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Multiplicity and uniqueness results for the singular nonlocal boundary value problem involving nonlinear integral conditions

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Abstract
In this paper, using fixed point index and the mixed monotone technique, we present some multiplicity and uniqueness results for the singular nonlocal boundary value problems involving nonlinear integral conditions. Our nonlinearity may be singular in its dependent variable and it is allowed to change sign.

1 Introduction

In this paper, we consider the existence of positive solutions of nonlinear nonlocal boundary value problem (BVP) of the form

$$-y'' = q(t)f\big(t, y(t)\big), \quad t \in (0,1) \tag{1.1}$$

with integral boundary conditions

$$y(0) = \alpha[y] = \int_0^1 \big(y(s)\big)^a \, dA(s), \qquad y(1) = \beta[y] = \int_0^1 \big(y(s)\big)^b \, dB(s) \tag{1.2}$$

involving Stieltjes integrals, $a \geq 0$, $b \geq 0$.

In [1], using the Leray-Schauder alternative, Z. Yang considered the problem

$$-y'' = f\big(y(t)\big), \quad t \in (0,1) \tag{1.3}$$

with integral boundary conditions

$$y(0) = \alpha[y] = \int_0^1 y(s)\, dA(s), \qquad y(1) = \beta[y] = \int_0^1 y(s)\, dB(s) \tag{1.4}$$

and discussed the existence and uniqueness of a positive solution for BVP (1.3)-(1.4) with a sign-changing nonlinearity f. C.S. Goodrich discussed (1.1) with nonlinear integral conditions

$$y(0) = H_1\left(\int_0^1 y(s)\, dA(s)\right) + \int_E H_2\big(s, y(s)\big)\, ds, \qquad y(1) = 0,$$

where $E \subseteq (0,1)$ is some measurable set (see [2]). Moreover, there are some interesting results when the measures are signed (see [3–5]). Using the mixed monotone technique, L. Kong considered

$$-y'' = \lambda f\big(t, y(t)\big), \quad t \in (0,1) \tag{1.5}$$

with (1.4) and discussed the uniqueness of positive solutions (see [4]). J.R.L. Webb discussed the multiplicity of positive solutions for BVP (1.5)-(1.4) when $f(t,y)$ is positive and continuous on $(0,1) \times [0,+\infty)$; note that f has no singularities at $y = 0$ (see [5]). Using the fixed point index, G. Infante discussed (1.1) with nonlinear integral boundary conditions (see [3]),

$$y(0) + H_1\left(\int_0^1 y(s)\,dA(s)\right) = 0, \qquad y(1) + u(\eta) = H_2\left(\int_0^1 y(s)\,dB(s)\right).$$

Inspired by the above works and [6–16], we consider the BVP (1.1)-(1.2) when f is singular at $y = 0$ and f may be sign changing. Using the fixed point index and the mixed monotone technique we establish some new existence results for the BVP (1.1)-(1.2).

Our paper is organized as follows. In Section 2, we present some lemmas and preliminaries. Section 3 discusses the existence of multiple positive solutions for BVP (1.1)-(1.2) when f is positive. In Section 4, we discuss the multiplicity of positive solutions for the semipositone BVP (1.1)-(1.2). In Section 5, using the mixed monotone technique, we discuss the uniqueness of a positive solution of BVP (1.1)-(1.2).

2 Preliminaries

Let $C[0,1] = \{y : [0,1] \to \mathbb{R} | y(t) \text{ is continuous on } [0,1]\}$ with norm $\|y\| = \max_{t\in[0,1]} |y(t)|$. It is easy to see that $C[0,1]$ is a Banach space. Define

$$P = \big\{y \in C[0,1] | y \text{ is concave on } [0,1] \text{ with } y(t) \geq 0 \text{ for all } t \in [0,1]\big\}. \tag{2.1}$$

It is easy to prove P is a cone of $C[0,1]$.

Lemma 2.1 (see [17]) *Let Ω be a bounded open set in a real Banach space E, P be a cone of E, $\theta \in \Omega$ and $A : \overline{\Omega} \cap P \to P$ be continuous and compact. Suppose $\lambda Ax \neq x$, $\forall x \in \partial\Omega \cap P$, $\lambda \in (0,1]$. Then*

$$i(A, \Omega \cap P, P) = 1. \tag{2.2}$$

Lemma 2.2 (see [17]) *Let Ω be a bounded open set in a real Banach space E, P be a cone of E, $\theta \in \Omega$ and $A : \overline{\Omega} \cap P \to P$ be continuous and compact. Suppose $Ax \nleq x$, $\forall x \in \partial\Omega \cap P$. Then*

$$i(A, \Omega \cap P, P) = 0. \tag{2.3}$$

Lemma 2.3 (see [18]) *Let $y \in P$ (defined in (2.1)). Then*

$$y(t) \geq t(1-t)\|y\| \quad \text{for } t \in [0,1]. \tag{2.4}$$

Now we present the following conditions for convenience:

(C_1) A and B are of bounded variation with positive measures, $0 < \int_0^1 dA(s)$, $a > 0$, $0 < \int_0^1 dB(s)$, $b > 0$,

(C_2)

$$\begin{cases} \text{there exists a function } \psi_1 \\ \text{continuous on } [0,1] \text{ and positive on } (0,1) \text{ such that} \\ f(t,y) \geq \psi_1(t) \text{ on } (0,1) \times (0,1], \end{cases} \tag{2.5}$$

(C_3)

$$q \in C(0,1), \qquad q > 0 \quad \text{on } (0,1) \quad \text{and} \quad \int_0^1 t(1-t)q(t)\,dt < \infty, \tag{2.6}$$

(C_4)

$$f : [0,1] \times (0,\infty) \to (0,\infty) \text{ is continuous}, \tag{2.7}$$

(C_5) there exists a continuous function $g : [0,1] \times (0,\infty) \times (0,\infty) \to (0,\infty)$ with $g(t,x,y)$ nondecreasing in x and nonincreasing in y and for $x > 0$ we have $f(t,x) = g(t,x,x)$. Moreover, there is a constant θ with $0 \leq \theta < 1$ such that

$$g\left(t, \lambda x, \frac{1}{\lambda} y\right) \geq \lambda^\theta g(t,x,y), \quad \forall x > 0, y > 0, 0 < \lambda < 1.$$

3 Multiplicity of positive solutions for singular boundary value problems with positive nonlinearities

In this section, we consider the existence of multiple positive solutions for BVP (1.1)-(1.2). To show that BVP (1.1)-(1.2) has a solution, for $y \in P$, define

$$\begin{aligned} (T_\epsilon y)(t) &= (1-t)\alpha[y] + t\beta[y] \\ &\quad + \int_0^1 k(t,s)q(s)f\bigl(s, \max\{\epsilon, y(s)\}\bigr)\,ds, \quad t \in [0,1], 1 \geq \epsilon > 0, \end{aligned} \tag{3.1}$$

where

$$k(t,s) = \begin{cases} (1-t)s, & 0 \leq s \leq t \leq 1, \\ (1-s)t, & 0 \leq t \leq s \leq 1. \end{cases}$$

Lemma 3.1 *Suppose* (C_1)-(C_4) *hold. Then* $T_\epsilon : P \to P$ *is continuous and completely continuous for all* $1 \geq \epsilon > 0$.

Proof It is easy to prove that T_ϵ is well defined and $(T_\epsilon y)(t) \geq 0$ for all $t \in P$. For $y \in P$, we have

$$\begin{cases} (T_\epsilon y)''(t) \leq 0 \quad \text{on } (0,1), \\ (T_\epsilon y)(0) = \alpha[y], \qquad (T_\epsilon y)(1) = \beta[y], \end{cases} \tag{3.2}$$

so

$$(T_\epsilon y)(t) \text{ is concave on } [0,1]. \tag{3.3}$$

Consequently, $T_\epsilon : P \to P$. A standard argument shows that $T_\epsilon : P \to P$ is continuous and completely continuous (see [4, 19, 20]). □

Lemma 3.2 *Suppose that* $\int_0^1 dA(s) > 0$ *and* $\int_0^1 dB(s) > 0$. *Then*

$$\int_0^1 s^a(1-s)^a\,dA(s) > 0, \qquad \int_0^1 s^b(1-s)^b\,dB(s) > 0.$$

The proof is trivial and we omit it.

Define

$$\begin{aligned} H = \bigl\{&x \in C\bigl([0,1],\mathbb{R}\bigr) \cap C^1\bigl([0,1),\mathbb{R}\bigr) \cap C\bigl((0,1),(0,+\infty)\bigr) \cap C^2\bigl((0,1),\mathbb{R}\bigr) | x \text{ satisfies} \\ &x''(t) + q(t)f\bigl(t, \max\bigl\{\epsilon, x(t)\bigr\}\bigr) = 0, 0 < t < 1, x(0) = \alpha[x], x(1) = \beta[x], \forall 1 \ge \epsilon > 0\bigr\}. \end{aligned}$$

Lemma 3.3 *If* $H \neq \emptyset$ *and* (C_2) *hold, there exists a* $\delta_0 > 0$ *such that*

$$x(t) \ge \delta_0, \quad \forall t \in [0,1], x \in H.$$

Proof Suppose $x \in H$. There are two cases to consider:

(1) $\|x\| > 1$. Lemma 2.3 implies that

$$x(t) \ge t(1-t)\|x\| \ge t(1-t), \quad t \in [0,1]. \tag{3.4}$$

(2) $0 < \|x\| \le 1$. Condition (C_2) guarantees that

$$\begin{aligned} x(t) &= (1-t)\alpha[x] + t\beta[x] + \int_0^1 k(t,s)q(s)f\bigl(s, \max\bigl\{\epsilon, x(s)\bigr\}\bigr)\,ds \\ &\ge \int_0^1 k(t,s)q(s)\psi_1(s)\,ds = \gamma_0(t), \quad t \in [0,1]. \end{aligned}$$

Since $\gamma_0''(t) \ge 0$ and $\gamma_0(0) = 0$ and $\gamma_0(1) = 0$, Lemma 2.3 implies that

$$\gamma_0(t) \ge t(1-t)\|\gamma_0\|, \quad \forall t \in [0,1]. \tag{3.5}$$

Let $\delta_1 = \min\{1, \|\gamma_0\|\}$. From (3.4) and (3.5), one has

$$x(t) \ge \delta_1 t(1-t), \quad \forall t \in [0,1].$$

Lemma 3.2 implies that

$$x(0) = \alpha[x] = \int_0^1 x^a(s)\,dA(s) \ge \delta_1^a \int_0^1 s^a(1-s)^a\,dA(s) > 0$$

and

$$x(1) = \beta[x] = \int_0^1 x^b(s)\,dB(s) \geq \delta_1^b \int_0^1 s^b(1-s)^b\,dB(s) > 0.$$

Set

$$\delta_0 = \min\left\{\delta_1^a \int_0^1 s^a(1-s)^a\,dA(s), \delta_1^b \int_0^1 s^b(1-s)^b\,dB(s)\right\}.$$

Since $x(t)$ is concave on $[0,1]$, we have

$$x(t) \geq \delta_0, \quad \forall t \in [0,1], x \in H.$$

The proof is complete. □

Lemma 3.4 *Suppose that there exists an* $\bar{a} \in (0, \frac{1}{2})$ *such that*

$$\lim_{y \to +\infty} \frac{f(t,y)}{y} = +\infty \tag{3.6}$$

uniformly on $[\bar{a}, 1-\bar{a}]$. *Then there exists an* $R' > 1$ *such that for all* $R \geq R'$

$$i(T_\epsilon, \Omega_R \cap P, P) = 0, \quad \forall 0 < \epsilon \leq 1.$$

Proof From (3.6), there exists a $R_1 > 1$ such that

$$f(t,y) \geq N^* y, \quad \forall y \geq R_1, \tag{3.7}$$

where

$$N^* > \frac{2}{\bar{a}^2 \int_{\bar{a}}^{1-\bar{a}} k(\bar{a},s)q(s)\,ds}.$$

Let $R' = \frac{R_1}{\bar{a}^2}$ and

$$\Omega_R = \left\{x \in C[0,1] \,|\, \|x\| < R\right\}, \quad \forall R \geq R'.$$

Now we show

$$T_\epsilon y \nleq y \quad \text{for } y \in P \cap \partial\Omega_R, \forall 0 < \epsilon \leq 1. \tag{3.8}$$

Suppose that there exists a $y_0 \in P \cap \partial\Omega_R$ with $T_\epsilon y_0 \leq y_0$. Then $\|y_0\| = R$. Also since $y_0(t)$ is concave on $[0,1]$ (since $y_0 \in P$) we have from Lemma 2.3 that $y_0(t) \geq t(1-t)\|y_0\| \geq t(1-t)R$ for $t \in [0,1]$. For $t \in [\bar{a}, 1-\bar{a}]$, one has

$$y_0(t) \geq \bar{a}^2 R \geq \bar{a}^2 R' = R_1, \quad \forall t \in [\bar{a}, 1-\bar{a}],$$

which together with (3.7) yields the result that

$$f\bigl(t, y_0(t)\bigr) \geq N^* y_0(t) \geq N^* \bar{a}^2 R, \quad \forall t \in [\bar{a}, 1-\bar{a}]. \tag{3.9}$$

Then we have, using (3.9),

$$\begin{aligned} y_0(\bar{a}) \geq T_\epsilon y_0(\bar{a}) &= (1-\bar{a})\alpha[y_0] + \bar{a}\beta[y_0] + \int_0^1 k(\bar{a},s)q(s)f\left(s, \max\left\{\epsilon, y_0(s)\right\}\right) ds \\ &\geq \int_{\bar{a}}^{1-\bar{a}} k(\bar{a},s)q(s)f\left(s, \max\left\{\epsilon, y_0(s)\right\}\right) ds \\ &= \int_{\bar{a}}^{1-\bar{a}} k(\bar{a},s)q(s)f\left(s, y_0(s)\right) ds \\ &\geq N^* R\bar{a}^2 \int_{\bar{a}}^{1-\bar{a}} k(\bar{a},s)q(s)\, ds \\ &> R = \|y_0\|, \end{aligned}$$

which is a contradiction. Hence (3.8) is true. Lemma 2.2 guarantees that

$$i(T_\epsilon, \Omega_R \cap P, P) = 0, \quad \forall 0 < \epsilon \leq 1.$$

The proof is complete. □

Lemma 3.5 *Suppose that* $\max\{a,b\} > 1$. *Then there exists an* $R' > 1$ *such that for all* $R \geq R'$

$$i(T_\epsilon, \Omega_R \cap P, P) = 0, \quad \forall 0 < \epsilon \leq 1.$$

Proof Since $\max\{a,b\} > 1$, without loss of generality, we suppose that $a > 1$. Let $R' > 1$ with $R'^{a-1} \int_0^1 s^a(1-s)^a\, dA(s) > 1$. Set

$$\Omega_R = \left\{x \in C[0,1] \mid \|x\| < R\right\}, \quad R \geq R'.$$

Now we show

$$T_\epsilon x \nleq x, \quad \forall x \in \partial\Omega_R \cap P, \forall 0 < \epsilon \leq 1.$$

In fact, suppose that $x_0 \in \partial\Omega_R \cap P$ and satisfies

$$T_\epsilon x_0 \leq x_0.$$

Lemma 2.3 guarantees that

$$x_0(t) \geq \|x_0\| t(1-t) \geq Rt(1-t), \quad t \in [0,1].$$

Then

$$\begin{aligned} R \geq x_0(0) &= \int_0^1 x_0^a(s)\, dA(s) \geq \int_0^1 \|x_0\|^a s^a(1-s)^a\, dA(s) \\ &= RR^{a-1} \int_0^1 s^a(1-s)^a\, dA(s) \geq RR'^{a-1} \int_0^1 s^a(1-s)^a\, dA(s) > R. \end{aligned}$$

This is a contradiction. Lemma 2.2 guarantees that

$$i(T_\epsilon, \Omega_R \cap P, P) = 0, \quad \forall R \geq R', \forall 0 < \epsilon \leq 1.$$

The proof is complete. □

Theorem 3.1 *Suppose* (C_1), (C_2), (C_3), *and* (C_4) *hold and the following conditions are satisfied*:

$$\begin{cases} 0 \leq f(t,y) \leq g(y) + h(y) \text{ on } [0,1] \times (0,\infty) \text{ with} \\ g > 0 \text{ continuous and nonincreasing on } (0,\infty), \\ h \geq 0 \text{ continuous on } [0,\infty), \text{ and } \frac{h}{g} \\ \text{nondecreasing on } (0,\infty) \end{cases} \tag{3.10}$$

and

$$\sup_{r\in(0,+\infty)} \frac{1}{1+\frac{h(r)}{g(r)}} \int_{c_0 \max\{r^a, r^b\}}^{r} \frac{dy}{g(y)} > b_0 \tag{3.11}$$

hold; *here*

$$\begin{aligned} c_0 &= \max\left\{\int_0^1 dA(s), \int_0^1 dB(s)\right\}, \\ b_0 &= \max\left\{2\int_0^{\frac{1}{2}} t(1-t)q(t)\,dt, 2\int_{\frac{1}{2}}^1 t(1-t)q(t)\,dt\right\}. \end{aligned} \tag{3.12}$$

Then BVP (1.1)-(1.2) *has at least one positive solution*.

Proof Choose $\epsilon > 0$ and $r > 0$ with $\epsilon < \min\{1, c_0\{r^a, r^b\}\}$ and

$$\frac{1}{1+\frac{h(r)}{g(r)}} \int_{c_0\{r^a, r^b\}}^{r} \frac{dy}{g(y)} > b_0. \tag{3.13}$$

Let

$$\Omega_1 = \{y \in C[0,1] \mid \|y\| < r\},$$

and T_ϵ is defined in (3.1). Lemma 3.1 guarantees that $T_\epsilon : P \to P$ is continuous and completely continuous.

Now we show that

$$y \neq \lambda T_\epsilon y, \quad \forall y \in \partial\Omega_1 \cap P, \lambda \in (0,1]. \tag{3.14}$$

Suppose that there is a $y_0 \in \partial\Omega_1 \cap P$ and $\lambda_0 \in [0,1]$ with $y_0 = \lambda_0 T_\epsilon y_0$, *i.e.*, y_0 satisfies

$$\begin{cases} y_0'' + \lambda_0 q(t) f(t, \max\{\epsilon, y_0(t)\}) = 0, \quad 0 < t < 1, \\ y_0(0) = \lambda_0 \alpha[y], \qquad y_0(1) = \lambda_0 \beta[y_0]. \end{cases} \tag{3.15}$$

Then $y_0''(t) \leq 0$ on $(0,1)$ and $y_0(0) = \lambda_0 \alpha[y_0] \leq r^a \int_0^1 dA(s) \leq c_0 r^a < r = \|y_0\|$, $y_0(1) = \lambda_0 \beta[y_0] \leq r^b \int_0^1 dB(s) \leq c_0 r^b < r = \|y_0\|$, which guarantees that there exists a $t_0 \in (0,1)$, $r = \|y_0\| = y_0(t_0)$ with $y'(t_0) = 0$ and $y_0'(t) \geq 0$ for all $t \in (0,t_0)$ and $y_0'(t) \leq 0$ for all $t \in (t_0,1)$. For $t \in (0,1)$, we have

$$-y_0''(t) \leq g\bigl(\max\{\epsilon, y_0(t)\}\bigr)\left\{1 + \frac{h(\max\{\epsilon, y_0(t)\})}{g(\max\{\epsilon, y_0(t)\})}\right\}q(t)$$
$$\leq g\bigl(\max\{\epsilon, y_0(t)\}\bigr)\left\{1 + \frac{h(r)}{g(r)}\right\}q(t). \tag{3.16}$$

Integrate from t ($t < t_0$) to t_0 to obtain

$$y_0'(t) \leq g\bigl(\max\{\epsilon, y_0(t)\}\bigr)\left\{1 + \frac{h(r)}{g(r)}\right\}\int_t^{t_0} q(s)\,ds \leq g\bigl(y_0(t)\bigr)\left\{1 + \frac{h(r)}{g(r)}\right\}\int_t^{t_0} q(s)\,ds$$

and then integrate from 0 to t_0 to obtain

$$\begin{aligned}\int_{\alpha[y_0]}^{y_0(t_0)} \frac{dy}{g(y)} &\leq \left\{1 + \frac{h(r)}{g(r)}\right\}\int_0^{t_0}\int_s^{t_0} q(\tau)\,d\tau\,ds \\ &= \left\{1 + \frac{h(r)}{g(r)}\right\}\int_0^{t_0} sq(s)\,ds \\ &\leq \left\{1 + \frac{h(r)}{g(r)}\right\}\frac{1}{1-t_0}\int_0^{t_0} s(1-s)q(s)\,ds \quad \left(\text{note } 1 \leq \frac{1-s}{1-t_0}, \forall s \in [t_0,1)\right),\end{aligned}$$

which together with $\alpha[y_0] \leq c_0 r^a \leq c_0 \max\{r^a, r^b\}$ yields the result that

$$\int_{c_0 \max\{r^\alpha, r^\beta\}}^{r} \frac{dy}{g(y)} \leq \int_{\alpha[y_0]}^{r} \frac{dy}{g(y)} \leq \left\{1 + \frac{h(r)}{g(r)}\right\}\frac{1}{1-t_0}\int_0^{t_0} s(1-s)q(s)\,ds. \tag{3.17}$$

Similarly if we integrate (3.16) from t_0 to t ($t \geq t_0$) and then from t_0 to 1 we obtain

$$\int_{c_0 \max\{r^a, r^b\}}^{r} \frac{du}{g(u)} \leq \left\{1 + \frac{h(r)}{g(r)}\right\}\frac{1}{t_0}\int_{t_0}^{1} s(1-s)q(s)\,ds. \tag{3.18}$$

Now (3.17) and (3.18) imply

$$\int_{c_0 \max\{r^a, r^b\}}^{r} \frac{du}{g(u)} \leq b_0\left\{1 + \frac{h(r)}{g(r)}\right\},$$

which contradicts (3.13). Therefore, (3.14) is true. Lemma 2.1 implies that

$$i(T_\epsilon, \Omega_1 \cap P, P) = 1, \tag{3.19}$$

which yields the result that there exists a $y_{1,\epsilon} \in \Omega_1 \cap P$ such that

$$T_\epsilon y_{1,\epsilon} = y_{1,\epsilon},$$

i.e., $H \neq \emptyset$ in Lemma 3.3. Moreover, Lemma 3.3 is true, which guarantees that there exists a $\delta_0 > 0$ such that

$$x(t) \geq \delta_0, \quad \forall t \in [0,1], x \in H. \tag{3.20}$$

Now let $\epsilon_0 = \frac{\delta_0}{2}$. From the above proof, there exists a $x_{1,\epsilon_0} \in \Omega_1 \cap P$ such that

$$T_{\epsilon_0} x_{1,\epsilon_0} = x_{1,\epsilon_0}.$$

Since $0 < \epsilon_0 \leq 1$, (3.20) implies that

$$x_{1,\epsilon_0}(t) \geq \delta_0 > \epsilon_0, \quad t \in [0,1]. \tag{3.21}$$

Moreover, since $x_{1,\epsilon_0}(t)$ satisfies

$$\begin{cases} x''_{1,\epsilon_0}(t) + q(t)f(t, \max\{\epsilon_0, x_{1,\epsilon_0}(t)\}) = 0, & 0 < t < 1, \\ x_{1,\epsilon_0}(0) = \alpha[x_{1,\epsilon_0}], & x_{1,\epsilon_0}(1) = \beta[x_{1,\epsilon_0}], \end{cases}$$

(3.21) guarantees that

$$\begin{cases} x''_{1,\epsilon_0}(t) + q(t)f(t, x_{1,\epsilon_0}(t)) = 0, & 0 < t < 1, \\ x_{1,\epsilon_0}(0) = \alpha[x_{1,\epsilon_0}], & x_{1,\epsilon_0}(1) = \beta[x_{1,\epsilon_0}]. \end{cases}$$

Thus, BVP (1.1)-(1.2) has at least one positive solution. The proof is complete. □

Theorem 3.2 *Suppose the conditions of Theorem* 3.1 *hold and there exists an* $\bar{a} \in (0, \frac{1}{2})$ *such that*

$$\lim_{y \to +\infty} \frac{f(t,y)}{y} = +\infty$$

uniformly on $[\bar{a}, 1-\bar{a}]$. *Then BVP* (1.1)-(1.2) *has at least two positive solutions*.

Proof Choose $r > 0$ as in (3.13), $\epsilon_0 > 0$ with $\epsilon_0 < \min\{\delta_0, 1, c_0\{r^a, r^b\}\}$, where δ_0 is defined in (3.20), and $R > \max\{r, R'\}$ in Lemma 3.4. Set

$$\Omega_1 = \{y \in C[0,1] \mid \|y\| < r\},$$
$$\Omega_2 = \{y \in C[0,1] \mid \|y\| < R\}.$$

From the proof of Theorem 3.1 and Lemma 3.4, we have

$$i(T_{\epsilon_0}, \Omega_1 \cap P, P) = 1$$

and

$$i(T_{\epsilon_0}, \Omega_2 \cap P, P) = 0,$$

which implies that

$$i\big(T_{\epsilon_0},(\Omega_2-\Omega_1)\cap P,P\big)=-1.$$

Thus, there exist $x_{1,\epsilon_0}\in\Omega_1\cap P$ and $x_{2,\epsilon_0}\in(\Omega_2-\Omega_1)\cap P$ such that

$$T_{\epsilon_0}x_{1,\epsilon_0}=x_{1,\epsilon_0},\qquad T_{\epsilon_0}x_{2,\epsilon_0}=x_{2,\epsilon_0}.$$

From the proof of Theorem 3.1, $x_{1,\epsilon_0}(t)$ and $x_{2,\epsilon_0}(t)$ are two positive solutions for BVP (1.1)-(1.2). The proof is complete. □

Theorem 3.3 *Suppose the conditions of Theorem* 3.1 *hold and* $\max\{a,b\}>1$. *Then BVP* (1.1)-(1.2) *has at least two positive solutions.*

Proof Choose $r>0$ as in (3.13), $\epsilon_0>0$ with $\epsilon_0<\min\{\delta_0,1,c_0\{r^a,r^b\}\}$, where δ_0 is defined in (3.20), and $R>\max\{r,R'\}$ in Lemma 3.5. Set

$$\Omega_1=\big\{y\in C[0,1]\,|\,\|y\|<r\big\},$$
$$\Omega_2=\big\{y\in C[0,1]\,|\,\|y\|<R\big\}.$$

From the proof of Theorem 3.1 and Lemma 3.5, we have

$$i(T_{\epsilon_0},\Omega_1\cap P,P)=1$$

and

$$i(T_{\epsilon_0},\Omega_2\cap P,P)=0,$$

which implies that

$$i\big(T_{\epsilon_0},(\Omega_2-\Omega_1)\cap P,P\big)=-1.$$

Thus, there exist $x_{1,\epsilon_0}\in\Omega_1\cap P$ and $x_{2,\epsilon_0}\in(\Omega_2-\Omega_1)\cap P$ such that

$$T_{\epsilon_0}x_{1,\epsilon_0}=x_{1,\epsilon_0},\qquad T_{\epsilon_0}x_{2,\epsilon_0}=x_{2,\epsilon_0}.$$

From the proof of Theorem 3.1, $x_{1,\epsilon_0}(t)$ and $x_{2,\epsilon_0}(t)$ are two positive solutions for BVP (1.1)-(1.2). The proof is complete. □

Example 3.1 Consider

$$y''(t)+\mu\big(y^{-\delta_1}(t)+y^{\delta_2}(t)\big)=0,\quad 0<t<1,\tag{3.22}$$

with

$$y(0)=\int_0^1 y^{\frac{1}{3}}(s)\,dA(s),\qquad y(1)=\int_0^1 y^{\frac{1}{2}}(s)\,dB(s),\quad dA(s)=\frac{1}{2}\,ds, dB(s)=\frac{1}{8}\,de^s,\tag{3.23}$$

where $\delta_1>0$, $\delta_2>1$.

Let $q(t) = \mu$, $f(t,y) = y^{-\delta_1} + y^{\delta_2}$, $g(y) = y^{-\delta_1}$, $h(y) = y^{\delta_2}$, $c_0 = \max\{\int_0^1 dA(s), \int_0^1 dB(s)\} = \frac{1}{2}$, $b_0 = \frac{1}{6}\mu$. It is easy to see that (C_1)-(C_4) and (3.10) hold. Since

$$\frac{1}{1+\frac{h(1)}{g(1)}}\int_{c_0\max\{1^{\frac{1}{3}},1^{\frac{1}{2}}\}}^{1}\frac{1}{g(y)}\,dy = \frac{1-(\frac{1}{2})^{\delta_1+1}}{2(1+\delta_1)},$$

letting $\mu_0 < 3\frac{1-(\frac{1}{2})^{\delta_1+1}}{2(1+\delta_1)}$, we have

$$\sup_{r\in(0,+\infty)}\frac{1}{1+\frac{h(r)}{g(r)}}\int_{c_0\max\{r^{\frac{1}{3}},r^{\frac{1}{2}}\}}^{r}\frac{1}{g(y)}\,dy > b_0$$

for all $\mu \le \mu_0$, which guarantees that (3.11) is true. Moreover, since

$$\lim_{y\to+\infty}\frac{f(t,y)}{y} = +\infty$$

uniformly on $[0,1]$, all the conditions of Theorem 3.2 hold, which implies that (3.22)-(3.23) has at least two positive solutions (for $\mu \le \mu_0$).

Example 3.2 Consider

$$y''(t) + \mu\bigl(y^{-\delta_1}(t) + y^{\delta_2}(t)\bigr) = 0, \quad 0 < t < 1, \tag{3.24}$$

with

$$y(0) = \int_0^1 y^3(s)\,dA(s), \qquad y(1) = \int_0^1 y^{\frac{1}{2}}(s)\,dB(s), \quad dA(s) = \frac{1}{2}\,ds, dB(s) = \frac{1}{8}\,de^s, \tag{3.25}$$

where $\delta_1 > 0$, $\delta_2 < 1$.

It is easy to see that all conditions of Theorem 3.3 hold, which implies that (3.24)-(3.25) has at least two positive solutions.

4 Multiplicity of positive solutions for the singular semipositone boundary value problem

In this section, we consider the case

$$f(t,y) = F(t,y) - \gamma(t), \quad t \in (0,1),$$

where the conditions (C_1), (C_3), (C_4) for $F(t,y)$ instead of $f(t,y)$ hold and $\gamma \in C((0,1),(0,+\infty))$ with

$$w(t) = \int_0^1 k(t,s)q(s)\gamma(s)\,ds < +\infty, \quad t \in [0,1], \qquad c_1 = \int_0^1 q(t)\gamma(t)\,dt < +\infty.$$

For $y \in P$, define

$$\begin{aligned}(T_\epsilon y)(t) &= (1-t)\alpha\bigl[[y-w]^*\bigr] + t\beta\bigl[[y-w]^*\bigr] \\ &\quad + \int_0^1 k(t,s)q(s)F\bigl(s,\max\bigl\{\epsilon,\bigl[y(s)-w(s)\bigr]^*\bigr\}\bigr)\,ds, \quad t \in [0,1], 0 < \epsilon \le 1, \end{aligned}\tag{4.1}$$

where $k(t,s)$ is defined in (3.1) and

$$\big[y(t)-w(t)\big]^* = \begin{cases} y(t)-w(t), & \text{if } y(t)-w(t) > 0, \\ 0, & \text{if } y(t)-w(t) \le 0. \end{cases}$$

Now we present the following condition for convenience:

$(\mathrm{C}_2)'$

$$\begin{cases} \text{there exists a function } \psi_{4c_1} \\ \text{continuous on } [0,1] \text{ and positive on } (0,1) \text{ such that} \\ F(t,y) \ge \psi_{4c_1}(t) \text{ on } (0,1)\times(0,4c_1] \text{ with} \\ \max_{t\in[0,1]}\int_0^1 k(t,s)q(s)\psi_{4c_1}(s)\,ds > 2c_1. \end{cases}$$

Define

$$\begin{aligned} H = \big\{ & x \in C\big([0,1],\mathbb{R}\big)\cap C^1\big([0,1),\mathbb{R}\big)\cap C\big((0,1),(0,+\infty)\big)\cap C^2\big((0,1),\mathbb{R}\big) \mid x \text{ satisfies} \\ & x''(t)+q(t)F\big(t,\max\big\{\epsilon,\big[x(t)-w(t)\big]^*\big\}\big)=0, 0<t<1, \\ & x(0)=\alpha\big[[x-w]^*\big], x(1)=\beta\big[[x-w]^*\big], \forall 1\ge\epsilon>0\big\}. \end{aligned}$$

Lemma 4.1 *If $H \ne \emptyset$ and $(\mathrm{C}_2)'$ hold, then there exists a $\delta_0 > 0$ such that*

$$\big[x(t)-w(t)\big]^* \ge \delta_0, \quad \forall t\in[0,1], x\in H.$$

Proof Suppose that $x \in H$. There are two cases to consider:

(1) $\|x\| \ge 4c_1$. Since

$$w(t) \le t(1-t)\int_0^1 q(s)\gamma(s)\,ds = c_1 t(1-t), \tag{4.2}$$

we have

$$w(t) \le \frac{1}{4}4c_1 t(1-t) \le \frac{1}{4}\|x\|t(1-t).$$

From Lemma 2.3, we have

$$x(t) \ge \|x\|t(1-t) \ge 4c_1 t(1-t), \quad \forall t\in(0,1),$$

which implies that

$$\begin{aligned} x(t)-w(t) &\ge \|x\|t(1-t) - \frac{1}{4}\|x\|t(1-t) = \frac{3}{4}\|x\|t(1-t) \\ &\ge \frac{3}{4}4c_1 t(1-t) = 3c_1 t(1-t), \quad t\in[0,1]. \end{aligned}$$

Hence

$$\big[x(t)-w(t)\big]^* \ge 3c_1 t(1-t), \quad t\in[0,1]$$

and so

$$x(0) = \alpha\big[[x-w]^*\big] \geq \int_0^1 \big(3c_1s(1-s)\big)^a\,dA(s), x(1) = \beta\big[[x-w]^*\big] \geq \int_0^1 \big(3c_1s(1-s)\big)^b\,dB(s).$$

The concavity of $x(t)$ implies that

$$x(t) \geq \min\left\{\int_0^1 \big(3c_1s(1-s)\big)^a\,dA(s), \int_0^1 \big(3c_1s(1-s)\big)^b\,dB(s)\right\} \stackrel{\text{def.}}{=} \delta_1.$$

Then

$$\big[x(t)-w(t)\big]^* \geq \frac{3}{4}x(t) \geq \frac{3}{4}\delta_1, \quad t \in [0,1]. \tag{4.3}$$

(2) $0 < \|x\| \leq 4c_1$. Condition $(\mathrm{C}_2)'$ guarantees that

$$\begin{aligned} \|x\| &\geq \max_{t\in[0,1]} \int_0^1 k(t,s)q(s)F\big(s, \max\big\{\epsilon, \big[x(s)-w(s)\big]^*\big\}\big)\,ds \\ &\geq \max_{t\in[0,1]} \int_0^1 k(t,s)q(s)\psi_{4c_1}(s)\,ds > 2c_1, \end{aligned}$$

which together with $x \in P$ implies that

$$x(t) \geq t(1-t)\|x\| \geq 2c_1t(1-t), \quad t \in [0,1]. \tag{4.4}$$

From (4.2) and (4.4), we have

$$w(t) \leq c_1t(1-t) = \frac{1}{2}2c_1t(1-t) \leq \frac{1}{2}x(t), \quad t \in [0,1],$$

and so

$$\big[x(t)-w(t)\big]^* \geq \frac{1}{2}x(t) \geq c_1t(1-t), \quad t \in [0,1].$$

Then

$$\begin{aligned} x(t) &= (1-t)\alpha\big[[x-w]^*\big] + t\beta\big[[x-w]^*\big] + \int_0^1 k(t,s)q(s)F\big(s, \max\big\{\epsilon, \big[x(s)-w(s)\big]^*\big\}\big)\,ds \\ &\geq (1-t)\int_0^1 \big(c_1s(1-s)\big)^a\,dA(s) + t\int_0^1 \big(c_1s(1-s)\big)^b\,dB(s) \\ &\geq \min_{t\in[0,1]}\left[(1-t)\int_0^1 \big(c_1s(1-s)\big)^a\,dA(s) + t\int_0^1 \big(c_1s(1-s)\big)^b\,dB(s)\right], \quad t \in [0,1], \end{aligned}$$

which implies

$$\begin{aligned} \big[x(t)-w(t)\big]^* &\geq \frac{1}{2}x(t) \\ &\geq \frac{1}{2}\min_{t\in[0,1]}\left[(1-t)\int_0^1 \big(c_1s(1-s)\big)^a\,dA(s) + t\int_0^1 \big(c_1s(1-s)\big)^b\,dB(s)\right] \\ &\stackrel{\text{def.}}{=} \delta_2. \end{aligned} \tag{4.5}$$

Let

$$\delta_0 = \min\left\{\delta_2, \frac{3}{4}\delta_1\right\}.$$

Now (4.3) and (4.5) guarantee that

$$\left[x(t) - w(t)\right]^* \geq \delta_0, \quad t \in [0,1].$$

The proof is complete. □

Lemma 4.2 *Suppose there exists an $\bar{a} \in (0, \frac{1}{2})$ such that*

$$\lim_{y \to +\infty} \frac{F(t,y)}{y} = +\infty \tag{4.6}$$

uniformly on $[\bar{a}, 1-\bar{a}]$. Then there exists an $R' > 0$ such that for all $R \geq R'$

$$i(T_\epsilon, \Omega_R \cap P, P) = 0, \quad \forall 0 < \epsilon \leq 1.$$

Proof From (4.6), there exists a $R_1 > \max\{1, 2c_1\}$ such that

$$F(t,y) \geq N^* y, \quad \forall y \geq R_1, \tag{4.7}$$

where

$$N^* > \frac{2}{\bar{a}^2 \int_{\bar{a}}^{1-\bar{a}} k(\bar{a},s)q(s)\,ds}.$$

Let $R' = \frac{2}{\bar{a}^2} R_1$ and

$$\Omega_R = \left\{x \in C[0,1] \mid \|x\| < R\right\}, \quad R \geq R'.$$

Now we show

$$T_\epsilon y \nleq y \quad \text{for } y \in P \cap \partial\Omega_R, 0 < \epsilon \leq 1. \tag{4.8}$$

Suppose that there exists a $y_0 \in P \cap \partial\Omega_R$ with $T_\epsilon y_0 \leq y_0$. Then $\|y_0\| = R$. Also since $y_0(t)$ is concave on $[0,1]$ (since $y_0 \in P$) we have from Lemma 2.3 that $y_0(t) \geq t(1-t)\|y_0\| \geq t(1-t)R$ for $t \in [0,1]$. For $t \in [\bar{a}, 1-\bar{a}]$, we have (notice $\|y_0\| = R \geq 2c_1$)

$$\left[y_0(t) - w(t)\right]^* \geq \frac{1}{2} y_0(t) \geq \frac{1}{2} R\bar{a}^2 \geq R_1, \quad \forall t \in [\bar{a}, 1-\bar{a}],$$

which together with (4.7) yields the result that

$$F\left(t, \max\left\{\epsilon, \left[y_0(t) - w(t)\right]^*\right\}\right) \geq N^*\left[y_0(t) - w(t)\right]^* \geq N^* \frac{1}{2} R\bar{a}^2, \quad \forall t \in [\bar{a}, 1-\bar{a}]. \tag{4.9}$$

Then we have, using (4.9),

$$\begin{aligned}
y_0(\bar{a}) &\ge T_\epsilon y_0(\bar{a}) \\
&= (1-\bar{a})\alpha\bigl[[y_0-w]^*\bigr] + \bar{a}\beta\bigl[[y_0-w]^*\bigr] \\
&\quad + \int_0^1 k(\bar{a},s)q(s)F\bigl(s,\max\bigl\{\epsilon,\bigl[y_0(s)-w(s)\bigr]^*\bigr\}\bigr)\,ds \\
&\ge \int_{\bar{a}}^{1-\bar{a}} k(\bar{a},s)q(s)F\bigl(s,\max\bigl\{\epsilon,\bigl[y_0(s)-w(s)\bigr]^*\bigr\}\bigr)\,ds \\
&= \int_{\bar{a}}^{1-\bar{a}} k(\bar{a},s)q(s)F\bigl(s,\bigl[y_0(s)-w(s)\bigr]^*\bigr)\,ds \\
&\ge N^*\frac{1}{2}R\bar{a}^2\int_{\bar{a}}^{1-\bar{a}} k(\bar{a},s)q(s)\,ds > R = \|y_0\|,
\end{aligned}$$

which is a contradiction. Hence (4.8) is true. Thus Lemma 2.2 guarantees that

$$i(T_\epsilon,\Omega_R\cap P,P)=0,\quad \forall 0<\epsilon\le 1.$$

The proof is complete. □

Lemma 4.3 *Suppose that* $\max\{a,b\}>1$. *Then there exists an* $R'>0$ *such that for all* $R\ge R'$

$$i(T_\epsilon,\Omega_R\cap P,P)=0,\quad \forall 0<\epsilon\le 1.$$

Proof Since $\max\{a,b\}>1$, without loss of generality, we suppose that $a>1$. Let $R'>\max\{1,2c_1\}$ with $\frac{1}{2^a}R'^{a-1}\int_0^1 s^a(1-s)^a\,dA(s)>1$. Set

$$\Omega_R=\bigl\{x\in C[0,1]\mid\|x\|<R\bigr\}.$$

Now we show that

$$T_\epsilon x\not\le x,\quad \forall x\in\partial\Omega_R\cap P,\forall 0<\epsilon\le 1.$$

In fact, suppose that $x_0\in\partial\Omega_R\cap P$ and satisfies

$$T_\epsilon x\le x.$$

Then $\|y_0\|=R$. Also since $y_0(t)$ is concave on $[0,1]$ (since $y_0\in P$) we have from Lemma 2.3 that $y_0(t)\ge t(1-t)\|y_0\|\ge t(1-t)R$ for $t\in[0,1]$. For $t\in[0,1]$ we have

$$\bigl[y_0(t)-w(t)\bigr]^*\ge\frac{1}{2}y_0(t)\ge\frac{1}{2}Rt(1-t).$$

Then

$$\begin{aligned}
R\ge y(0)&=\int_0^1\bigl(\bigl[y_0(s)-w(s)\bigr]^*\bigr)^a\,dA(s)\ge\int_0^1\left(\frac{1}{2}\|y_0\|s(1-s)\right)^a dA(s) \\
&=\frac{1}{2^a}RR^{a-1}\int_0^1 s^a(1-s)^a\,dA(s)\ge\frac{1}{2^a}RR'^{a-1}\int_0^1 s^a(1-s)^a\,dA(s)>R.
\end{aligned}$$

This is a contradiction. Lemma 2.3 guarantees that

$$i(T_\epsilon, \Omega_R \cap P, P) = 0, \quad \forall 0 < \epsilon \leq 1.$$

The proof is complete. □

Theorem 4.1 *Suppose* (C_1), $(C_2)'$, (C_3), *and* (C_4) *hold and the following conditions are satisfied*:

$$\begin{cases} 0 \leq F(t,y) \leq g(y) + h(y) \text{ on } [0,1] \times (0,\infty) \text{ with} \\ g > 0 \text{ continuous and nonincreasing on } (0,\infty), \\ h \geq 0 \text{ continuous on } [0,\infty), \text{ and } \frac{h}{g} \\ \text{nondecreasing on } (0,\infty), \end{cases} \tag{4.10}$$

and

$$\sup_{r\in(2c_1,+\infty)} \frac{1}{1+\frac{h(r)}{g(r)}} \int_{c_0 \max\{r^a, r^b\}}^{r} \frac{dy}{g(\frac{1}{2}y)} > b_0 \tag{4.11}$$

holds; *here*

$$\begin{aligned} c_0 &= \max\left\{ \int_0^1 dA(s), \int_0^1 dB(s) \right\}, \\ b_0 &= \max\left\{ 2\int_0^{\frac{1}{2}} t(1-t)q(t)\,dt, 2\int_{\frac{1}{2}}^1 t(1-t)q(t)\,dt \right\}. \end{aligned} \tag{4.12}$$

Then BVP (1.1)-(1.2) *has at least one positive solution*.

Proof From (4.11), choose $r > 2c_1$, $\epsilon > 0$ with $\epsilon < \min\{1, \frac{1}{2}c_0 \max\{r^a, r^b\}\}$ with

$$\frac{1}{1+\frac{h(r)}{g(r)}} \int_{c_0 \max\{r^a, r^b\}}^{r} \frac{dy}{g(\frac{1}{2}y)} > b_0. \tag{4.13}$$

Let

$$\Omega_1 = \{ y \in C[0,1] \mid \|y\| < r \}.$$

Let T_ϵ be defined as in (4.1). Lemma 3.1 guarantees that $T_\epsilon : P \to P$ is continuous and completely continuous.

Now we show that

$$y \neq \lambda T_\epsilon y, \quad \forall y \in \partial\Omega_1 \cap P, \lambda \in [0,1]. \tag{4.14}$$

Suppose that there is a $y_0 \in \partial\Omega_1 \cap P$ and $\lambda_0 \in [0,1]$ with $y_0 = \lambda_0 T_\epsilon y_0$. Since $y_0(t) \geq t(1-t)\|y_0\| \geq t(1-t)2c_1$ and $w(t) = \int_0^1 k(t,s)q(s)\gamma(s)\,ds \leq t(1-t)\int_0^1 q(s)\gamma(s)\,ds = c_1 t(1-t) = \frac{c_1}{\|y_0\|} t(1-t)\|y_0\| \leq \frac{1}{2}y_0(t)$, we have

$$y_0(t) - w(t) \geq \frac{1}{2}y_0(t), \quad t \in [0,1].$$

Since y_0 satisfies

$$\begin{cases} y_0'' + \lambda_0 q(t)F(t, \max\{\epsilon, [y_0(t) - w(t)]^*\}) = 0, & 0 < t < 1, \\ y_0(0) = \lambda_0\alpha[[y_0 - w]^*], & y_0(1) = \lambda_0\beta[[y_0 - w]^*], \end{cases} \tag{4.15}$$

$y_0(0) = \lambda_0\alpha[[y_0 - w]^*] \leq r^a \int_0^1 dA(s) \leq c_0 r^a < r = \|y_0\|$ and $y_0(1) = \lambda_0\beta[[y_0 - w]^*] \leq r^b \int_0^1 dB(s) \leq c_0 r^b < r = \|y_0\|$, there exists a $t_0 \in (0,1)$ such that $y_0'(t_0) = 0$ and $y_0'(t) \geq 0$ on $(0, t_0)$, $y_0'(t) \leq 0$ on $(t_0, 1)$. For $t \in (0,1)$, it is easy to see that

$$\begin{aligned} -y_0''(t) &\leq g\bigl(\max\{\epsilon, [y_0(t) - w(t)]^*\}\bigr)\left\{1 + \frac{h(\max\{\epsilon, [y_0(t) - w(t)]^*\})}{g(\max\{\epsilon, [y_0(t) - w(t)]^*\})}\right\} q(t) \\ &\leq g\bigl(\max\{\epsilon, [y_0(t) - w(t)]^*\}\bigr)\left\{1 + \frac{h(r)}{g(r)}\right\} q(t). \end{aligned} \tag{4.16}$$

Integrate from t to t_0 to obtain

$$\begin{aligned} y_0'(t) &\leq g\bigl(\max\{\epsilon, [y_0(t) - w(t)]^*\}\bigr)\left\{1 + \frac{h(r)}{g(r)}\right\} \int_t^{t_0} q(s)\,ds \\ &\leq g\left(\max\left\{\epsilon, \frac{1}{2}y_0(t)\right\}\right)\left\{1 + \frac{h(r)}{g(r)}\right\} \int_t^{t_0} q(s)\,ds \\ &\leq g\left(\frac{1}{2}y_0(t)\right)\left\{1 + \frac{h(r)}{g(r)}\right\} \int_t^{t_0} q(s)\,ds \end{aligned}$$

and then integrate from 0 to t_0 to obtain

$$\int_{\alpha[[y_0-w]^*]}^{y_0(t_0)} \frac{dy}{g(\frac{1}{2}y)} \leq \left\{1 + \frac{h(r)}{g(r)}\right\} \int_0^{t_0} sq(s)\,ds \leq \left\{1 + \frac{h(r)}{g(r)}\right\} \frac{1}{1 - t_0} \int_0^{t_0} s(1-s)\,ds,$$

which together with $\alpha[[y_0 - w]^*] \leq \alpha[y_0] \leq c_0 \max\{r^a, r^b\}$ yields

$$\int_{c_0 \max\{r^a, r^\beta\}}^{r} \frac{dy}{g(\frac{1}{2}y)} \leq \int_{\alpha[[y_0-w]^*]}^{r} \frac{dy}{g(\frac{1}{2}y)} \leq \left\{1 + \frac{h(r)}{g(r)}\right\} \frac{1}{1 - t_0} \int_0^{t_0} s(1-s)\,ds. \tag{4.17}$$

Similarly if we integrate (4.16) from t_0 to t ($t \geq t_0$) and then from t_0 to 1 we obtain

$$\int_{c_0 \max\{r^a, r^b\}}^{r} \frac{du}{g(\frac{1}{2}u)} \leq \left\{1 + \frac{h(r)}{g(r)}\right\} \frac{1}{t_0} \int_{t_0}^{1} s(1-s)q(s)\,ds. \tag{4.18}$$

Now (4.17) and (4.18) imply

$$\int_{c_0 \max\{r^a, r^b\}}^{r} \frac{du}{g(\frac{1}{2}u)} \leq b_0 \left\{1 + \frac{h(r)}{g(r)}\right\},$$

which contradicts (4.13). Then (4.14) is true. Lemma 2.1 implies that

$$i(T_\epsilon, \Omega_1 \cap P, P) = 1. \tag{4.19}$$

Thus, there exists an $x_\epsilon \in P \cap \Omega_1$ such that $T_\epsilon x_\epsilon = x_\epsilon$, which yields the result that $H \neq \emptyset$ in Lemma 4.1 and there is a $\delta_0 > 0$ such that

$$\left[x(t) - w(t)\right]^* \geq \delta_0, \quad \forall x \in H. \tag{4.20}$$

Let $\epsilon_0 = \frac{1}{2}\delta_0$ and $T_{\epsilon_0} x_{\epsilon_0} = x_{\epsilon_0}$. Obviously $x_{\epsilon_0} \in H$ and $[x_{\epsilon_0}(t) - w(t)]^* \geq \delta_0$. From

$$x_{\epsilon_0}(t) = (1-t)\alpha\left[[x_{\epsilon_0} - w]^*\right] + t\beta\left[[x_{\epsilon_0} - w]^*\right] + \int_0^1 k(t,s)q(s)F\left(s, \left[\left[x_{\epsilon_0}(s) - w(s)\right]^*\right]\right) ds,$$

we have

$$x_{\epsilon_0}(t) = (1-t)\alpha[x_{\epsilon_0} - w] + t\beta[x_{\epsilon_0} - w] + \int_0^1 k(t,s)q(s)F\left(s, x_{\epsilon_0}(s) - w(s)\right) ds, \quad t \in [0,1].$$

Let $y(t) = x_{\epsilon_0}(t) - w(t)$, $t \in [0,1]$. It is easy to see that $y(t)$ is a positive solution of BVP (1.1)-(1.2). The proof is complete. □

Theorem 4.2 *Suppose the conditions of Theorem* 4.1 *hold and there exists an* $\bar{a} \in (0, \frac{1}{2})$ *such that*

$$\lim_{y \to +\infty} \frac{F(t,y)}{y} = +\infty$$

uniformly on $[\bar{a}, 1 - \bar{a}]$. *Then BVP* (1.1)-(1.2) *has at least two positive solutions.*

Proof Choose r as in (4.13), $\epsilon_0 > 0$ with $\epsilon_0 < \min\{\delta_0, 1, \frac{1}{2}c_0 \max\{r^a, r^b\}\}$, where δ_0 is defined in (4.20), and $R > \max\{1, r\}$ in Lemma 4.2. Set

$$\Omega_1 = \left\{y \in C[0,1] \,|\, \|y\| < r\right\},$$
$$\Omega_2 = \left\{y \in C[0,1] \,|\, \|y\| < R\right\}.$$

From the proof of Theorem 4.1 and Lemma 4.2, we have

$$i(T_{\epsilon_0}, \Omega_1 \cap P, P) = 1$$

and

$$i(T_{\epsilon_0}, \Omega_2 \cap P, P) = 0,$$

which implies that

$$i\left(T_{\epsilon_0}, (\Omega_2 - \Omega_1) \cap P, P\right) = -1.$$

Thus, there exist $x_{1,\epsilon_0} \in \Omega_1 \cap P$ and $x_{2,\epsilon_0} \in (\Omega_2 - \Omega_1) \cap P$ such that

$$T_{\epsilon_0} x_{1,\epsilon_0} = x_{1,\epsilon_0}, \qquad T_{\epsilon_0} x_{2,\epsilon_0} = x_{2,\epsilon_0}.$$

Let $y_{1,\epsilon_0}(t) = x_{1,\epsilon_0}(t) - w(t)$ and $y_{2,\epsilon_0}(t) = x_{2,\epsilon_0}(t) - w(t)$ for all $t \in [0,1]$. It is easy to see that $y_{1,\epsilon_0}(t)$ and $y_{2,\epsilon_0}(t)$ are two positive solutions for BVP (1.1)-(1.2). The proof is complete. □

Theorem 4.3 *Suppose the conditions of Theorem* 4.1 *hold and* $\max\{a,b\} > 1$. *Then BVP* (1.1)-(1.2) *has at least two positive solutions.*

Proof Choose r as in (4.13), $\epsilon_0 > 0$ with $\epsilon_0 < \min\{\delta_0, 1, \frac{1}{2}c_0 \max\{r^a, r^b\}\}$, where δ_0 is defined in (4.20), and $R > \max\{1, r\}$ in Lemma 4.3. Set

$$\Omega_1 = \{y \in C[0,1] \mid \|y\| < r\},$$
$$\Omega_2 = \{y \in C[0,1] \mid \|y\| < R\}.$$

From the proof of Theorem 4.1 and Lemma 4.3, we have

$$i(T_{\epsilon_0}, \Omega_1 \cap P, P) = 1$$

and

$$i(T_{\epsilon_0}, \Omega_2 \cap P, P) = 0,$$

which implies that

$$i\big(T_{\epsilon_0}, (\Omega_2 - \Omega_1) \cap P, P\big) = -1.$$

Thus, there exist $x_{1,\epsilon_0} \in \Omega_1 \cap P$ and $x_{2,\epsilon_0} \in (\Omega_2 - \Omega_1) \cap P$ such that

$$T_{\epsilon_0} x_{1,\epsilon_0} = x_{1,\epsilon_0}, \qquad T_{\epsilon_0} x_{2,\epsilon_0} = x_{2,\epsilon_0}.$$

Let $y_{1,\epsilon_0}(t) = x_{1,\epsilon_0}(t) - w(t)$ and $y_{2,\epsilon_0}(t) = x_{2,\epsilon_0}(t) - w(t)$ for all $t \in [0,1]$. It is easy to see that $y_{1,\epsilon_0}(t)$ and $y_{2,\epsilon_0}(t)$ are two positive solutions for BVP (1.1)-(1.2). The proof is complete. □

Example 4.1 Consider

$$y''(t) + \frac{7}{64}\left(y^{-2}(t) + y^{\delta_1}(t) - \frac{1}{1000}\frac{1}{t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}\right) = 0, \quad 0 < t < 1, \tag{4.21}$$

$$y(0) = \int_0^1 y^{\frac{1}{3}}(s)\,dA(s),$$
$$y(1) = \int_0^1 \big(y(s)\big)^{\frac{1}{2}}\,dB(s), \quad dA(s) = \frac{1}{2}\,ds, dB(s) = \frac{1}{2}\,d\sin s, \tag{4.22}$$

where $\delta_1 > 1$.

Let $q(t) = \frac{7}{64}$, $F(t,y) = y^{-2} + y^{\delta_1}$, $g(y) = y^{-2}$, $h(y) = y^{\delta_1}$, $c_0 = \max\{\int_0^1 dA(s), \int_0^1 dB(s)\} = \frac{1}{2}$, $b_0 = \frac{7}{384}$, $\gamma(t) = \frac{1}{1000}\frac{1}{t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}$, $c_1 = \int_0^1 q(s)\gamma(s)\,ds \le \frac{1}{500}$. It is easy to see that (C_1), (C_3)-(C_4)

and (4.10) hold, and since $F(t,y) \geq \frac{1}{(4c_1)^2} \geq (125)^2$ for $(t,y) \in [0,1] \times (0, 4c_1]$ and

$$\max_{t\in[0,1]} \int_0^1 k(t,s)q(s)(125)^2\,ds = \frac{7}{64}125^2 \max_{t\in[0,1]} \int_0^1 k(t,s)\,ds > 2c_1,$$

we find that $(C_2)'$ is true.

Since

$$\frac{1}{1+\frac{h(1)}{g(1)}} \int_{c_0 \max\{1^{\frac{1}{3}},1^{\frac{1}{2}}\}}^1 \frac{1}{g(\frac{1}{2}y)}\,dy = \frac{1-\frac{1}{8}}{24} = \frac{7}{192},$$

we have

$$\sup_{r\in(2c_1,+\infty)} \frac{1}{1+\frac{h(r)}{g(r)}} \int_{c_0 \max\{r^{\frac{1}{3}},1^{\frac{1}{2}}\}}^r \frac{1}{g(\frac{1}{2}y)}\,dy > b_0,$$

which guarantees that (4.11) is true. Moreover, since

$$\lim_{y\to+\infty} \frac{F(t,y)}{y} = +\infty$$

uniformly on $[0,1]$, all the conditions of Theorem 4.2 hold. Then (4.21)-(4.22) has at least two positive solutions.

Example 4.2 Consider

$$y''(t) + \frac{7}{64}\left(y^{-2}(t) + y^{\delta_1}(t) - \frac{1}{1000}\frac{1}{t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}\right) = 0, \quad 0 < t < 1, \tag{4.23}$$

$$\begin{aligned} &y(0) = \int_0^1 y^3(s)\,dA(s), \\ &y(1) = \int_0^1 \big(y(s)\big)^{\frac{1}{2}}\,dB(s), \quad dA(s) = \frac{1}{2}\,ds, dB(s) = \frac{1}{2}\,d\sin s, \end{aligned} \tag{4.24}$$

where $0 < \delta_1 \leq 1$.

Since $a = 3 > 1$, using Theorem 4.3, we see that (4.23)-(4.24) has at least two positive solutions.

5 Uniqueness of positive solutions for the singular boundary value problem

In this section, we consider the uniqueness of positive solution for BVP (1.1)-(1.2).

Lemma 5.1 (see [20]) *Suppose that E is a Banach space with a normal and solid cone* $P \subseteq E$ *and* $A : \mathring{P} \times \mathring{P} \to \mathring{P}$ *is a mixed monotone operator. Moreover, suppose there is a constant* θ *with* $0 \leq \theta < 1$ *such that*

$$A\left(tx, \frac{1}{t}y\right) \geq t^\theta A(x,y), \quad \forall x, y \in \mathring{P}, 0 < t < 1.$$

Then A has a unique fixed point in $\mathring{P}$.

Theorem 5.1 *Suppose that* (C_1), (C_3), (C_4), (C_5) *hold and* $1 > \max\{a, b\} > 0$. *Then BVP* (1.1)-(1.2) *has a unique positive solution.*

Proof It is easy to see that P defined by (2.1) is a normal and solid cone. For $x, y \in \mathring{P}$, define

$$A(x,y)(t) = (1-t)\alpha[x] + t\beta[y] + \int_0^1 k(t,s)q(s)g\bigl(s, x(s), y(s)\bigr)\,ds, \quad t \in [0,1],$$

where g is given in (C_5). Since $x, y \in \mathring{P}$, we have $\min_{t\in[0,1]} x(t) > 0$ and $\min_{t\in[0,1]} y(t) > 0$. Then ($C_1$) guarantees that

$$\int_0^1 x^a(s)\,dA(s) > 0, \qquad \int_0^1 y^b(s)\,dB(s) > 0,$$

which implies that

$$\min_{t\in[0,1]}\bigl[(1-t)\alpha[x] + t\beta[y]\bigr] > 0.$$

Therefore, $A(x,y) \in \mathring{P}$.

Let $\theta_1 = \max\{\theta, \max\{a,b\}\}$. For $0 < \lambda < 1$ and $x \in \mathring{P}$, $y \in \mathring{P}$, from (C_5), we have

$$\begin{aligned} A\left(\lambda x, \frac{1}{\lambda}y\right)(t) &= (1-t)\alpha[\lambda x] + t\beta[\lambda y] + \int_0^1 k(t,s)g\left(s, \lambda x(s), \frac{1}{\lambda}y(s)\right)ds \\ &\geq (1-t)\lambda^a\alpha[x] + t\lambda^b\beta[y] + \lambda^\theta \int_0^1 k(t,s)g\bigl(s, x(s), y(s)\bigr)\,ds \\ &\geq (1-t)\lambda^{\theta_1}\alpha[x] + t\lambda^{\theta_1}\beta[y] + \lambda^{\theta_1}\int_0^1 k(t,s)g\bigl(s, x(s), y(s)\bigr)\,ds \\ &= \lambda^{\theta_1}A(x,y)(t), \quad \forall t \in [0,1]. \end{aligned}$$

From Lemma 5.1, A has a unique fixed point x^* in $\mathring{P}$, which satisfies

$$\begin{aligned} x^*(t) &= (1-t)\alpha\bigl[x^*\bigr] + t\beta\bigl[x^*\bigr] + \int_0^1 k(t,s)q(s)g\bigl(s, x^*(s), x^*(s)\bigr)\,ds \\ &= (1-t)\alpha\bigl[x^*\bigr] + t\beta\bigl[x^*\bigr] + \int_0^1 k(t,s)q(s)f\bigl(s, x^*(s)\bigr)\,ds, \quad t \in [0,1]. \end{aligned}$$

Then $x^*(t)$ is the unique positive solution of BVP (1.1)-(1.2). The proof is complete. □

Example 5.1 Consider

$$y''(t) + \frac{7}{64}\bigl(y^{-\delta_1}(t) + y^{\delta_2}(t)\bigr) = 0, \quad 0 < t < 1, \tag{5.1}$$

$$\begin{aligned} y(0) &= \int_0^1 y^{\delta_3}(s)\,dA(s), \\ y(1) &= \int_0^1 \bigl(y(s)\bigr)^{\delta_4}\,dA(s), \quad dA(s) = d\sin s^2, dB(s) = de^s, \end{aligned} \tag{5.2}$$

where $0 < \max\{\delta_1, \delta_2, \delta_3, \delta_4\} < 1$.

It is easy to see that all conditions of Theorem 5.1 hold, which guarantees that (5.1)-(5.2) has a unique positive solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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