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On the maximum principle for elliptic operators in weighted spaces

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Abstract

We establish a maximum principle for subsolutions of second order elliptic equations. In particular, we consider some linear operators with leading coefficients locally VMO, while the other coefficients and the boundary conditions involve a suitable weight function.

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1 Introduction

It is well known that *a priori* estimates and uniqueness results, which are necessary in the proof of the well-posedness for boundary value problems for elliptic equations in nondivergence form, are based on Aleksandrov type estimates, *i.e.*, on estimates for the maximum of a solution in terms of the L^n -norm of the right-hand side.

If Ω is a bounded domain in \mathbb{R}^n ($n > 2$) and

$$L = \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a, \quad (1.1)$$

is a uniformly elliptic operator in Ω , the classical result of AD Aleksandrov states that if $u \in C^0(\bar{\Omega}) \cap W^{2,n}(\Omega)$, with $u \leq 0$ in $\partial\Omega$, verifies $Lu \geq f$, where $a_i, a, f \in L^n(\Omega)$ ($a \leq 0$), then

$$\sup_{\Omega} u \leq c \|f\|_{L^n(\Omega)}, \quad (1.2)$$

where $c \in \mathbb{R}_+$ depends only on n , Ω , $\|a_i\|_{L^n(\Omega)}$ and on the ellipticity constant.

There have been various directions of developments and extensions of Aleksandrov estimate. For example, maximum principles have been established in different types of boundary problems, such as in the stationary oblique derivative problem or in the stationary Venttsel' problem. Another direction of development of the Aleksandrov ideas is the extension of maximum estimates to equations with lower order coefficients and right-hand sides in other function classes (for example, in spaces with anisotropic norms or weighted spaces). In particular, a large number of works is devoted to the weakening of requirements for the right-hand side of the equation considered (see, for example, [1] and its large bibliography).

In this framework, it is well known that additional hypotheses on the leading coefficients are necessary to obtain the estimates. Several authors have obtained estimates for the maximum of a solution through the L^p -norms of the right-hand side ($p > n/2$) under different conditions on the leading coefficients.

For instance, if Ω is an arbitrary open subset of \mathbb{R}^n and $p \in]n/2, +\infty[$, a bound of type (1.2) and a consequent uniqueness result can be found in [2]. In fact, it has been proved that, if the coefficients a_{ij} are bounded and locally VMO, the coefficients a_i, a satisfy suitable summability conditions, and $\text{ess sup}_\Omega a < 0$, then for any solution u of the problem

$$\begin{cases} u \in W_{\text{loc}}^{2,p}(\Omega) \cap C^0(\bar{\Omega}), \\ Lu \geq f, \quad f \in L_{\text{loc}}^p(\Omega), \\ u|_{\partial\Omega} \leq 0, \\ \limsup_{|x| \rightarrow +\infty} u(x) \leq 0 \quad \text{if } \Omega \text{ is unbounded,} \end{cases} \tag{1.3}$$

there exist a ball $B \subset\subset \Omega$ and a constant $c \in \mathbb{R}_+$ such that

$$\sup_{\Omega} u \leq c \left(\int_B |f^-|^p dx \right)^{\frac{1}{p}}, \tag{1.4}$$

where f^- is the negative part of f ,

$$\int_B |f^-|^p dx = \frac{1}{|B|} \int_B |f^-|^p dx,$$

and c depends on n, p , on the ellipticity constant and on the regularity of the coefficients of L .

If the boundary of a domain has various singularities, as for example corners or edges, then, in accordance with the linear theory, it is natural to assume that the lower order coefficients and the right-hand side of the equation belong to some weighted spaces L^p , where the weight is usually a power of the distance function from the ‘singular set’ on the boundary of domain. In these cases, the estimates on the solutions are obtained in terms of such weight function.

For instance, if ρ is a bounded weight function related to the distance function from a non-empty subset S_ρ of the boundary of an arbitrary domain Ω , not necessarily bounded and regular (see Section 2 for the definition of such weight function), in [3] has been studied a problem similar to the problem (1.3) with boundary conditions and data related to the weight function ρ . In particular, if $s \in \mathbb{R}, S_\rho = \partial\Omega$, the coefficients a_{ij} are bounded and locally VMO, the coefficients a_i, a belong to suitable weighted spaces L^∞ , in [3] the author has proved that the solution u of the problem

$$\begin{cases} u \in W_{\text{loc}}^{2,p}(\Omega), \\ Lu \geq f, \quad f \in L_{\text{loc}}^p(\Omega), \\ \limsup_{x \rightarrow x_0} \rho^s(x)u(x) \leq 0, \quad \forall x_0 \in \partial\Omega, \\ \limsup_{|x| \rightarrow +\infty} \rho^s(x)u(x) \leq 0 \quad \text{if } \Omega \text{ is unbounded,} \end{cases} \tag{1.5}$$

verifies the estimate

$$\sup_{x \in \Omega} \rho^s(x)u(x) \leq c \left(\int_B |\rho^{s+2} f^-|^p dx \right)^{\frac{1}{p}}, \tag{1.6}$$

where $B \subset \subset \Omega$ is an open ball and the constant $c \in \mathbb{R}_+$ depends on n, p, s, ρ , on the ellipticity constant and on the regularity of the coefficients of L . As a consequence, some uniqueness results are also obtained. Results of this type are also established in [4] under the more general hypothesis $\emptyset \neq S_\rho \subset \partial\Omega$, but for an operator L with coefficients $a_i = 0$.

The aim of this paper is to improve the above quoted results in [3] by obtaining a similar estimate under much weaker assumptions. In particular, the main difference lies in the hypotheses on the coefficients a_i, a which are not supposed to belong to weighted spaces L^∞ but just to appropriate weighted Sobolev spaces $K_t^r(\Omega)$ (see Section 2 for the definition of such weighted spaces), which strictly contain the weighted spaces L^∞ . Moreover, as in [4], we consider the more general hypothesis $\emptyset \neq S_\rho \subset \partial\Omega$.

2 Notation

In this section we introduce some notation used throughout this paper. Moreover, we recall the definitions of a class of weight functions and of some function spaces in which the coefficients of our operator will be chosen.

Let A be a Lebesgue measurable subset of \mathbb{R}^n and let $\Sigma(A)$ be the collection of all Lebesgue measurable subsets of A . If $F \in \Sigma(A)$, we denote by $|F|$ the Lebesgue measure of F and by $\mathcal{D}(F)$ the class of restrictions to F of functions $\zeta \in C_0^\infty(\mathbb{R}^n)$ with $\bar{F} \cap \text{supp } \zeta \subseteq F$. Moreover, if $X(F)$ is a space of functions defined on F , we denote by $X_{\text{loc}}(F)$ the class of all functions $g : F \rightarrow \mathbb{R}$ such that $\zeta g \in X(F)$ for all $\zeta \in \mathcal{D}(F)$. Furthermore, for $g \in L^p(A)$ ($p \geq 1$), we put

$$\omega^p[g, A](t) = \sup_{\substack{E \in \Sigma(A) \\ |E| \leq t}} \|g\|_{L^p(E)}, \quad t \in \mathbb{R}_+.$$

Since $\omega^p[g, A](t)$ is a decreasing function and $\lim_{t \rightarrow 0} \omega^p[g, A](t) = 0$, we can refer to $\omega^p[g, A]$ as the *modulus of continuity* of g in $L^p(A)$.

Let Ω be an open subset of \mathbb{R}^n , $n \geq 2$. We denote by $\mathcal{A}(\Omega)$ the class of measurable weight functions $\rho : \Omega \rightarrow \mathbb{R}_+$ such that

$$\gamma^{-1}\rho(y) \leq \rho(x) \leq \gamma\rho(y), \quad \forall y \in \Omega, \forall x \in \Omega \cap B(y, \rho(y)), \tag{2.1}$$

where $\gamma \in \mathbb{R}_+$ is independent of x and y , and $B(y, \rho(y))$ is the open ball of radius $\rho(y)$ centered at y .

We remark that $\mathcal{A}(\Omega)$ contains the class of all functions $\rho : \Omega \rightarrow \mathbb{R}_+$ which are Lipschitz continuous in Ω with Lipschitz constant less than 1.

Typical examples of functions $\rho \in \mathcal{A}(\Omega)$ are the function

$$x \in \Omega \rightarrow 1 + a|x|, \quad a \in]0, 1[,$$

if $\Omega = \mathbb{R}^n$ and, if $\Omega \neq \mathbb{R}^n$ and S is a nonempty subset of $\partial\Omega$, the function

$$x \in \Omega \rightarrow a \cdot \text{dist}(x, S), \quad a \in]0, 1[.$$

For any $\rho \in \mathcal{A}(\Omega)$ we put

$$S_\rho = \{z \in \partial\Omega \mid \rho(x) \leq |x - z| \forall x \in \Omega\}. \tag{2.2}$$

We recall that the set S_ρ is a closed subset of $\partial\Omega$ and

$$z \in S_\rho \iff \lim_{x \rightarrow z} \rho(x) = 0$$

(see [5]).

It is well known that

$$\rho \in L^\infty_{\text{loc}}(\bar{\Omega}), \quad \rho^{-1} \in L^\infty_{\text{loc}}(\bar{\Omega} \setminus S_\rho), \tag{2.3}$$

and, if $S_\rho \neq \emptyset$ [5, 6],

$$\rho(x) \leq \text{dist}(x, S_\rho), \quad \forall x \in \Omega. \tag{2.4}$$

Let $\rho \in \mathcal{A}(\Omega)$. For $k \in \mathbb{N}_0$, $1 \leq p \leq +\infty$ and $s \in \mathbb{R}$, we denote by $W_s^{k,p}(\Omega)$ the space of distributions u on Ω such that $\rho^{s+|\alpha|-k} \partial^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq k$. We observe that $W_s^{k,p}(\Omega)$ is a Banach space with the norm defined by

$$\|u\|_{W_s^{k,p}(\Omega)} = \sum_{|\alpha| \leq k} \|\rho^{s+|\alpha|-k} \partial^\alpha u\|_{L^p(\Omega)}.$$

Moreover, it is separable if $1 \leq p < +\infty$, reflexive if $1 < p < +\infty$, and, in particular, $W_s^{k,2}(\Omega)$ is an Hilbert space. We put $W_s^{0,p}(\Omega) = L_s^p(\Omega)$, and we observe that the space $C_0^\infty(\Omega)$ is dense in $L_s^p(\Omega)$ (see [7, 8]).

A more detailed account of properties of the above defined weighted Sobolev spaces can be found in [7, 9] and [8].

For any $x \in \Omega$, we put

$$\Omega(x) = \Omega \cap B(x, \rho(x)). \tag{2.5}$$

Let $\rho \in \mathcal{A}(\Omega)$. For $1 \leq p < +\infty$ and $s \in \mathbb{R}$, we denote by $K_s^p(\Omega)$ the class of functions $g \in L^\infty_{\text{loc}}(\bar{\Omega} \setminus S_\rho)$ such that

$$\|g\|_{K_s^p(\Omega)} = \sup_{x \in \Omega} (\rho^{s-\frac{n}{p}}(x) \|g\|_{L^p(\Omega(x))}) < +\infty. \tag{2.6}$$

Obviously $K_s^p(\Omega)$ is a Banach space with the norm defined by (2.6). It is easy to prove that the space $L_s^\infty(\Omega)$ is a subset of $K_s^p(\Omega)$ (see [10]). Thus, we can define a new space of functions $\tilde{K}_s^p(\Omega)$ as the closure of $L_s^\infty(\Omega)$ in $K_s^p(\Omega)$.

We recall the following characterization of the above defined space (see [10]):

$$g \in \tilde{K}_s^p(\Omega) \iff g \in K_s^p(\Omega) \text{ and } \lim_{t \rightarrow 0} \left(\sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x \in \Omega} \frac{|\Omega(x) \cap E|}{\rho^n(x)} \leq t}} \|g \chi_E\|_{K_s^p(\Omega)} \right) = 0,$$

where χ_E denotes the characteristic function of the set E .

Therefore, we define *modulus of continuity* of g in $\tilde{K}_s^p(\Omega)$ as a map $\tilde{\omega}_s^p[g] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that [11]

$$\begin{aligned} \sup_{E \in \Sigma(\Omega)} \|g \chi_E\|_{K_s^p(\Omega)} &\leq \tilde{\omega}_s^p[g](t), \\ \sup_{x \in \Omega} \frac{|\Omega(x) \cap E|}{\rho^n(x)} &\leq t \end{aligned} \tag{2.7}$$

$$\lim_{t \rightarrow 0} \tilde{\omega}_s^p[g](t) = 0.$$

Further properties of above mentioned function spaces can be found in [5, 10], and [11].

If Ω has the property

$$|\Omega(x, r)| \geq Ar^n, \quad \forall x \in \Omega, \forall r \in]0, 1], \tag{2.8}$$

where $\Omega(x, r) = B(x, r) \cap \Omega$ and A is a positive constant independent of x and r , it is possible to consider the space $BMO(\Omega, t)$ ($t \in \mathbb{R}_+$) composed by all functions $g \in L^1_{loc}(\bar{\Omega})$ such that

$$[g]_{BMO(\Omega, t)} = \sup_{\substack{x \in \Omega \\ r \in]0, t]}} \int_{\Omega(x, r)} \left| g - \int_{\Omega(x, r)} g \right| < +\infty,$$

where

$$\int_{\Omega(x, r)} g = |\Omega(x, r)|^{-1} \int_{\Omega(x, r)} g.$$

If $g \in BMO(\Omega) = BMO(\Omega, t_A)$, with

$$t_A = \sup_{t \in \mathbb{R}_+} \left(\sup_{\substack{x \in \Omega \\ r \in]0, t]}} \frac{r^n}{|\Omega(x, r)|} \leq \frac{1}{A} \right),$$

we will say that $g \in VMO(\Omega)$ if $[g]_{BMO(\Omega, t)} \rightarrow 0$ for $t \rightarrow 0^+$. A function $\eta[g] : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is called a *modulus of continuity* of g in $VMO(\Omega)$ if

$$[g]_{BMO(\Omega, t)} \leq \eta[g](t), \quad \forall t \in \mathbb{R}_+, \quad \lim_{t \rightarrow 0^+} \eta[g](t) = 0.$$

We say that $g \in VMO_{loc}(\Omega)$ if $(\zeta g)_o \in VMO(\mathbb{R}^n)$ for any $\zeta \in C_0^\infty(\Omega)$, where $(\zeta g)_o$ denotes the zero extension of ζg outside of Ω . A more detailed account of properties of the above defined spaces $BMO(\Omega)$ and $VMO(\Omega)$ can be found in [12].

We conclude this section introducing a class of applications needed in the sequel.

From now on we consider $\rho \in \mathcal{A}(\Omega) \cap L^\infty(\Omega)$ and we suppose that the following condition on ρ holds:

(h₀) there exists a function $\sigma \in \mathcal{A}(\Omega) \cap C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$ which is equivalent to ρ and such that

$$|\partial^\alpha \sigma(x)| \leq c_\alpha \sigma^{1-|\alpha|}(x), \quad \forall x \in \Omega, \forall \alpha \in \mathbb{N}_0^n,$$

where c_α is independent of x (see [6]).

We observe that the condition (h_0) holds, for example, if Ω is an unbounded open set with the cone property, or if the open set Ω has not the cone property but the weight function ρ is equivalent to the function $\text{dist}(\cdot, \partial\Omega)$ (see [6]).

Let us fix $g \in C_0^\infty(\mathbb{R}_+)$ satisfying the conditions

$$0 \leq g \leq 1, \quad g(t) = 1 \quad \text{if } t \geq 1, \quad g(t) = 0 \quad \text{if } t \leq \frac{1}{2}. \tag{2.9}$$

For each $k \in \mathbb{N}$, we put

$$\eta_k(x) = \frac{1}{k} \zeta_k(x) + (1 - \zeta_k(x)) \cdot \sigma(x), \quad x \in \Omega,$$

where $\zeta_k(x) = g(k\sigma(x))$, $x \in \Omega$. Obviously, $\eta_k \in C^\infty(\Omega)$ for any $k \in \mathbb{N}$ and

$$\eta_k(x) = \begin{cases} \frac{1}{k} & \text{if } x \in \bar{\Omega}_k, \\ \sigma(x) & \text{if } x \in \Omega \setminus \Omega_{2k}, \end{cases}$$

where

$$\Omega_k = \left\{ x \in \Omega \mid \sigma(x) > \frac{1}{k} \right\}. \tag{2.10}$$

Moreover, for $k \in \mathbb{N}$, it is easy to prove that

$$\sigma(x) \leq \eta_k(x) \leq 2\sigma(x), \quad x \in \Omega \setminus \bar{\Omega}_k, \tag{2.11}$$

$$c'_k \sigma(x) \leq \eta_k(x) \leq \sigma(x), \quad x \in \Omega_k, \tag{2.12}$$

$$(\eta_k(x))_x \leq c_1 (\sigma(x))_x, \quad x \in \Omega, \tag{2.13}$$

$$(\eta_k(x))_{xx} \leq c_2 \frac{(\sigma(x))_x^2 + \sigma(x) \cdot (\sigma(x))_{xx}}{\sigma(x)}, \quad x \in \Omega, \tag{2.14}$$

where $c'_k \in \mathbb{R}_+$ depends on k and σ , and $c_1, c_2 \in \mathbb{R}_+$ depend only on n . Furthermore, for any $s \in \mathbb{R}$, we have

$$\frac{(\eta_k^s(x))_x}{\eta_k^s(x)} \leq c_3 \frac{(\eta_k(x))_x}{\sigma(x)}, \quad x \in \Omega, \tag{2.15}$$

$$\frac{(\eta_k^s(x))_{xx}}{\eta_k^s(x)} \leq c_3 \frac{(\eta_k(x))_x^2 + \eta_k(x) \cdot (\eta_k(x))_{xx}}{\sigma^2(x)}, \quad x \in \Omega, \tag{2.16}$$

where $c_3 \in \mathbb{R}_+$ depends on s and n .

3 Hypotheses and preliminary results

Suppose that Ω has the property (2.8) and let $p > n/2$. Consider in Ω the differential operator \tilde{L} defined by

$$\tilde{L} = \sum_{ij=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n d_i \frac{\partial}{\partial x_i} + d,$$

with the following assumptions on the coefficients:

$$\begin{aligned}
 (h_1) \quad & \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO_{loc}(\Omega), & i, j = 1, \dots, n, \\ \exists \nu_o, \nu \in \mathbb{R}_+ : \sum_{i,j=1}^n \|a_{ij}\|_{L^\infty(\Omega)} \leq \nu_o, \\ \sum_{i,j=1}^n \xi_i \xi_j a_{ij} \geq \nu |\xi|^2 & \text{a.e. in } \Omega, \forall \xi \in \mathbb{R}^n, \end{cases} \\
 (i_1) \quad & \begin{cases} d_i = a_i + \tilde{d}_i, & a_i \in \tilde{K}_1^r(\Omega), \tilde{d}_i \in L_1^\infty(\Omega), i = 1, \dots, n, \\ d = a + \tilde{d}, & a \in \tilde{K}_2^p(\Omega), \tilde{d} \in L_2^\infty(\Omega), \\ d \leq 0 & \text{a.e. in } \Omega, \end{cases}
 \end{aligned}$$

where $r > n$ if $p \leq n$, and $r = p$ if $p > n$.

Fixing $x_o \in \Omega$ and $\tau \in \mathbb{R}_+$ such that $\tau \leq \sigma(x_o)$, we put $B = B(x_o, \tau)$ and $B^* = B(x_o, 1)$.

We observe that under assumptions (h_1) and (i_1) , the operator \tilde{L} from $W^{2,p}(B)$ into $L^p(B)$ is bounded and the following estimate holds:

$$\|\tilde{L}u\|_{L^p(B)} \leq c' \|u\|_{W^{2,p}(B)}, \quad \forall u \in W^{2,p}(B),$$

where $c' \in \mathbb{R}_+$ depends on $n, p, r, \rho, \nu_o, \|a_i\|_{L^r(B)}, \|\tilde{d}_i\|_{L^\infty(B)}, \|a\|_{L^p(B)}, \|\tilde{d}\|_{L^\infty(B)}$.

Let v be a solution of the problem

$$\begin{cases} v \in W^{2,p}(B), \\ \tilde{L}v \geq h, & h \in L^p(B), \\ v|_{\partial B} \leq 0. \end{cases} \tag{3.1}$$

We want to prove a bound for the solution v of the above problem (see Lemma 3.1 below), which will be the primary technical tool in the proof of our main result (see the next Section). In order to use a classical result of Vitanza (see, Theorem 2.1 in [13]) it is necessary to make an appropriate change of variables which allows to transform the operator \tilde{L} into a differential operator \tilde{L}^* whose lower order coefficients, in particular, belonging to Lebesgue spaces and their moduli of continuity can be estimated by moduli of continuity of the corresponding coefficients of \tilde{L} . To this aim, let us consider the map $T : B \rightarrow B^*$ defined by

$$T(x) = x_o + \frac{x - x_o}{\tau}. \tag{3.2}$$

Clearly

$$z = T(x) \iff x = x_o + \tau(z - x_o) = T^{-1}(z).$$

For any function g defined on B , we set

$$g^* = g \circ T^{-1}. \tag{3.3}$$

Using the equivalence between ρ and σ it is easy to prove that $\tau d_i^* \in L^r(B^*)$ for any $i = 1, \dots, n$ and $\tau^2 d^* \in L^p(B^*)$; moreover,

$$\|\tau d_i^*\|_{L^r(B^*)} \leq c_1 (\|a_i\|_{K_1^r(\Omega)} + \|\tilde{d}_i\|_{L_1^\infty(\Omega)}^{\frac{1}{r}}), \quad i = 1, \dots, n, \tag{3.4}$$

and

$$\|\tau^2 d^*\|_{L^p(B^*)} \leq c_2 (\|a\|_{K_2^p(\Omega)} + \|\tilde{d}\|_{L_2^\infty(\Omega)}^{\frac{1}{p}}), \tag{3.5}$$

where $c_1 \in \mathbb{R}_+$ depends on n, ρ, r and $c_2 \in \mathbb{R}_+$ depends on n, ρ, p .

On the other hand, for any $E^* \in \Sigma(B^*)$ and $t \in \mathbb{R}_+$, we have $|E^*| \leq t$ if and only if $\frac{|E|}{\tau^n} \leq t$ where $E = \{x \in B \mid x = T^{-1}(z), z \in E^*\}$. Thus, we obtain

$$\begin{aligned} \omega^r[\tau d_i^*, B^*](t) &\leq \sup_{\substack{E \in \Sigma(B) \\ \frac{|E|}{\sigma^n(x_0)} \leq t}} \|\tau^{1-\frac{r}{p}} d_i\|_{L^p(E)}, \quad i = 1, \dots, n, \\ \omega^p[\tau^2 d^*, B^*](t) &\leq \sup_{\substack{E \in \Sigma(B) \\ \frac{|E|}{\sigma^n(x_0)} \leq t}} \|\tau^{2-\frac{p}{p}} d\|_{L^p(E)}. \end{aligned} \tag{3.6}$$

Using again the equivalence between ρ and σ and (2.7), from (3.6) we also deduce

$$\omega^r[\tau d_i^*, B^*](t) \leq c_3 (\tilde{\omega}_1^r[a_i](t) + \|\tilde{d}_i\|_{L_1^\infty(\Omega)}^{\frac{1}{r}} \cdot t^{\frac{1}{r}}), \quad i = 1, \dots, n, \tag{3.7}$$

and

$$\omega^p[\tau^2 d^*, B^*](t) \leq c_4 (\tilde{\omega}_2^p[a](t) + \|\tilde{d}\|_{L_2^\infty(\Omega)}^{\frac{1}{p}} \cdot t^{\frac{1}{p}}), \tag{3.8}$$

where $c_3 \in \mathbb{R}_+$ depends on ρ, r and $c_4 \in \mathbb{R}_+$ depends on ρ, p .

We are now able to prove the requested *a priori* bound.

Lemma 3.1 *Suppose that the conditions (h₁) and (i₁) hold. Let v be a solution of the problem (3.1). Then there exists c_o ∈ ℝ₊ such that*

$$\sup_B v \leq c_o \cdot \tau^{2-\frac{n}{p}} \|h^-\|_{L^p(B)}, \tag{3.9}$$

where c_o depends on $n, p, r, \rho, \nu, \nu_0, [p(a_{ij})]_{BMO(\mathbb{R}^n, \cdot)}, \|a_i\|_{K_1^r(\Omega)}, \|a\|_{K_2^p(\Omega)}, \|\tilde{d}_i\|_{L_1^\infty(\Omega)}, \|\tilde{d}\|_{L_2^\infty(\Omega)}, \tilde{\omega}_1^r[a_i], \tilde{\omega}_2^p[a]$, and where $p(a_{ij})$ are the extensions of a_{ij} to \mathbb{R}^n in $L^\infty(\mathbb{R}^n) \cap VMO(\mathbb{R}^n)$ for any $i, j = 1, \dots, n$.

Proof Let $v \in W^{2,p}(B)$. Taking into account the definitions (3.2) and (3.3), it is easily seen that

$$\begin{aligned} (\tilde{L}v)^* &= \sum_{i,j=1}^n a_{ij}^*(v_{x_i x_j})^* + \sum_{i=1}^n d_i^*(v_{x_i})^* + d^* v^* \\ &= \tau^{-2} \sum_{i,j=1}^n a_{ij}^* v_{z_i z_j}^* + \tau^{-1} \sum_{i=1}^n d_i^* v_{z_i}^* + d^* v^*, \end{aligned}$$

and hence

$$\tau^2 (\tilde{L}v)^* = \tilde{L}^* v^*,$$

where

$$\tilde{L}^* = \sum_{i,j=1}^n a_{ij}^* \frac{\partial^2}{\partial z_i \partial z_j} + \tau \sum_{i=1}^n a_i^* \frac{\partial}{\partial z_i} + \tau^2 d^*.$$

Let us denote by $p(a_{ij})$ the extensions of a_{ij} to \mathbb{R}^n such that

$$p(a_{ij}) \in L^\infty(\mathbb{R}^n) \cap VMO(\mathbb{R}^n), \quad i, j = 1, \dots, n \tag{3.10}$$

(for the existence of such functions see Theorem 5.1 in [12]). Since

$$p(a_{ij})^* \in L^\infty(\mathbb{R}^n) \cap VMO(\mathbb{R}^n), \quad p(a_{ij})^*|_{B^*} = a_{ij}^*, \quad i, j = 1, \dots, n, \tag{3.11}$$

we have

$$a_{ij}^* \in L^\infty(B^*) \cap VMO(B^*), \quad i, j = 1, \dots, n. \tag{3.12}$$

Moreover, from assumptions (h_1) , (i_1) , and (3.4), (3.5) it follows that

$$\begin{cases} a_{ij}^* = a_{ji}^*, & i, j = 1, \dots, n, \\ \sum_{i,j=1}^n \xi_i \xi_j a_{ij}^* \geq \nu |\xi|^2 & \text{a.e. in } B^*, \forall \xi \in \mathbb{R}^n, \\ \tau d_i^* \in L^r(B^*), \quad i = 1, \dots, n, \quad \tau^2 d^* \in L^p(B^*), \quad d^* \leq 0 \text{ a.e. in } B^*, \end{cases} \tag{3.13}$$

where r and p are as in hypothesis (i_1) .

Consider now the following problem:

$$\begin{cases} \tilde{L}^* w = g, & g \in L^p(B^*), \\ w \in W^{2,p}(B^*) \cap \dot{W}^{1,p}(B^*). \end{cases} \tag{3.14}$$

Putting together (3.11) and (3.13) with Theorem 2.1 of [13] if $n \geq 3$ or with Theorem 3.5 of [14] if $n = 2$, it follows that there exists a unique solution w of (3.14) satisfying the estimate

$$\|w\|_{W^{2,p}(B^*)} \leq K \|g\|_{L^p(B^*)}, \tag{3.15}$$

where $K \in \mathbb{R}_+$ depends on $n, p, \nu, \nu_0, [p(a_{ij})^*]_{BMO(\mathbb{R}^n, \cdot)}, \|\tau \cdot d_i^*\|_{L^r(B^*)}, \|\tau^2 \cdot d^*\|_{L^p(B^*)}, \omega^r[\tau d_i^*, B^*], \omega^p[\tau^2 d^*, B^*]$.

Thus from (3.15) and classical Sobolev embedding theorems (see Lemma 5.15 in [15]) we deduce that there exists $K_1 \in \mathbb{R}_+$, depending on the same parameters as K , such that

$$\max_{B^*} |w| \leq K_1 \|g\|_{L^p(B^*)}, \tag{3.16}$$

and hence for each $z \in B^*$ there is a function $G(z, \cdot) \in L^{p'}(B^*)$ ($1/p + 1/p' = 1$) such that

$$w(z) = - \int_{B^*} G(z, y) \cdot g(y) \, dy. \tag{3.17}$$

The map $G(z, \cdot)$ is the Green function for the operator \tilde{L}^* in B^* and it has the following properties:

$$\int_{B^*} G(z, y) \cdot \tilde{g}(y) dy \geq 0, \quad \forall \tilde{g} \in L^p(B^*), \tilde{g} \geq 0, \tag{3.18}$$

$$\|G(z, \cdot)\|_{L^{p'}(B^*)} \leq K_1. \tag{3.19}$$

Setting $g = \tilde{L}^*v^*$ in (3.14), we find that the function $w - v^*$, belonging to $W^{2,p}(B^*)$, is a solution of the following problem:

$$\begin{cases} \tilde{L}^*(w - v^*) = 0 & \text{in } B^*, \\ (w - v^*)|_{\partial B^*} = -v^*|_{\partial B^*} \geq 0. \end{cases} \tag{3.20}$$

Moreover, from (3.12), (3.13) and Lemma 3.1 of [16] (see also Lemma 3.1 of [2] for the case $n \geq 3$) it follows that $w - v^* \geq 0$ in B^* . Finally, applying (3.17) with $g = \tilde{L}^*v^* = \tau^2(\tilde{L}v)^*$ and using (3.18) and (3.19) we obtain

$$\begin{aligned} v^*(z) &\leq - \int_{B^*} G(z, y) \cdot \tau^2(\tilde{L}v)^*(y) dy \\ &\leq -\tau^2 \int_{B^*} G(z, y) \cdot h^*(y) dy \leq -2\tau^2 \int_{B^*} G(z, y) \cdot (h^*)^-(y) dy \\ &\leq 2\tau^2 \|G(z, \cdot)\|_{L^{p'}(B^*)} \cdot \|(h^*)^-\|_{L^p(B^*)} \leq 2\tau^2 \cdot K_1 \|(h^*)^-\|_{L^p(B^*)}, \quad \forall z \in B^*. \end{aligned} \tag{3.21}$$

From (3.21), converting back to the x -variables ($z = T(x)$), we easily deduce the estimate (3.9). □

4 Main results

In this section we use the previous result to prove a bound for the solution of our main problem.

Consider in Ω the differential operator L defined by

$$L = \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a,$$

and put

$$L_o = \sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Suppose that the leading coefficients of operator L satisfy the assumption (h_1) while the lower order coefficients verify the following condition:

$$(h_2) \quad \begin{cases} a_i \in \tilde{K}_1^r(\Omega), & i = 1, \dots, n, \\ a \in \tilde{K}_2^p(\Omega), \\ \exists a_o \in \mathbb{R}_+ : \text{ess sup}_{\Omega} \sigma^2 a = -a_o, \end{cases}$$

where r and p are as in hypothesis (i₁). Moreover, assume that the following condition on ρ holds:

$$(h_3) \quad \lim_{k \rightarrow +\infty} \left(\sup_{\Omega \setminus \Omega_k} ((\sigma(x))_x + \sigma(x)(\sigma(x))_{xx}) \right) = 0,$$

where Ω_k is defined in (2.10). For an example of function ρ whose regularizing function σ satisfy (h₃) we can refer to [17].

We introduce now a class of mappings needed in the sequel. Let us fix a function $\alpha \in C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$ which is equivalent to $\text{dist}(\cdot, \partial\Omega)$ (for more details on the existence of such an α see, for instance, Theorem 2, Chapter IV in [18] and Lemma 3.6.1 in [19]). Hence, for any $m \in \mathbb{N}$ we define the functions

$$\psi_m : x \in \bar{\Omega} \rightarrow g(m\alpha(x)) \left(1 - g\left(\frac{|x|}{2m}\right) \right),$$

where $g \in C^\infty(\bar{\mathbb{R}}_+)$ verifies (2.9). It is easy to prove that each ψ_m belongs to $C_0^\infty(\Omega)$ and

$$0 \leq \psi_m \leq 1, \quad \text{supp } \psi_m \subseteq E_{2m}, \quad (\psi_m)|_{E_m} = 1,$$

where

$$E_m = \left\{ x \in \Omega : |x| < m, \alpha(x) > \frac{1}{m} \right\}.$$

Remark 4.1 From hypothesis (h₁) and Lemma 4.2 in [12] it follows that for any $m \in \mathbb{N}$ the functions $(\psi_m a_{ij})_o$ (obtained as extensions of $\psi_m a_{ij}$ to \mathbb{R}^n with zero values out of Ω) belong to $VMO(\mathbb{R}^n)$ and

$$[(\psi_m a_{ij})_o]_{BMO(\mathbb{R}^n, t)} \leq [\psi_m a_{ij}]_{BMO(\Omega, t)},$$

for t small enough.

Now we are able to prove our main result.

Theorem 4.2 *Suppose that conditions (h₁), (h₂), (h₃) hold. Fixing $s \in \mathbb{R}$, let u be a solution of the problem*

$$\begin{cases} u \in W_{\text{loc}}^{2,p}(\Omega), \\ Lu \geq f, \quad f \in L_{\text{loc}}^p(\Omega), \\ \limsup_{x \rightarrow x_o} \sigma^s(x)u(x) \leq 0, \quad \forall x_o \in \partial\Omega, \\ \limsup_{|x| \rightarrow +\infty} \sigma^s(x)u(x) \leq 0 \quad \text{if } \Omega \text{ is unbounded.} \end{cases} \quad (4.1)$$

Then there exist an open ball $B \subset\subset \Omega$ and a constant $c \in \mathbb{R}_+$ such that

$$\sup_{x \in \Omega} \sigma^s(x)u(x) \leq c \left(\int_B |\sigma^{s+2} f^-|^p dx \right)^{\frac{1}{p}}, \quad (4.2)$$

where c depends on $n, p, r, \rho, \nu, \nu_0, a_o, \eta[\psi_m a_{ij}]$ ($m \in \mathbb{N}$), $\|a_i\|_{K_1^r(\Omega)}$, $\|a\|_{K_2^p(\Omega)}$, $\tilde{\omega}_1^r[a_i]$, $\tilde{\omega}_2^p[a]$.

Proof Without loss of generality it can be assumed that $\sup_{\Omega} \sigma^s(x)u(x) > 0$. For any $k \in \mathbb{N}$, we put

$$w_k(x) = \eta_k^s(x)u(x), \quad x \in \Omega. \tag{4.3}$$

Thus, from the last two conditions of (4.1) and from (2.11), (2.12) and (4.3) it follows that there exists $y_k \in \Omega$ such that $\sup_{\Omega} w_k(x) = w_k(y_k)$. Moreover, taking into account the classical Sobolev embedding theorem (see Theorem 5.4 in [15]), there exists $R_k \in]0, \text{dist}(y_k, \partial\Omega)[$ such that $w_k(x) > 0$ for all $x \in B(y_k, R_k)$.

Let $\lambda, \alpha_k, \alpha_o \in \mathbb{R}_+$, with $\alpha_o > 1$ (which will be suitably chosen later), such that

$$\alpha_k = \alpha_o \sigma(y_k), \quad \lambda \leq 1, \quad \lambda \alpha_k \leq \min\{R_k, \sigma(y_k)\}. \tag{4.4}$$

For simplicity of notation, for each $k \in \mathbb{N}$, we denote by B_k the open ball $B(y_k, \lambda \alpha_k)$.

Let us set

$$\varphi_k(x) = \begin{cases} 1 + \lambda^2 - \frac{|x-y_k|^2}{\alpha_k^2}, & x \in B_k, \\ 1, & x \in \Omega \setminus B_k. \end{cases} \tag{4.5}$$

It is easily seen that

$$1 \leq \varphi_k \leq 1 + \lambda^2 \leq 2. \tag{4.6}$$

Moreover, for $x \in B_k$

$$(\varphi_k)_{x_i} \leq \frac{2\lambda}{\alpha_k}, \quad (\varphi_k)_{x_i} \cdot (\varphi_k)_{x_j} \leq \frac{4\lambda^2}{\alpha_k^2}, \quad i, j = 1, \dots, n, \tag{4.7}$$

$$(\varphi_k)_{x_i x_j} = 0 \quad \text{if } i \neq j, \quad (\varphi_k)_{x_i x_j} = -\frac{2}{\alpha_k^2} \quad \text{if } i = j. \tag{4.8}$$

Consider now the function v_k defined by

$$v_k(x) = \varphi_k(x)w_k(x) - w_k(y_k), \quad x \in B_k. \tag{4.9}$$

Clearly

$$(v_k)_{|\partial B_k} = (w_k)_{|\partial B_k} - w_k(y_k) \leq 0, \quad v_k(y_k) = \lambda^2 w_k(y_k). \tag{4.10}$$

The first step of the proof is to show that there exists $k_o \in \mathbb{N}$ such that, for any $k \geq k_o$, each function v_k is a solution of a problem of type (3.1), where the coefficients of associated differential operator verify the assumptions of Lemma 3.1.

For any $k \in \mathbb{N}$, it is easy to prove

$$\begin{aligned} L_o w_k - u L_o \eta_k^s - 2 \sum_{i,j=1}^n a_{ij}(\eta_k^s)_{x_j} u_{x_i} + \sum_{i=1}^n a_i(\eta_k^s)_{x_i} \\ - u \sum_{i=1}^n a_i(\eta_k^s)_{x_i} + a \eta_k^s u = \eta_k^s L u, \quad x \in \Omega. \end{aligned} \tag{4.11}$$

Since

$$(\eta_k^s)_{x_j} u_{x_i} = (\eta_k^s u)_{x_i} \cdot \frac{(\eta_k^s)_{x_j}}{\eta_k^s} - \frac{(\eta_k^s)_{x_i} \cdot (\eta_k^s)_{x_j}}{(\eta_k^s)^2} \cdot (\eta_k^s u), \quad i, j = 1, \dots, n, \tag{4.12}$$

and u is a solution of problem (4.1), from (4.11) we deduce

$$L_o w_k + \sum_{i=1}^n b_i^k (w_k)_{x_i} + b^k w_k \geq g^k \quad \text{in } \Omega, \tag{4.13}$$

where we have put

$$b_i^k = a_i - 2 \sum_{j=1}^n a_{ij} \frac{(\eta_k^s)_{x_j}}{\eta_k^s}, \quad i = 1, \dots, n, \tag{4.14}$$

$$b^k = a + 2 \sum_{i,j=1}^n a_{ij} \frac{(\eta_k^s)_{x_i} (\eta_k^s)_{x_j}}{(\eta_k^s)^2} - \sum_{i,j=1}^n a_{ij} \frac{(\eta_k^s)_{x_i x_j}}{\eta_k^s}, \tag{4.15}$$

$$g^k = \eta_k^s f + w_k \sum_{i=1}^n a_i \frac{(\eta_k^s)_{x_i}}{\eta_k^s}. \tag{4.16}$$

We observe that using the hypotheses (h_0) , (h_1) , (h_2) , the equivalence between ρ and σ , and (2.11)-(2.16), we easily get

$$\begin{cases} (a_{ij} \cdot \frac{(\eta_k^s)_{x_j}}{\eta_k^s}) \in L^\infty(\Omega), & i, j = 1, \dots, n, \\ a_{ij} \cdot \frac{(\eta_k^s)_{x_i} (\eta_k^s)_{x_j}}{(\eta_k^s)^2}, a_{ij} \cdot \frac{(\eta_k^s)_{x_i x_j}}{\eta_k^s} \in L^\infty(\Omega), & i, j = 1, \dots, n, \\ g^k \in L^p_{\text{loc}}(\Omega). \end{cases} \tag{4.17}$$

Using now the estimate (4.13), it is easily seen that

$$\begin{aligned} & L_o(\varphi_k w_k) - w_k L_o \varphi_k - 2 \sum_{i,j=1}^n a_{ij} (\varphi_k)_{x_j} (w_k)_{x_i} \\ & + \sum_{i=1}^n b_i^k (\varphi_k w_k)_{x_i} - \sum_{i=1}^n b_i^k (\varphi_k)_{x_i} w_k + b^k \varphi_k w_k \\ & = \varphi_k \left(L_o w_k + \sum_{i=1}^n b_i^k (w_k)_{x_i} + b^k w_k \right) \geq \varphi_k g^k \quad \text{in } B_k. \end{aligned} \tag{4.18}$$

This last inequality can be rewritten as

$$\begin{aligned} & L_o(\varphi_k w_k) + \sum_{i=1}^n d_i^k (\varphi_k w_k)_{x_i} + d^k \varphi_k w_k \\ & \geq \varphi_k g^k + \sum_{i=1}^n b_i^k (\varphi_k)_{x_i} w_k \quad \text{in } B_k, \end{aligned} \tag{4.19}$$

where we have set

$$d_i^k = b_i^k - 2 \sum_{j=1}^n a_{ij} \frac{(\varphi_k)_{x_j}}{\varphi_k}, \quad i = 1, \dots, n, \tag{4.20}$$

$$d^k = b^k + 2 \sum_{i,j=1}^n a_{ij} \frac{(\varphi_k)_{x_i} (\varphi_k)_{x_j}}{(\varphi_k)^2} - \sum_{i,j=1}^n a_{ij} \frac{(\varphi_k)_{x_i x_j}}{\varphi_k}. \tag{4.21}$$

Hence, putting together (4.9) with (4.19) we get

$$L_o v_k + \sum_{i=1}^n d_i^k (v_k)_{x_i} + d^k v_k \geq h^k \quad \text{in } B_k, \tag{4.22}$$

where

$$h^k = \varphi_k g^k + w_k \sum_{i=1}^n b_i^k (\varphi_k)_{x_i} - d^k w_k (y_k). \tag{4.23}$$

Observe that using the hypotheses (h₁), (h₂), and (4.17), (4.5)-(4.8), it is easy to prove that, for any $k \in \mathbb{N}$, the coefficients d_i^k (for $i = 1, \dots, n$) and d^k satisfy the first two conditions of assumption (i₁) and the function $h^k \in L^p(B_k)$. We show now that, for a suitable choice of the constant α_o , there exists $k_o \in \mathbb{N}$ such that for any $k \geq k_o$ the coefficients d^k verify also the last condition of (i₁). To this aim, we firstly observe that using again hypotheses (h₁), (h₂), and (2.15), (2.16), from (4.15) we obtain

$$b^k \leq -\frac{a_o}{\sigma^2} + \frac{c_1}{\sigma^2} [(\eta_k)_x^2 + \eta_k (\eta_k)_{xx}], \quad \text{a.e. in } \Omega, \tag{4.24}$$

where $c_1 \in \mathbb{R}_+$ depends on v_o , n and s .

Thus, from (4.4), (2.11)-(2.14) and hypothesis (h₃) it follows that there exists $k_o \in \mathbb{N}$ such that for any $k \geq k_o$ we get

$$b^k \leq -\frac{a_o}{2\sigma^2}, \quad \text{a.e. in } \Omega. \tag{4.25}$$

Now, for $k \geq k_o$, putting together (4.25) with (4.21) and using the assumption (h₁), the properties (4.6)-(4.8) and (4.4), we obtain

$$d^k \leq -\frac{a_o}{2\sigma^2} + \frac{8v_o \lambda^2}{\alpha_k^2} + \frac{2v_o}{\alpha_k^2} \leq \left[-\frac{a_o}{2 \cdot \gamma^2} + \frac{10v_o}{\alpha_o^2} \right] \sigma^{-2}(y_k), \quad \text{a.e. in } B_k. \tag{4.26}$$

Hence, fixing α_o such that

$$\frac{1}{\alpha_o^2} \leq \frac{a_o}{40v_o \cdot \gamma^2} \tag{4.27}$$

from (4.26) it follows that for each $k \geq k_o$

$$d^k \leq -\frac{a_o}{4\gamma^4 \sigma^2(x)}, \quad \text{a.e. in } B_k. \tag{4.28}$$

Putting together (4.28) with (4.25) and observing that $d^k = b^k$ in $\Omega \setminus B_k$, we deduce that $d^k \leq 0$ a.e. in Ω . The above considerations together with (4.4), (4.9), (4.10), and (4.22) show that for any $k \geq k_0$ the problem

$$\begin{cases} v_k \in W^{2,p}(B_k), \\ L_0 v_k + \sum_{i=1}^n a_i^k (v_k)_{x_i} + d^k v_k \geq h^k, & h^k \in L^p(B_k), \\ v_k|_{\partial B_k} \leq 0 \end{cases} \quad (4.29)$$

satisfy the assumptions of Lemma 3.1. Therefore, there exists a constant $c_1 \in \mathbb{R}_+$ depending on $n, p, r, \rho, \nu, \nu_0, [p(a_{ij})]_{BMO(\mathbb{R}^n, \cdot)}, \|a_i\|_{K_1^1(\Omega)}, \|a\|_{K_2^p(\Omega)}, \tilde{\omega}_1^r[a_i], \tilde{\omega}_2^p[a]$ such that

$$\sup_{B_k} v_k \leq c_1 (\lambda \alpha_k)^{2-\frac{n}{p}} \| (h^k)^- \|_{L^p(B_k)}. \quad (4.30)$$

By (4.10), the last bound with $x = y_k$ becomes

$$\lambda^2 w_k(y_k) \leq c_1 (\lambda \alpha_k)^{2-\frac{n}{p}} \| (h^k)^- \|_{L^p(B_k)}. \quad (4.31)$$

Now, in order to obtain the estimate (4.2), we have to provide a lower bound for the function h^k in terms of the data f . First of all, we observe that, using the definitions (4.14) and (4.16), we can rewrite (4.23) as

$$\begin{aligned} h^k &= \varphi_k \eta_k^s f + w_k \sum_{i=1}^n a_i \left(\frac{(\eta_k^s)_{x_i}}{\eta_k^s} \cdot \varphi_k + (\varphi_k)_{x_i} \right) \\ &\quad - 2w_k \sum_{i,j=1}^n a_{ij} \frac{(\eta_k^s)_{x_j}}{\eta_k^s} \cdot (\varphi_k)_{x_i} - d^k w_k(y_k). \end{aligned} \quad (4.32)$$

On the other hand, by assumption (h₁), and by (2.15), (4.7), and (4.4) we easily obtain

$$\left| 2 \cdot \sum_{i,j=1}^n a_{ij} \frac{(\eta_k^s)_{x_j}}{\eta_k^s} \cdot (\varphi_k)_{x_i} \right| \leq \frac{c_2}{\sigma^2(x)} \cdot (\eta_k)_{x_i}, \quad (4.33)$$

where $c_2 \in \mathbb{R}_+$ depends on ν_0, n, s, ρ, a_0 . Thus, using (2.13) and hypothesis (h₃) it follows that there exists $k_1 \geq k_0$, with $k_1 \in \mathbb{N}$, such that for any $k \geq k_1$

$$\left| 2 \cdot \sum_{i,j=1}^n a_{ij} \frac{(\eta_k^s)_{x_j}}{\eta_k^s} \cdot (\varphi_k)_{x_i} \right| \leq \frac{a_0}{4\gamma^4 \sigma^2(x)}. \quad (4.34)$$

Putting together (4.34) and (4.28) with (4.32) we obtain

$$h^k \geq \varphi_k \eta_k^s f + w_k \sum_{i=1}^n a_i \left(\frac{(\eta_k^s)_{x_i}}{\eta_k^s} \cdot \varphi_k + (\varphi_k)_{x_i} \right). \quad (4.35)$$

Taking into account (4.35), from (4.31) we get

$$w_k(y_k) \leq c_1 \lambda^{-\frac{n}{p}} \alpha_k^{2-\frac{n}{p}} \| (\eta_k^s f)^- \|_{L^p(B_k)} + \Psi_1^k + \Psi_2^k, \quad (4.36)$$

where we have put

$$\Psi_1^k = c_1 \lambda^{-\frac{n}{p}} \alpha_k^{2-\frac{n}{p}} \left\| w_k \sum_{i=1}^n a_i (\varphi_k)_{x_i} \right\|_{L^p(B_k)}, \tag{4.37}$$

and

$$\Psi_2^k = c_1 \lambda^{-\frac{n}{p}} \alpha_k^{2-\frac{n}{p}} \left\| w_k \sum_{i=1}^n a_i \frac{(\eta_k^s)_{x_i}}{\eta_k^s} \cdot \varphi_k \right\|_{L^p(B_k)}. \tag{4.38}$$

To end the proof, we give some upper bounds for the functions Ψ_1^k and Ψ_2^k (with $k \geq k_1$). First of all, observe that using (4.7) and Hölder's inequality in (4.37) we obtain

$$\Psi_1^k \leq c_3 \lambda^{1-\frac{n}{r}} \alpha_k^{1-\frac{n}{r}} w_k(y_k) \left\| \sum_{i=1}^n a_i \right\|_{L^r(B_k)}, \tag{4.39}$$

where $c_3 \in \mathbb{R}_+$ depends on the same parameters as c_1 . Using now (4.4), the equivalence on ρ and σ , we get

$$\Psi_1^k \leq c_4 \lambda^{1-\frac{n}{r}} \alpha_o^{1-\frac{n}{r}} w_k(y_k) \left\| \sum_{i=1}^n a_i \right\|_{K_1^r(\Omega)}, \tag{4.40}$$

where $c_4 \in \mathbb{R}_+$ depends on the same parameters as c_1 . If we choose λ such that

$$\lambda^{1-\frac{n}{r}} \leq \frac{1}{4c_4 \alpha_o^{1-n/r} \left\| \sum_{i=1}^n a_i \right\|_{K_1^r(\Omega)}} \tag{4.41}$$

from (4.40), for $k \geq k_1$, we get

$$\Psi_1^k \leq \frac{w_k(y_k)}{4}. \tag{4.42}$$

Arguing similarly we obtain, for each $k \geq k_1$, the following bound on the function Ψ_2^k :

$$\Psi_2^k \leq c_5 \lambda^{-\frac{n}{p}} \alpha_o^{2-\frac{n}{p}} w_k(y_k) \left\| \sum_{i=1}^n a_i \right\|_{K_1^r(\Omega)} \sup_{\Omega \setminus \Omega_k} (\eta_k)_x, \tag{4.43}$$

where $c_5 \in \mathbb{R}_+$ depends on the same parameters as c_1 and on s . Thus, using again (2.13) and assumption (h₃), we see that there exists $k_2 \geq k_1$, with $k_2 \in \mathbb{N}$, such that for $k \geq k_2$ we get

$$\Psi_2^k \leq \frac{w_k(y_k)}{4}. \tag{4.44}$$

Finally, chosen $k = k_2$, putting together (4.42) and (4.44) with (4.36) and using (4.4), (2.11), and (2.12) it follows that

$$w_{k_2}(y_{k_2}) \leq c_6 (\lambda \alpha_{k_2})^{-\frac{n}{p}} \left\| \sigma^{2+s} f^- \right\|_{L^p(B_{k_2})}, \tag{4.45}$$

where $c_6 \in \mathbb{R}_+$ depends on the same parameters as c_1 and on a_o . Taking into account (4.3) and using again (2.11) and (2.12), from (4.45) we get

$$\sup_{\Omega} \sigma^s(x)u(x) \leq c_7 \left(\int_{B_{k_2}} |\sigma^{2+s} f^-|^p \right)^{1/p}, \quad (4.46)$$

where $c_7 \in \mathbb{R}_+$ depends on the same parameters as c_1 and on a_o .

Finally, if we choose

$$p(a_{ij}|_{B_{k_2}}) = (\psi_{m_o} a_{ij})_o, \quad (4.47)$$

where $m_o \in \mathbb{N}$ is such that $\psi_{m_o}|_{B_{k_2}} = 1$, the estimate (4.2) follows from (4.46), (4.47), and Remark 4.1. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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