wave defined by* (4.12). *Then we have*

$$\left\|\partial_x^k(u - \tilde{v})(t)\right\|_{L^2} \le C(1 + t)^{-\frac{5}{4} - \frac{k}{2}}. \tag{4.15}$$

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