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Nonlinear second-order impulsive q-difference Langevin equation with boundary conditions

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Abstract

In this paper, we discuss the existence and uniqueness of solutions for Langevin impulsive q-difference equations with boundary conditions. Our study relies on Banach's and Schaefer's fixed point theorems. Illustrative examples are also presented.

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Keywords: q_k-derivative; q_k-integral; impulsive q_k-difference equation; existence; uniqueness; Langevin equation

1 Introduction and preliminaries

In recent years, the boundary value problems of fractional order differential equations have emerged as an important area of research, since these problems have applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, aerodynamics, viscoelasticity and damping, electro-dynamics of complex medium, wave propagation, blood flow phenomena, *etc.* [1–5]. Many researchers have studied the existence theory for nonlinear fractional differential equations with a variety of boundary conditions, for instance, see the papers [6–18], and the references therein.

The Langevin equation (first formulated by Langevin in 1908) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [19]. For some new developments on the fractional Langevin equation, see, for example, [20–27].

Nowadays there is a significant increase of activities in the area of q-calculus due to its applications in various fields such as mathematics, mechanics, and physics. The book by Kac and Cheung [28] covers many of the fundamental aspects of the quantum calculus. A variety of new results can be found in the papers [29–41] and the references cited therein.

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Recent development in this field has been motivated by many applied problems, such as control theory, population dynamics and medicine. For some recent works on the theory of impulsive differential equations, we refer the interested reader to the monographs [42–44].

Recently in [45] the notions of q_k-derivative and q_k-integral on finite intervals were introduced. Let us recall here these notions. For a fixed $k \in \mathbb{N} \cup \{0\}$ let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and $0 < q_k < 1$ be a constant. We define q_k-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows.

Definition 1.1 Assume $f : J_k \to \mathbb{R}$ is a continuous function and let $t \in J_k$. Then the expression

$$\begin{aligned} D_{q_k} f(t) &= \frac{f(t) - f(q_k t + (1 - q_k)t_k)}{(1 - q_k)(t - t_k)}, \quad t \neq t_k, \\ D_{q_k} f(t_k) &= \lim_{t \to t_k} D_{q_k} f(t), \end{aligned} \tag{1.1}$$

is called the q_k-derivative of function f at t.

We say that f is q_k-differentiable on J_k provided $D_{q_k} f(t)$ exists for all $t \in J_k$. Note that if $t_k = 0$ and $q_k = q$ in (1.1), then $D_{q_k} f = D_q f$, where D_q is the well-known q-derivative of the function $f(t)$ defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}. \tag{1.2}$$

In addition, we should define the higher q_k-derivative of functions.

Definition 1.2 Let $f : J_k \to \mathbb{R}$ be a continuous function, we call the second-order q_k-derivative $D_{q_k}^2 f$ provided $D_{q_k} f$ is q_k-differentiable on J_k with $D_{q_k}^2 f = D_{q_k}(D_{q_k} f) : J_k \to \mathbb{R}$. Similarly, we define higher order q_k-derivative $D_{q_k}^n : J_k \to \mathbb{R}$.

The q_k-integral is defined as follows.

Definition 1.3 Assume $f : J_k \to \mathbb{R}$ is a continuous function. Then the q_k-integral is defined by

$$\int_{t_k}^{t} f(s)\, d_{q_k} s = (1 - q_k)(t - t_k) \sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n t + \bigl(1 - q_k^n\bigr)t_k\bigr) \tag{1.3}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$ then the definite q_k-integral is defined by

$$\begin{aligned} \int_{a}^{t} f(s)\, d_{q_k} s &= \int_{t_k}^{t} f(s)\, d_{q_k} s - \int_{t_k}^{a} f(s)\, d_{q_k} s \\ &= (1 - q_k)(t - t_k) \sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n t + \bigl(1 - q_k^n\bigr)t_k\bigr) \\ &\quad - (1 - q_k)(a - t_k) \sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n a + \bigl(1 - q_k^n\bigr)t_k\bigr). \end{aligned}$$

Note that if $t_k = 0$ and $q_k = q$, then (1.3) reduces to q-integral of a function $f(t)$, defined by $\int_0^t f(s)\, d_q s = (1 - q)t \sum_{n=0}^{\infty} q^n f(q^n t)$ for $t \in [0, \infty)$.

For the basic properties of the q_k-derivative and q_k-integral we refer to [45].

In this paper we combine all the above subjects and investigate the nonlinear second-order impulsive q_k-difference Langevin equation with boundary conditions of the form

$$\begin{cases} D_{q_k}(D_{q_k}+\lambda)x(t)=f(t,x(t)), \quad t\in J, t\neq t_k, \\ \Delta x(t_k)=I_k(x(t_k)), \quad k=1,2,\ldots,m, \\ D_{q_k}x(t_k^+)-D_{q_{k-1}}x(t_k)=I_k^*(x(t_k)), \quad k=1,2,\ldots,m, \\ \alpha x(0)+\beta D_{q_0}x(0)=x(T), \qquad \gamma x(0)+\eta D_{q_0}x(0)=D_{q_m}x(T), \end{cases} \tag{1.4}$$

where $0=t_0<t_1<t_2<\cdots<t_k<\cdots<t_m<t_{m+1}=T$, $f:J\times\mathbb{R}\to\mathbb{R}$ is a continuous function, λ is a given constant, $I_k, I_k^*\in C(\mathbb{R},\mathbb{R})$, $\Delta x(t_k)=x(t_k^+)-x(t_k)$ for $k=1,2,\ldots,m$, $x(t_k^+)=\lim_{h\to 0+}x(t_k+h)$, $0<q_k<1$ for $k=0,1,2,\ldots,m$, and α, β, γ, η are given constants.

The rest of this paper is organized as follows. In Section 2, we present a preliminary result which will be used in this paper. In Section 3, we will consider the existence results for problem (1.4) while in Section 4, we will give examples to illustrate our main results.

2 An auxiliary lemma

In this section, we present an auxiliary lemma which will be used throughout this paper. Let $J=[0,T]$, $J_0=[t_0,t_1]$, $J_k=(t_k,t_{k+1}]$ for $k=1,2,\ldots,m$.

Lemma 2.1 *Let $\lambda T(\eta+\beta\lambda-\alpha)\neq(\alpha-1)(\eta-1)+\gamma(T-\beta)$. The unique solution of problem* (1.4) *is given by*

$$\begin{aligned} x(t) = {} & \frac{\delta_1+\delta_2 t}{\Omega}\Biggl\{\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\bigl(r,x(r)\bigr)\,d_{q_{k-1}}r\,d_{q_{k-1}}s-\lambda\int_{t_{k-1}}^{t_k}x(s)\,d_{q_{k-1}}s+I_k\bigl(x(t_k)\bigr)\biggr) \\ & +\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k} f\bigl(s,x(s)\bigr)\,d_{q_{k-1}}s+I_k^*\bigl(x(t_k)\bigr)+\lambda I_k\bigl(x(t_k)\bigr)\biggr)(T-t_k) \\ & +\int_{t_m}^{T}\int_{t_m}^{s} f\bigl(r,x(r)\bigr)\,d_{q_m}r\,d_{q_m}s-\lambda\int_{t_m}^{T}x(s)\,d_{q_m}s\Biggr\} \\ & +\frac{\delta_3+\delta_4 t}{\Omega}\Biggl\{\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k} f\bigl(s,x(s)\bigr)\,d_{q_{k-1}}s+I_k^*\bigl(x(t_k)\bigr)+\lambda I_k\bigl(x(t_k)\bigr)\biggr) \\ & +\int_{t_m}^{T} f\bigl(s,x(s)\bigr)\,d_{q_m}s\Biggr\} \\ & +\sum_{0<t_k<t}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\bigl(r,x(r)\bigr)\,d_{q_{k-1}}r\,d_{q_{k-1}}s-\lambda\int_{t_{k-1}}^{t_k}x(s)\,d_{q_{k-1}}s+I_k\bigl(x(t_k)\bigr)\biggr) \\ & +\sum_{0<t_k<t}\biggl(\int_{t_{k-1}}^{t_k} f\bigl(s,x(s)\bigr)\,d_{q_{k-1}}s+I_k^*\bigl(x(t_k)\bigr)+\lambda I_k\bigl(x(t_k)\bigr)\biggr)(t-t_k) \\ & +\int_{t_k}^{t}\int_{t_k}^{s} f\bigl(r,x(r)\bigr)\,d_{q_k}r\,d_{q_k}s-\lambda\int_{t_k}^{t}x(s)\,d_{q_k}s, \end{aligned} \tag{2.1}$$

with $\sum_{0<0}(\cdot)=0$, where

$$\Omega=(\alpha-1)(\eta-1)-\lambda T(\eta+\beta\lambda-\alpha)+\gamma(T-\beta),$$

$$\delta_1=\eta-1+\beta\lambda, \qquad \delta_2=\lambda(\eta+\beta\lambda-1-\alpha)+1-\gamma,$$

$$\delta_3=T-\beta, \qquad \delta_4=\alpha-1-\beta\lambda.$$

Proof For $t \in J_0$ using q_0-integral for the first equation of (1.4), we get

$$D_{q_0}x(t) = D_{q_0}x(0) + \lambda x(0) + \int_0^t f\big(s, x(s)\big)\, d_{q_0}s - \lambda x(t).$$

Setting $x(0) = A$ and $D_{q_0}x(0) = B$, we have

$$D_{q_0}x(t) = \lambda A + B + \int_0^t f\big(s, x(s)\big)\, d_{q_0}s - \lambda x(t), \tag{2.2}$$

which leads to

$$D_{q_0}x(t_1) = \lambda A + B + \int_0^{t_1} f\big(s, x(s)\big)\, d_{q_0}s - \lambda x(t_1). \tag{2.3}$$

For $t \in J_0$ we obtain by q_0-integrating (2.2),

$$x(t) = A + (\lambda A + B)t + \int_0^t \int_0^s f\big(r, x(r)\big)\, d_{q_0}r\, d_{q_0}s - \lambda \int_0^t x(s)\, d_{q_0}s.$$

In particular, for $t = t_1$

$$x(t_1) = A + (\lambda A + B)t_1 + \int_0^{t_1} \int_0^s f\big(r, x(r)\big)\, d_{q_0}r\, d_{q_0}s - \lambda \int_0^{t_1} x(s)\, d_{q_0}s. \tag{2.4}$$

For $t \in J_1 = (t_1, t_2]$, q_1-integrating (1.4), we have

$$D_{q_1}x(t) = D_{q_1}x\big(t_1^+\big) + \lambda x\big(t_1^+\big) + \int_{t_1}^t f\big(s, x(s)\big)\, d_{q_1}s - \lambda x(t).$$

From the second impulsive equations of (1.4), we have

$$\begin{aligned} D_{q_1}x(t) = \lambda A + B &+ \int_0^{t_1} f\big(s, x(s)\big)\, d_{q_0}s + I_1^*\big(x(t_1)\big) \\ &+ \lambda I_1\big(x(t_1)\big) + \int_{t_1}^t f\big(s, x(s)\big)\, d_{q_1}s - \lambda x(t). \end{aligned} \tag{2.5}$$

Applying q_1-integral to (2.5) for $t \in J_1$, we obtain

$$\begin{aligned} x(t) = x\big(t_1^+\big) &+ \left[\lambda A + B + \int_0^{t_1} f\big(s, x(s)\big)\, d_{q_0}s + I_1^*\big(x(t_1)\big) + \lambda I_1\big(x(t_1)\big)\right](t - t_1) \\ &+ \int_{t_1}^t \int_{t_1}^s f\big(r, x(r)\big)\, d_{q_1}r\, d_{q_1}s - \lambda \int_{t_1}^t x(s)\, d_{q_1}s. \end{aligned} \tag{2.6}$$

Using the second impulsive equation of (1.4) with (2.4) and (2.6), one has

$$\begin{aligned} x(t) = A + (\lambda A + B)t_1 &+ \int_0^{t_1} \int_0^s f\big(r, x(r)\big)\, d_{q_0}r\, d_{q_0}s - \lambda \int_0^{t_1} x(s)\, d_{q_0}s + I_1\big(x(t_1)\big) \\ &+ \left[\lambda A + B + \int_0^{t_1} f\big(s, x(s)\big)\, d_{q_0}s + I_1^*\big(x(t_1)\big) + \lambda I_1\big(x(t_1)\big)\right](t - t_1) \end{aligned}$$

$$
\begin{aligned}
&+ \int_{t_1}^{t} \int_{t_1}^{s} f\big(r, x(r)\big)\, d_{q_1} r\, d_{q_1} s - \lambda \int_{t_1}^{t} x(s)\, d_{q_1} s \\
&= A + (\lambda A + B)t + \int_0^{t_1} \int_0^{s} f\big(r, x(r)\big)\, d_{q_0} r\, d_{q_0} s - \lambda \int_0^{t_1} x(s)\, d_{q_0} s + I_1\big(x(t_1)\big) \\
&+ \left[\int_0^{t_1} f\big(s, x(s)\big)\, d_{q_0} s + I_1^*\big(x(t_1)\big) + \lambda I_1\big(x(t_1)\big) \right](t - t_1) \\
&+ \int_{t_1}^{t} \int_{t_1}^{s} f\big(r, x(r)\big)\, d_{q_1} r\, d_{q_1} s - \lambda \int_{t_1}^{t} x(s)\, d_{q_1} s.
\end{aligned}
$$

Repeating the above process, for $t \in J$, we get

$$
\begin{aligned}
x(t) &= A + (\lambda A + B)t \\
&+ \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f\big(r, x(r)\big)\, d_{q_{k-1}} r\, d_{q_{k-1}} s - \lambda \int_{t_{k-1}}^{t_k} x(s)\, d_{q_{k-1}} s + I_k\big(x(t_k)\big) \right) \\
&+ \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f\big(s, x(s)\big)\, d_{q_{k-1}} s + I_k^*\big(x(t_k)\big) + \lambda I_k\big(x(t_k)\big) \right)(t - t_k) \\
&+ \int_{t_k}^{t} \int_{t_k}^{s} f\big(r, x(r)\big)\, d_{q_k} r\, d_{q_k} s - \lambda \int_{t_k}^{t} x(s)\, d_{q_k} s.
\end{aligned} \tag{2.7}
$$

For $t = T$, we get

$$
\begin{aligned}
x(T) &= (1 + \lambda T)A + BT \\
&+ \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f\big(r, x(r)\big)\, d_{q_{k-1}} r\, d_{q_{k-1}} s - \lambda \int_{t_{k-1}}^{t_k} x(s)\, d_{q_{k-1}} s + I_k\big(x(t_k)\big) \right) \\
&+ \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} f\big(s, x(s)\big)\, d_{q_{k-1}} s + I_k^*\big(x(t_k)\big) + \lambda I_k\big(x(t_k)\big) \right)(T - t_k) \\
&+ \int_{t_m}^{T} \int_{t_m}^{s} f\big(r, x(r)\big)\, d_{q_m} r\, d_{q_m} s - \lambda \int_{t_m}^{T} x(s)\, d_{q_m} s.
\end{aligned} \tag{2.8}
$$

It is easy to see that

$$
\begin{aligned}
D_{q_k} x(t) &= \lambda A + B \\
&+ \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f\big(s, x(s)\big)\, d_{q_{k-1}} s + I_k^*\big(x(t_k)\big) + \lambda I_k\big(x(t_k)\big) \right) \\
&+ \int_{t_k}^{t} f\big(s, x(s)\big)\, d_{q_k} s - \lambda x(t).
\end{aligned}
$$

For $t = T$ and using $x(T) = \alpha A + \beta B$, we have

$$
\begin{aligned}
D_{q_m} x(T) &= \lambda A + B \\
&+ \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} f\big(s, x(s)\big)\, d_{q_{k-1}} s + I_k^*\big(x(t_k)\big) + \lambda I_k\big(x(t_k)\big) \right) \\
&+ \int_{t_m}^{T} f\big(s, x(s)\big)\, d_{q_m} s - \lambda x(T)
\end{aligned}
$$

$$\begin{aligned}
&= (1-\alpha)\lambda A + (1-\lambda\beta)B \\
&\quad + \sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big) + \lambda I_k\big(x(t_k)\big)\right) \\
&\quad + \int_{t_m}^{T} f\big(s,x(s)\big)\,d_{q_m}s. \qquad (2.9)
\end{aligned}$$

Applying the boundary conditions of (1.4) with (2.8) and (2.9), it follows that

$$\begin{aligned}
A &= \frac{\eta+\lambda\beta-1}{\Omega}\Bigg\{\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(r,x(r)\big)\,d_{q_{k-1}}r\,d_{q_{k-1}}s\right. \\
&\quad \left. - \lambda\int_{t_{k-1}}^{t_k} x(s)\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) \\
&\quad + \sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big) + \lambda I_k\big(x(t_k)\big)\right)(T-t_k) \\
&\quad + \int_{t_m}^{T}\int_{t_m}^{s} f\big(r,x(r)\big)\,d_{q_m}r\,d_{q_m}s - \lambda\int_{t_m}^{T} x(s)\,d_{q_m}s\Bigg\} \\
&\quad - \frac{\beta-T}{\Omega}\Bigg\{\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big) + \lambda I_k\big(x(t_k)\big)\right) + \int_{t_m}^{T} f\big(s,x(s)\big)\,d_{q_m}s\Bigg\}
\end{aligned}$$

and

$$\begin{aligned}
B &= \frac{\alpha-1-\lambda T}{\Omega}\Bigg\{\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big) + \lambda I_k\big(x(t_k)\big)\right) \\
&\quad + \int_{t_m}^{T} f\big(s,x(s)\big)\,d_{q_m}s\Bigg\} \\
&\quad - \frac{\gamma-1+\alpha\lambda}{\Omega}\Bigg\{\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\big(r,x(r)\big)\,d_{q_{k-1}}r\,d_{q_{k-1}}s\right. \\
&\quad \left. - \lambda\int_{t_{k-1}}^{t_k} x(s)\,d_{q_{k-1}}s + I_k\big(x(t_k)\big)\right) \\
&\quad + \sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k} f\big(s,x(s)\big)\,d_{q_{k-1}}s + I_k^*\big(x(t_k)\big) + \lambda I_k\big(x(t_k)\big)\right)(T-t_k) \\
&\quad + \int_{t_m}^{T}\int_{t_m}^{s} f\big(r,x(r)\big)\,d_{q_m}r\,d_{q_m}s - \lambda\int_{t_m}^{T} x(s)\,d_{q_m}s\Bigg\}.
\end{aligned}$$

Substituting the values of A and B into (2.7), we get (2.1) as required. The proof is completed. □

3 Main results

Let $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R}: x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1,2,\ldots,m\}$. $PC(J,\mathbb{R})$ is a Banach space with the norm $\|x\|_{PC} = \sup\{|x(t)|; t \in J\}$.

From Lemma 2.1, we define an operator $\mathcal{S}: PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$\begin{aligned}(\mathcal{S}x)(t) = {}& \frac{\delta_1+\delta_2 t}{\Omega}\Biggl\{\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\bigl(r,x(r)\bigr)\,d_{q_{k-1}}r\,d_{q_{k-1}}s \\ & - \lambda\int_{t_{k-1}}^{t_k} x(s)\,d_{q_{k-1}}s + I_k\bigl(x(t_k)\bigr)\biggr) \\ & + \sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k} f\bigl(s,x(s)\bigr)\,d_{q_{k-1}}s + I_k^*\bigl(x(t_k)\bigr) + \lambda I_k\bigl(x(t_k)\bigr)\biggr)(T-t_k) \\ & + \int_{t_m}^{T}\int_{t_m}^{s} f\bigl(r,x(r)\bigr)\,d_{q_m}r\,d_{q_m}s - \lambda\int_{t_m}^{T} x(s)\,d_{q_m}s\Biggr\} \\ & + \frac{\delta_3+\delta_4 t}{\Omega}\Biggl\{\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k} f\bigl(s,x(s)\bigr)\,d_{q_{k-1}}s + I_k^*\bigl(x(t_k)\bigr) + \lambda I_k\bigl(x(t_k)\bigr)\biggr) \\ & + \int_{t_m}^{T} f\bigl(s,x(s)\bigr)\,d_{q_m}s\Biggr\} \\ & + \sum_{0<t_k<t}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\bigl(r,x(r)\bigr)\,d_{q_{k-1}}r\,d_{q_{k-1}}s - \lambda\int_{t_{k-1}}^{t_k} x(s)\,d_{q_{k-1}}s + I_k\bigl(x(t_k)\bigr)\biggr) \\ & + \sum_{0<t_k<t}\biggl(\int_{t_{k-1}}^{t_k} f\bigl(s,x(s)\bigr)\,d_{q_{k-1}}s + I_k^*\bigl(x(t_k)\bigr) + \lambda I_k\bigl(x(t_k)\bigr)\biggr)(t-t_k) \\ & + \int_{t_k}^{t}\int_{t_k}^{s} f\bigl(r,x(r)\bigr)\,d_{q_k}r\,d_{q_k}s - \lambda\int_{t_k}^{t} x(s)\,d_{q_k}s, \end{aligned} \tag{3.1}$$

where constants δ_1, δ_2, δ_3, δ_4, and Ω are defined as in Lemma 2.1. It should be noticed that problem (1.4) has solutions if and only if the operator $\mathcal{S}$ has fixed points.

Our first result is an existence and uniqueness result for the impulsive boundary value problem (1.4) by using the Banach contraction mapping principle.

For convenience, we set

$$\begin{aligned}\Lambda_1 = {}& \left(\frac{|\delta_1|+|\delta_2|T+|\Omega|}{|\Omega|}\right)\Biggl[L_1\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}} + |\lambda|\sum_{k=1}^{m+1}(t_k-t_{k-1}) \\ & + mL_2 + \sum_{k=1}^{m}\bigl(L_1(t_k-t_{k-1}) + L_3 + |\lambda|L_2\bigr)(T-t_k)\Biggr] \\ & + \left(\frac{|\delta_3|+|\delta_4|T}{|\Omega|}\right)\bigl[L_1T + mL_3 + m|\lambda|L_2\bigr] \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\Lambda_2 = {}& \left(\frac{|\delta_1|+|\delta_2|T+|\Omega|}{|\Omega|}\right)\Biggl[K_1\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}} + mK_2 \\ & + \sum_{k=1}^{m}\bigl(K_1(t_k-t_{k-1}) + K_3 + |\lambda|K_2\bigr)(T-t_k)\Biggr] \\ & + \left(\frac{|\delta_3|+|\delta_4|T}{|\Omega|}\right)\bigl[K_1T + mK_3 + m|\lambda|K_2\bigr]. \end{aligned} \tag{3.3}$$

Theorem 3.1 *Assume that the following conditions hold*:

(H_1) $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ *is a continuous function and there exists a constant* $L_1>0$ *such that*

$$\left|f(t,x)-f(t,y)\right|\le L_1|x-y|$$

for each $t\in J$ *and* $x,y\in\mathbb{R}$.

(H_2) *The functions* $I_k, I_k^*:\mathbb{R}\to\mathbb{R}$ *are continuous and there exist constants* $L_2, L_3>0$ *such that*

$$\left|I_k(x)-I_k(y)\right|\le L_2|x-y| \quad \text{and} \quad \left|I_k^*(x)-I_k^*(y)\right|\le L_3|x-y|$$

for each $x,y\in\mathbb{R}$, $k=1,2,\dots,m$.

If

$$\Lambda_1\le\delta<1, \tag{3.4}$$

where Λ_1 *is defined by* (3.2), *then the impulsive* q_k*-difference Langevin boundary value problem* (1.4) *has a unique solution on* J.

Proof Firstly, we transform the impulsive q_k-difference Langevin boundary value problem (1.4) into a fixed point problem, $x=\mathcal{S}x$, where the operator $\mathcal{S}$ is defined by (3.1). Applying the Banach contraction mapping principle, we shall show that $\mathcal{S}$ has a fixed point which is the unique solution of the boundary value problem (1.4).

Let K_1, K_2, and K_3 be nonnegative constants such that $K_1=\sup_{t\in J}|f(t,0)|$, $K_2=\sup\{|I_k(0)|:k=1,2,\dots,m\}$, and $K_3=\sup\{|I_k^*(0)|:k=1,2,\dots,m\}$. We choose a suitable constant ρ by

$$\rho\ge\frac{\Lambda_2}{1-\varepsilon},$$

where $\delta\le\varepsilon<1$ and Λ_2 defined by (3.3). Now, we will show that $\mathcal{S}B_\rho\subset B_\rho$, where a set B_ρ is defined as $B_\rho=\{x\in PC(J,\mathbb{R}):\|x\|\le\rho\}$. For $x\in B_\rho$, we have

$$\begin{aligned}
\|\mathcal{S}x\| \le \sup_{t\in J}\Biggl\{ & \frac{|\delta_1|+|\delta_2|t}{|\Omega|}\Biggl\{\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl|f\bigl(r,x(r)\bigr)\bigr|\,d_{q_{k-1}}r\,d_{q_{k-1}}s \\
& + |\lambda|\int_{t_{k-1}}^{t_k}\bigl|x(s)\bigr|\,d_{q_{k-1}}s+\bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr) \\
& + \sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,d_{q_{k-1}}s+\bigl|I_k^*\bigl(x(t_k)\bigr)\bigr|+|\lambda|\bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr)(T-t_k) \\
& + \int_{t_m}^{T}\int_{t_m}^{s}\bigl|f\bigl(r,x(r)\bigr)\bigr|\,d_{q_m}r\,d_{q_m}s+|\lambda|\int_{t_m}^{T}\bigl|x(s)\bigr|\,d_{q_m}s\Biggr\} \\
& + \frac{|\delta_3|+|\delta_4|t}{|\Omega|}\Biggl\{\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,d_{q_{k-1}}s+\bigl|I_k^*\bigl(x(t_k)\bigr)\bigr|+|\lambda|\bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr) \\
& + \int_{t_m}^{T}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,d_{q_m}s\Biggr\}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\left|f\big(r,x(r)\big)\right|d_{q_{k-1}}r\,d_{q_{k-1}}s+|\lambda|\int_{t_{k-1}}^{t_k}\left|x(s)\right|d_{q_{k-1}}s+\left|I_k\big(x(t_k)\big)\right|\right)\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\left|f\big(s,x(s)\big)\right|d_{q_{k-1}}s+\left|I_k^*\big(x(t_k)\big)\right|+|\lambda|\left|I_k\big(x(t_k)\big)\right|\right)(t-t_k)\\
&+\int_{t_k}^{t}\int_{t_k}^{s}\left|f\big(r,x(r)\big)\right|d_{q_k}r\,d_{q_k}s+|\lambda|\int_{t_k}^{t}\left|x(s)\right|d_{q_k}s\Bigg\}\\
\le{}&\frac{|\delta_1|+|\delta_2|T}{|\Omega|}\Bigg\{\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\left(\left|f\big(r,x(r)\big)-f(r,0)\right|+\left|f(r,0)\right|\right)d_{q_{k-1}}r\,d_{q_{k-1}}s\right.\\
&\left.+|\lambda|\int_{t_{k-1}}^{t_k}\|x\|\,d_{q_{k-1}}s+\left(\left|I_k\big(x(t_k)\big)-I_k(0)\right|+\left|I_k(0)\right|\right)\right)\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\left(\left|f\big(s,x(s)\big)-f(s,0)\right|+\left|f(s,0)\right|\right)d_{q_{k-1}}s\right.\\
&+\left(\left|I_k^*\big(x(t_k)\big)-I_k^*(0)\right|+\left|I_k^*(0)\right|\right)\\
&\left.+|\lambda|\left(\left|I_k\big(x(t_k)\big)-I_k\big(x(0)\big)\right|+\left|I_k\big(x(0)\big)\right|\right)\right)(T-t_k)\\
&+\int_{t_m}^{T}\int_{t_m}^{s}\left(\left|f\big(r,x(r)\big)-f(r,0)\right|+\left|f(r,0)\right|\right)d_{q_m}r\,d_{q_m}s+|\lambda|\int_{t_m}^{T}\|x\|\,d_{q_m}s\Bigg\}\\
&+\frac{|\delta_3|+|\delta_4|T}{|\Omega|}\Bigg\{\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\left(\left|f\big(s,x(s)\big)-f(s,0)\right|+\left|f(s,0)\right|\right)d_{q_{k-1}}s\right.\\
&\left.+\left(\left|I_k^*\big(x(t_k)\big)-I_k^*(0)\right|+\left|I_k^*(0)\right|\right)+|\lambda|\left(\left|I_k\big(x(t_k)\big)-I_k\big(x(0)\big)\right|+\left|I_k\big(x(0)\big)\right|\right)\right)\\
&+\int_{t_m}^{T}\left(\left|f\big(s,x(s)\big)-f(s,0)\right|+\left|f(s,0)\right|\right)d_{q_m}s\Bigg\}\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\left(\left|f\big(r,x(r)\big)-f(r,0)\right|+\left|f(r,0)\right|\right)d_{q_{k-1}}r\,d_{q_{k-1}}s\right.\\
&\left.+|\lambda|\int_{t_{k-1}}^{t_k}\|x\|\,d_{q_{k-1}}s+\left(\left|I_k\big(x(t_k)\big)-I_k\big(x(0)\big)\right|+\left|I_k\big(x(0)\big)\right|\right)\right)\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\left(\left|f\big(s,x(s)\big)-f(s,0)\right|+\left|f(s,0)\right|\right)d_{q_{k-1}}s\right.\\
&+\left(\left|I_k^*\big(x(t_k)\big)-I_k^*(0)\right|+\left|I_k^*(0)\right|\right)\\
&\left.+|\lambda|\left(\left|I_k\big(x(t_k)\big)-I_k\big(x(0)\big)\right|+\left|I_k\big(x(0)\big)\right|\right)\right)(T-t_k)\\
&+\int_{t_m}^{T}\int_{t_m}^{s}\left(\left|f\big(r,x(r)\big)-f(r,0)\right|+\left|f(r,0)\right|\right)d_{q_m}r\,d_{q_m}s+|\lambda|\int_{t_m}^{T}\|x\|\,d_{q_m}s\\
\le{}&\frac{|\delta_1|+|\delta_2|T}{|\Omega|}\Bigg\{\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(L_1\rho+K_1)\,d_{q_{k-1}}r\,d_{q_{k-1}}s+|\lambda|\int_{t_{k-1}}^{t_k}\rho\,d_{q_{k-1}}s\right.\\
&\left.+(L_2\rho+K_2)\right)+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}(L_1\rho+K_1)\,d_{q_{k-1}}s+(L_3\rho+K_3)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ |\lambda|(L_2\rho + K_2)\bigg)(T - t_k) + \int_{t_m}^{T}\int_{t_m}^{s}(L_1\rho + K_1)\,d_{q_m}r\,d_{q_m}s + |\lambda|\int_{t_m}^{T}\rho\,d_{q_m}s\Bigg\} \\
&+ \frac{|\delta_3| + |\delta_4|T}{|\Omega|}\Bigg\{\sum_{k=1}^{m}\bigg(\int_{t_{k-1}}^{t_k}(L_1\rho + K_1)\,d_{q_{k-1}}s + (L_3\rho + K_3) + |\lambda|(L_2\rho + K_2)\bigg) \\
&+ \int_{t_m}^{T}(L_1\rho + K_1)\,d_{q_m}s\Bigg\} + \sum_{k=1}^{m}\bigg(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(L_1\rho + K_1)\,d_{q_{k-1}}r\,d_{q_{k-1}}s \\
&+ |\lambda|\int_{t_{k-1}}^{t_k}\rho\,d_{q_{k-1}}s + (L_2\rho + K_2)\bigg) + \sum_{k=1}^{m}\bigg(\int_{t_{k-1}}^{t_k}(L_1\rho + K_1)\,d_{q_{k-1}}s \\
&+ (L_3\rho + K_3) + |\lambda|(L_2\rho + K_2)\bigg)(T - t_k) \\
&+ \int_{t_m}^{T}\int_{t_m}^{s}(L_1\rho + K_1)\,d_{q_m}r\,d_{q_m}s + |\lambda|\int_{t_m}^{T}\rho\,d_{q_m}s \\
= {}& \Lambda_1\rho + \Lambda_2 \le (\delta + 1 - \varepsilon)\rho \le \rho,
\end{aligned}
$$

which implies that $\mathcal{S}B_\rho \subset B_\rho$.

For any $x, y \in PC(J, \mathbb{R})$ and for each $t \in J$, we have

$$
\begin{aligned}
&\big|\mathcal{S}x(t) - \mathcal{S}y(t)\big| \\
&\le \frac{|\delta_1| + |\delta_2|t}{|\Omega|}\Bigg\{\sum_{k=1}^{m}\bigg(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\big|f\big(r, x(r)\big) - f\big(r, y(r)\big)\big|\,d_{q_{k-1}}r\,d_{q_{k-1}}s \\
&+ |\lambda|\int_{t_{k-1}}^{t_k}\big|x(s) - y(s)\big|\,d_{q_{k-1}}s + \big|I_k\big(x(t_k)\big) - I_k\big(y(t_k)\big)\big|\bigg) \\
&+ \sum_{k=1}^{m}\bigg(\int_{t_{k-1}}^{t_k}\big|f\big(s, x(s)\big) - f\big(s, y(s)\big)\big|\,d_{q_{k-1}}s \\
&+ \big|I_k^*\big(x(t_k)\big) - I_k^*\big(y(t_k)\big)\big| + |\lambda|\big|I_k\big(x(t_k)\big) - I_k\big(y(t_k)\big)\big|\bigg)(T - t_k) \\
&+ \int_{t_m}^{T}\int_{t_m}^{s}\big|f\big(r, x(r)\big) - f\big(r, y(r)\big)\big|\,d_{q_m}r\,d_{q_m}s + |\lambda|\int_{t_m}^{T}\big|x(s) - y(s)\big|\,d_{q_m}s\Bigg\} \\
&+ \frac{|\delta_3| + |\delta_4|t}{|\Omega|}\Bigg\{\sum_{k=1}^{m}\bigg(\int_{t_{k-1}}^{t_k}\big|f\big(s, x(s)\big) - f\big(s, y(s)\big)\big|\,d_{q_{k-1}}s \\
&+ \big|I_k^*\big(x(t_k)\big) - I_k^*\big(y(t_k)\big)\big| + |\lambda|\big|I_k\big(x(t_k)\big) - I_k\big(y(t_k)\big)\big|\bigg) \\
&+ \int_{t_m}^{T}\big|f\big(s, x(s)\big) - f\big(s, y(s)\big)\big|\,d_{q_m}s\Bigg\} \\
&+ \sum_{0<t_k<t}\bigg(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\big|f\big(r, x(r)\big) - f\big(r, y(r)\big)\big|\,d_{q_{k-1}}r\,d_{q_{k-1}}s \\
&+ |\lambda|\int_{t_{k-1}}^{t_k}\big|x(s) - y(s)\big|\,d_{q_{k-1}}s + \big|I_k\big(x(t_k)\big) - I_k\big(y(t_k)\big)\big|\bigg) \\
&+ \sum_{0<t_k<t}\bigg(\int_{t_{k-1}}^{t_k}\big|f\big(s, x(s)\big) - f\big(s, y(s)\big)\big|\,d_{q_{k-1}}s + \big|I_k^*\big(x(t_k)\big) - I_k^*\big(y(t_k)\big)\big|
\end{aligned}
$$

$$
\begin{aligned}
&\quad + |\lambda| \big| I_k\big(x(t_k)\big) - I_k\big(y(t_k)\big)\big| \Big)(t - t_k) + \int_{t_k}^{t} \int_{t_k}^{s} \big| f\big(r, x(r)\big) - f\big(r, y(r)\big)\big| \, d_{q_k} r \, d_{q_k} s \\
&\quad + |\lambda| \int_{t_k}^{t} \big| x(s) - y(s)\big| \, d_{q_k} s \\
&\le \frac{|\delta_1| + |\delta_2| T}{|\Omega|} \|x - y\| \left\{ \sum_{k=1}^{m} \left(L_1 \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + |\lambda|(t_k - t_{k-1}) + L_2 \right) \right. \\
&\quad \left. + \sum_{k=1}^{m} \big(L_1(t_k - t_{k-1}) + L_3 + |\lambda| L_2 \big)(T - t_k) + L_1 \frac{(T - t_m)^2}{1 + q_m} + |\lambda|(T - t_m) \right\} \\
&\quad + \frac{|\delta_3| + |\delta_4| T}{|\Omega|} \|x - y\| \left\{ \sum_{k=1}^{m} \big(L_1(t_k - t_{k-1}) + L_3 + |\lambda| L_2 \big) + L_1(T - t_m) \right\} \\
&\quad + \|x - y\| \sum_{k=1}^{m} \left(L_1 \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + |\lambda|(t_k - t_{k-1}) + L_2 \right) \\
&\quad + \|x - y\| \sum_{k=1}^{m} \big(L_1(t_k - t_{k-1}) + L_3 + |\lambda| L_2 \big)(T - t_k) + L_1 \frac{(T - t_m)^2}{1 + q_m} \|x - y\| \\
&\quad + |\lambda|(T - t_m)\|x - y\| \\
&= \Lambda_1 \|x - y\|,
\end{aligned}
$$

which implies that $\|\mathcal{S}x - \mathcal{S}y\| \le \Lambda_1 \|x - y\|$. As $\Lambda_1 < 1$, $\mathcal{S}$ is a contraction. Therefore, by the Banach contraction mapping principle, we find that $\mathcal{S}$ has a fixed point which is the unique solution of problem (1.4). □

The second existence result is based on Schaefer's fixed point theorem.

Theorem 3.2 *Assume that the following conditions hold*:

(H_3) $f : J \times \mathbb{R} \to \mathbb{R}$ *is a continuous function and there exists a constant* $M_1 > 0$ *such that*

$$\big| f(t, x)\big| \le M_1$$

for each $t \in J$ *and all* $x \in \mathbb{R}$.

(H_4) *The functions* $I_k, I_k^* : \mathbb{R} \to \mathbb{R}$ *are continuous and there exist constants* $M_2, M_3 > 0$ *such that*

$$\big| I_k(x)\big| \le M_2 \quad \text{and} \quad \big| I_k^*(x)\big| \le M_3$$

for all $x \in \mathbb{R}$, $k = 1, 2, \ldots, m$.

If

$$\frac{|\delta_1| + |\delta_2| T + |\Omega|}{|\Omega|} |\lambda| T < 1, \tag{3.5}$$

then the impulsive q_k*-difference Langevin boundary value problem* (1.4) *has at least one solution on* J.

Proof We shall use Schaefer's fixed point theorem to prove that the operator $\mathcal{S}$ defined by (3.1) has a fixed point. We divide the proof into four steps.

Step 1: *Continuity of* $\mathcal{S}$.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC(J,\mathbb{R})$. Since f is a continuous function on $J \times \mathbb{R}$ and I_k, I_k^* are continuous functions on $\mathbb{R}$ for $k = 1, 2, \dots, m$, we have

$$f\big(t,x_n(t)\big) \to f\big(t,x(t)\big), \qquad I_k\big(x_n(t_k)\big) \to I_k\big(x(t_k)\big) \quad \text{and} \quad I_k^*\big(x_n(t_k)\big) \to I_k^*\big(x(t_k)\big)$$

for $k = 1, 2, \dots, m$, as $n \to \infty$.

Then, for each $t \in J$, we get

$$\begin{aligned}
&\big|(\mathcal{S}x_n)(t) - (\mathcal{S}x)(t)\big| \\
&\quad\le \frac{|\delta_1| + |\delta_2|t}{|\Omega|}\Bigg\{\sum_{k=1}^{m}\bigg(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\big|f\big(r,x_n(r)\big) - f\big(r,x(r)\big)\big|\,d_{q_{k-1}}r\,d_{q_{k-1}}s \\
&\qquad + \lambda\int_{t_{k-1}}^{t_k}\big|x_n(s) - x(s)\big|\,d_{q_{k-1}}s + \big|I_k\big(x_n(t_k)\big) - I_k\big(x(t_k)\big)\big|\bigg) \\
&\qquad + \sum_{k=1}^{m}\bigg(\int_{t_{k-1}}^{t_k}\big|f\big(s,x_n(s)\big) - f\big(s,x(s)\big)\big|\,d_{q_{k-1}}s + \big|I_k^*\big(x_n(t_k)\big) - I_k^*\big(x(t_k)\big)\big| \\
&\qquad + |\lambda|\big|I_k\big(x_n(t_k)\big) - I_k\big(x(t_k)\big)\big|\bigg)(T - t_k) \\
&\qquad + \int_{t_m}^{T}\int_{t_m}^{s}\big|f\big(r,x_n(r)\big) - f\big(r,x(r)\big)\big|\,d_{q_m}r\,d_{q_m}s + |\lambda|\int_{t_m}^{T}\big|x_n(s) - x(s)\big|\,d_{q_m}s\Bigg\} \\
&\qquad + \frac{|\delta_3| + |\delta_4|t}{\Omega}\Bigg\{\sum_{k=1}^{m}\bigg(\int_{t_{k-1}}^{t_k}\big|f\big(s,x_n(s)\big) - f\big(s,x(s)\big)\big|\,d_{q_{k-1}}s \\
&\qquad + \big|I_k^*\big(x_n(t_k)\big) - I_k^*\big(x(t_k)\big)\big| \\
&\qquad + |\lambda|\big|I_k\big(x_n(t_k)\big) - I_k\big(x(t_k)\big)\big|\bigg) + \int_{t_m}^{T}\big|f\big(s,x_n(s)\big) - f\big(s,x(s)\big)\big|\,d_{q_m}s\Bigg\} \\
&\qquad + \sum_{0<t_k<t}\bigg(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\big|f\big(r,x_n(r)\big) - f\big(r,x(r)\big)\big|\,d_{q_{k-1}}r\,d_{q_{k-1}}s \\
&\qquad + |\lambda|\int_{t_{k-1}}^{t_k}\big|x_n(s) - x(s)\big|\,d_{q_{k-1}}s + \big|I_k\big(x_n(t_k)\big) - I_k\big(x(t_k)\big)\big|\bigg) \\
&\qquad + \sum_{0<t_k<t}\bigg(\int_{t_{k-1}}^{t_k}\big|f\big(s,x_n(s)\big) - f\big(s,x(s)\big)\big|\,d_{q_{k-1}}s + \big|I_k^*\big(x_n(t_k)\big) - I_k^*\big(x(t_k)\big)\big| \\
&\qquad + |\lambda|\big|I_k\big(x_n(t_k)\big) - I_k\big(x(t_k)\big)\big|\bigg)(t - t_k) \\
&\qquad + \int_{t_k}^{t}\int_{t_k}^{s}\big|f\big(r,x_n(r)\big) - f\big(r,x(r)\big)\big|\,d_{q_k}r\,d_{q_k}s + |\lambda|\int_{t_k}^{t}\big|x_n(s) - x(s)\big|\,d_{q_k}s,
\end{aligned}$$

which gives $\|\mathcal{S}x_n - \mathcal{S}x\| \to 0$ as $n \to \infty$. This means that $\mathcal{S}$ is continuous.

Step 2: $\mathcal{S}$ *maps bounded sets into bounded sets in* $PC(J,\mathbb{R})$.

Let us prove that for any $\rho^* > 0$, there exists a positive constant σ such that for each $x \in B_{\rho^*} = \{x \in PC(J,\mathbb{R}) : \|x\| \leq \rho^*\}$, we have $\|\mathcal{S}x\| \leq \sigma$. For any $x \in B_{\rho^*}$, we have

$$\begin{aligned}
&\bigl|(\mathcal{S}x)(t)\bigr| \\
&\quad\leq \frac{|\delta_1| + |\delta_2|T}{|\Omega|}\Biggl\{\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl|f\bigl(r,x(r)\bigr)\bigr|\,d_{q_{k-1}}r\,d_{q_{k-1}}s + |\lambda|\int_{t_{k-1}}^{t_k}\bigl|x(s)\bigr|\,d_{q_{k-1}}s \\
&\qquad + \bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr) + \sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,d_{q_{k-1}}s + \bigl|I_k^*\bigl(x(t_k)\bigr)\bigr| \\
&\qquad + |\lambda|\bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr)(T - t_k) + \int_{t_m}^{T}\int_{t_m}^{s}\bigl|f\bigl(r,x(r)\bigr)\bigr|\,d_{q_m}r\,d_{q_m}s + |\lambda|\int_{t_m}^{T}\bigl|x(s)\bigr|\,d_{q_m}s\Biggr\} \\
&\qquad + \frac{|\delta_3| + |\delta_4|T}{|\Omega|}\Biggl\{\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,d_{q_{k-1}}s \\
&\qquad + \bigl|I_k^*\bigl(x(t_k)\bigr)\bigr| + |\lambda|\bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr) + \int_{t_m}^{T}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,d_{q_m}s\Biggr\} \\
&\qquad + \sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl|f\bigl(r,x(r)\bigr)\bigr|\,d_{q_{k-1}}r\,d_{q_{k-1}}s + |\lambda|\int_{t_{k-1}}^{t_k}\bigl|x(s)\bigr|\,d_{q_{k-1}}s + \bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr) \\
&\qquad + \sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\bigl|f\bigl(s,x(s)\bigr)\bigr|\,d_{q_{k-1}}s + \bigl|I_k^*\bigl(x(t_k)\bigr)\bigr| + |\lambda|\bigl|I_k\bigl(x(t_k)\bigr)\bigr|\biggr)(T - t_k) \\
&\qquad + \int_{t_m}^{T}\int_{t_m}^{s}\bigl|f\bigl(r,x(r)\bigr)\bigr|\,d_{q_m}r\,d_{q_m}s + |\lambda|\int_{t_m}^{T}\bigl|x(s)\bigr|\,d_{q_m}s \\
&\quad\leq \frac{|\delta_1| + |\delta_2|T}{|\Omega|}\Biggl\{\sum_{k=1}^{m}\biggl(M_1\frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + \rho^*|\lambda|(t_k - t_{k-1}) + M_2\biggr) \\
&\qquad + \sum_{k=1}^{m}\bigl(M_1(t_k - t_{k-1}) + M_3 + |\lambda|M_2\bigr)(T - t_k) + M_1\frac{(T - t_m)^2}{1 + q_m} + \rho^*|\lambda|(T - t_m)\Biggr\} \\
&\qquad + \frac{|\delta_3| + |\delta_4|T}{|\Omega|}\Biggl\{\sum_{k=1}^{m}\bigl(M_1(t_k - t_{k-1}) + M_3 + |\lambda|M_2\bigr) + M_1(T - t_m)\Biggr\} \\
&\qquad + \sum_{k=1}^{m}\biggl(M_1\frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + \rho^*|\lambda|(t_k - t_{k-1}) + M_2\biggr) + \rho^*|\lambda|(T - t_m) \\
&\qquad + \sum_{k=1}^{m}\bigl(M_1(t_k - t_{k-1}) + M_3 + |\lambda|M_2\bigr)(T - t_k) + M_1\frac{(T - t_m)^2}{1 + q_m} := \sigma.
\end{aligned}$$

Hence, we deduce that $\|\mathcal{S}x\| \leq \sigma$.

Step 3: *$\mathcal{S}$ maps bounded sets into equicontinuous sets of $PC(J,\mathbb{R})$.*

Let $\tau_1, \tau_2 \in J_i = (t_i, t_{i+1}]$ for some $i \in \{0, 1, 2, \ldots, m\}$, $\tau_1 < \tau_2$, B_{ρ^*} be a bounded set of $PC(J,\mathbb{R})$ as in Step 2, and let $x \in B_{\rho^*}$. Then we have

$$\begin{aligned}
&\bigl|(\mathcal{S}x)(\tau_2) - (\mathcal{S}x)(\tau_1)\bigr| \\
&\quad\leq \frac{|\delta_2||\tau_2 - \tau_1|}{|\Omega|}\Biggl\{\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}\bigl|f\bigl(r,x(r)\bigr)\bigr|\,d_{q_{k-1}}r\,d_{q_{k-1}}s
\end{aligned}$$

$$
\begin{aligned}
&+ |\lambda| \int_{t_{k-1}}^{t_k} |x(s)|\, d_{q_{k-1}} s + |I_k(x(t_k))| \Big) \\
&+ \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} |f(s,x(s))|\, d_{q_{k-1}} s + |I_k^*(x(t_k))| + |\lambda| |I_k(x(t_k))| \right)(T - t_k) \\
&+ \int_{t_m}^{T} \int_{t_m}^{s} |f(r,x(r))|\, d_{q_m} r\, d_{q_m} s + |\lambda| \int_{t_m}^{T} |x(s)|\, d_{q_m} s \Big\} \\
&+ \frac{|\delta_4||\tau_2 - \tau_1|}{|\Omega|} \left\{ \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} |f(s,x(s))|\, d_{q_{k-1}} s + |I_k^*(x(t_k))| + |\lambda| |I_k(x(t_k))| \right) \right. \\
&\left. + \int_{t_m}^{T} |f(s,x(s))|\, d_{q_m} s \right\} + |\lambda| \left| \int_{t_i}^{\tau_2} x(s)\, d_{q_i} s - \int_{t_i}^{\tau_1} x(s)\, d_{q_i} s \right| \\
&+ |\tau_2 - \tau_1| \sum_{k=1}^{i} \left(\int_{t_{k-1}}^{t_k} |f(s,x(s))|\, d_{q_{k-1}} s + |I_k^*(x(t_k))| + |\lambda| |I_k(x(t_k))| \right) \\
&+ \left| \int_{t_i}^{\tau_2} \int_{t_i}^{s} f(r,x(r))\, d_{q_i} r\, d_{q_i} s - \int_{t_i}^{\tau_1} \int_{t_i}^{s} f(r,x(r))\, d_{q_i} r\, d_{q_i} s \right| \\
\le{}& \frac{|\delta_2||\tau_2 - \tau_1|}{|\Omega|} \left\{ \sum_{k=1}^{m} \left(M_1 \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + \rho^* |\lambda| (t_k - t_{k-1}) + M_2 \right) \right. \\
&\left. + \sum_{k=1}^{m} \big(M_1 (t_k - t_{k-1}) + M_3 + |\lambda| M_2 \big)(T - t_k) + M_1 \frac{(T - t_m)^2}{1 + q_m} + \rho^* |\lambda| (T - t_m) \right\} \\
&+ \frac{|\delta_4||\tau_2 - \tau_1|}{|\Omega|} \left\{ \sum_{k=1}^{m} \big(M_1 (t_k - t_{k-1}) + M_3 + |\lambda| M_2 \big) + M_1 (T - t_m) \right\} \\
&+ |\tau_2 - \tau_1| \rho^* |\lambda| + |\tau_2 - \tau_1| \sum_{k=1}^{i} \big(M_1 (t_k - t_{k-1}) + M_3 + |\lambda| M_2 \big) \\
&+ |\tau_2 - \tau_1| M_1 \frac{(\tau_2 + \tau_1 + 2t_i)}{1 + q_i}.
\end{aligned}
$$

The right-hand side of the above inequality is independent of x and tends to zero as $\tau_1 \to \tau_2$. As a consequence of Steps 1 to 3, together with the Arzelá-Ascoli theorem, we deduce that $\mathcal{S} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ is completely continuous.

Step 4: *We show that the set*

$$
E = \{ x \in PC(J,\mathbb{R}) : x = \kappa \mathcal{S} x \text{ for some } 0 < \kappa < 1 \}
$$

is bounded.

Let $x \in E$. Then $x(t) = \kappa(\mathcal{S}x)(t)$ for some $0 < \kappa < 1$. Thus, for each $t \in J$, we have

$$
\begin{aligned}
x(t) &= \kappa(\mathcal{S}x)(t) \\
&= \frac{\kappa(\delta_1 + \delta_2 t)}{\Omega} \left\{ \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f(r,x(r))\, d_{q_{k-1}} r\, d_{q_{k-1}} s \right. \right. \\
&\quad \left. - \lambda \int_{t_{k-1}}^{t_k} x(s)\, d_{q_{k-1}} s + I_k(x(t_k)) \right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k} f\bigl(s,x(s)\bigr)\,d_{q_{k-1}}s+I_k^*\bigl(x(t_k)\bigr)+\lambda I_k\bigl(x(t_k)\bigr)\biggr)(T-t_k)\\
&+\int_{t_m}^{T}\int_{t_m}^{s} f\bigl(r,x(r)\bigr)\,d_{q_m}r\,d_{q_m}s-\lambda\int_{t_m}^{T}x(s)\,d_{q_m}s\biggr\}\\
&+\frac{\kappa(\delta_3+\delta_4 t)}{\Omega}\biggl\{\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k} f\bigl(s,x(s)\bigr)\,d_{q_{k-1}}s+I_k^*\bigl(x(t_k)\bigr)+\lambda I_k\bigl(x(t_k)\bigr)\biggr)\\
&+\int_{t_m}^{T} f\bigl(s,x(s)\bigr)\,d_{q_m}s\biggr\}+\kappa\sum_{0<t_k<t}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} f\bigl(r,x(r)\bigr)\,d_{q_{k-1}}r\,d_{q_{k-1}}s\\
&-\lambda\int_{t_{k-1}}^{t_k}x(s)\,d_{q_{k-1}}s+I_k\bigl(x(t_k)\bigr)\biggr)\\
&+\kappa\sum_{0<t_k<t}\biggl(\int_{t_{k-1}}^{t_k} f\bigl(s,x(s)\bigr)\,d_{q_{k-1}}s+I_k^*\bigl(x(t_k)\bigr)+\lambda I_k\bigl(x(t_k)\bigr)\biggr)(t-t_k)\\
&+\kappa\int_{t_k}^{t}\int_{t_k}^{s} f\bigl(r,x(r)\bigr)\,d_{q_k}r\,d_{q_k}s-\kappa\lambda\int_{t_k}^{t}x(s)\,d_{q_k}s.
\end{aligned}
$$

This implies by (H_3) and (H_4) that for each $t\in J$, we have

$$
\begin{aligned}
\|x\| \le{}& \frac{|\delta_1|+|\delta_2|T}{|\Omega|}\biggl\{\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} M_1\,d_{q_{k-1}}r\,d_{q_{k-1}}s\\
&+|\lambda|\int_{t_{k-1}}^{t_k}\bigl|x(s)\bigr|\,d_{q_{k-1}}s+M_2\biggr)\\
&+\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k} M_1\,d_{q_{k-1}}s+M_3+|\lambda|M_2\biggr)(T-t_k)\\
&+\int_{t_m}^{T}\int_{t_m}^{s} M_1\,d_{q_m}r\,d_{q_m}s+|\lambda|\int_{t_m}^{T}\bigl|x(s)\bigr|\,d_{q_m}s\biggr\}\\
&+\frac{|\delta_3|+|\delta_4|T}{|\Omega|}\biggl\{\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k} M_1\,d_{q_{k-1}}s+M_3+|\lambda|M_2\biggr)+\int_{t_m}^{T}M_1\,d_{q_m}s\biggr\}\\
&+\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s} M_1\,d_{q_{k-1}}r\,d_{q_{k-1}}s+|\lambda|\int_{t_{k-1}}^{t_k}\bigl|x(s)\bigr|\,d_{q_{k-1}}s+M_2\biggr)\\
&+\sum_{k=1}^{m}\biggl(\int_{t_{k-1}}^{t_k} M_1\,d_{q_{k-1}}s+M_3+|\lambda|M_2\biggr)(T-t_k)\\
&+\int_{t_m}^{T}\int_{t_m}^{s} M_1\,d_{q_m}r\,d_{q_m}s+|\lambda|\int_{t_m}^{T}\bigl|x(s)\bigr|\,d_{q_m}s\\
\le{}& \frac{|\delta_1|+|\delta_2|T+|\Omega|}{|\Omega|}\biggl\{M_1\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+|\lambda|\|x\|T+mM_2\\
&+\sum_{k=1}^{m}\bigl(M_1(t_k-t_{k-1})+M_3+|\lambda|M_2\bigr)(T-t_k)\biggr\}\\
&+\frac{|\delta_3|+|\delta_4|T}{|\Omega|}\bigl\{M_1T+mM_3+m|\lambda|M_2\bigr\}.
\end{aligned}
$$

Setting

$$\begin{aligned}\Gamma = {}& \frac{|\delta_1| + |\delta_2|T + |\Omega|}{|\Omega|}\Biggl\{M_1 \sum_{k=1}^{m+1} \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} + mM_2 \\ &+ \sum_{k=1}^{m}\bigl(M_1(t_k - t_{k-1}) + M_3 + |\lambda|M_2\bigr)(T - t_k)\Biggr\} \\ &+ \frac{|\delta_3| + |\delta_4|T}{|\Omega|}\bigl\{M_1 T + mM_3 + m|\lambda|M_2\bigr\},\end{aligned}$$

we have

$$\|x\| \leq \frac{|\delta_1| + |\delta_2|T + |\Omega|}{|\Omega|}|\lambda|\|x\|T + \Gamma,$$

which yields

$$\|x\| \leq \frac{\Gamma}{1 - \frac{|\delta_1|+|\delta_2|T+|\Omega|}{|\Omega|}|\lambda|T} := M.$$

This shows that the set E is bounded. As a consequence of Schaefer's fixed point theorem, we conclude that $\mathcal{S}$ has a fixed point which is a solution of the impulsive q_k-difference Langevin boundary value problem (1.4). □

4 Examples

Example 4.1 Consider the following boundary value problem for the second-order impulsive q_k-difference Langevin equation:

$$\begin{cases} D_{(\frac{2k+1}{5k+2})^{\frac{1}{2}}}(D_{(\frac{2k+1}{5k+2})^{\frac{1}{2}}} + \frac{1}{10})x(t) = \frac{t}{e^{2t}(10+t)^2} \cdot \frac{|x(t)|}{(1+|x(t)|)}, & t \in J = [0,1], t \neq t_k, \\ \Delta x(t_k) = \frac{|x(t_k)|}{9(9+|x(t_k)|)}, \quad t_k = \frac{k}{10}, k = 1,2,\ldots,9, \\ D_{(\frac{2k+1}{5k+2})^{\frac{1}{2}}} x(t_k^+) - D_{(\frac{2k-1}{5k-3})^{\frac{1}{2}}} x(t_k) = \frac{1}{8}\tan^{-1}(\frac{1}{10}x(t_k)), & t_k = \frac{k}{10}, k = 1,2,\ldots,9, \\ \frac{1}{7}x(0) + \frac{2}{9}D_{\frac{1}{\sqrt{2}}}x(0) = x(1), \qquad \frac{2}{7}x(0) + \frac{1}{9}D_{\frac{1}{\sqrt{2}}}x(0) = D_{\sqrt{\frac{19}{47}}}x(1). \end{cases} \tag{4.1}$$

Here $q_k = \sqrt{(2k+1)/(5k+2)}$, $k = 0,1,2,\ldots,9$, $m = 9$, $T = 1$, $\lambda = 1/10$, $\alpha = 1/7$, $\beta = 2/9$, $\gamma = 2/7$, $\eta = 1/9$, $f(t,x) = (t|x(t)|)/(e^{2t}(10+t)^2(1+|x(t)|))$, $I_k(x) = |x|/(9(9+|x|))$, and $I_k^*(x) = (1/8)\tan^{-1}(x/10)$. Since

$$\begin{aligned}&\bigl|f(t,x) - f(t,y)\bigr| \leq (1/100)|x-y|, \\ &\bigl|I_k(x) - I_k(y)\bigr| \leq (1/81)|x-y| \quad \text{and} \quad \bigl|I_k^*(x) - I_k^*(y)\bigr| \leq (1/80)|x-y|,\end{aligned}$$

then (H_1) and (H_2) are satisfied with $L_1 = (1/100)$, $L_2 = (1/81)$, $L_3 = (1/80)$. We can find that $\Omega = 3{,}103/3{,}150$, $\delta_1 = (-13)/15$, $\delta_2 = 46/75$, $\delta_3 = 7/9$, $\delta_4 = (-277)/315$ and thus

$$\Lambda_1 \approx 0.920497882 < 1.$$

Hence, by Theorem 3.1, the boundary value problem (4.1) has a unique solution on $[0,1]$.

Example 4.2 Consider the following boundary value problem for the second-order impulsive q_k-difference Langevin equation:

$$\begin{cases} D_{(\frac{k+1}{3k+4})^2}(D_{(\frac{k+1}{3k+4})^2} + \frac{1}{5})x(t) = \frac{3t^3}{(4+x^2)^{\frac{1}{2}}}, & t \in J = [0,1], t \neq t_k, \\ \Delta x(t_k) = \frac{2k\cos^2 \pi t}{k+t^2|x(t_k)|}, & t_k = \frac{k}{10}, k = 1,2,\ldots,9, \\ D_{(\frac{k+1}{3k+4})^2} x(t_k^+) - D_{(\frac{k}{3k+1})^2} x(t_k) = \frac{4\sin((\pi t)/2)}{3k+|x(t_k)|\cos^2 2t}, & t_k = \frac{k}{10}, k = 1,2,\ldots,9, \\ \frac{1}{4}x(0) + \frac{1}{5}D_{\frac{1}{16}} x(0) = x(1), & \frac{2}{9}x(0) + \frac{1}{7}D_{\frac{1}{16}} x(0) = D_{\frac{100}{961}} x(1). \end{cases} \tag{4.2}$$

Here $q_k = ((k+1)/(3k+4))^2$, $k = 0,1,2,\ldots,9$, $m = 9$, $T = 1$, $\lambda = 1/5$, $\alpha = 1/4$, $\beta = 1/5$, $\gamma = 2/9$, $\eta = 1/7$, $f(t,x) = ((3t^3)/(4+x^2)^{1/2})$, $I_k(x) = ((2k\cos^2 \pi t)/(k+t^2|x|))$, and $I_k^*(x) = ((4\sin((\pi t)/2))/(3k+|x|\cos^2 2t))$. Clearly,

$$\left|f(t,x)\right| = \left|\frac{3t^3}{(4+x^2)^{\frac{1}{2}}}\right| \leq \frac{3}{2}, \qquad \left|I_k(x)\right| = \left|\frac{2k\cos^2 \pi t}{k+t^2|x|}\right| \leq 2$$

and

$$\left|I_k^*(x)\right| = \left|\frac{4\sin((\pi t)/2)}{3k+|x|\cos^2 2t}\right| \leq \frac{4}{3}.$$

We can find that

$$\frac{|\delta_1| + |\delta_2|T + |\Omega|}{|\Omega|}|\lambda|T = \frac{13{,}958}{26{,}273} < 1,$$

where $\Omega = (\alpha-1)(\eta-1) - \lambda T(\eta+\beta\lambda-\alpha) + \gamma(T-\beta) = 26{,}273/31{,}500$, $\delta_1 = \eta - 1 + \beta\lambda = (-143)/175$ and $\delta_2 = \lambda(\eta+\beta\lambda-1-\alpha) + 1 - \gamma = 17{,}777/31{,}500$.

Hence, by Theorem 3.2, the boundary value problem (4.2) has at least one solution on $[0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally in this article. They read and approved the final manuscript.

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