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# Wave-breaking criterion and global solution for a generalized periodic coupled Camassa-Holm system

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**Abstract**
The local well-posedness for a generalized periodic coupled Camassa-Holm system is established in the Sobolev space $H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$ with $s > \frac{7}{2}$. A wave-breaking criterion of strong solutions is acquired in the Sobolev space $H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$ with $s > \frac{3}{2}$ by employing the localization analysis in the transport equation theory and a sufficient condition of global existence for the system is derived in the Sobolev space $H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$ with $s > 3$.
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## 1 Introduction

In this article, we consider a generalized periodic coupled Camassa-Holm system on the circle $\mathbf{S}$ with $\mathbf{S} = \mathbf{R}/\mathbf{Z}$ (the circle of unit length):

$$\begin{cases} m_t + um_x + 2u_x m + \eta\bar{\eta}_x = 0, & t > 0, x \in \mathbf{R}, \\ \eta_t + (u\eta)_x = 0, & t > 0, x \in \mathbf{R}, \\ m(0,x) = m_0(x), & x \in \mathbf{R}, \\ \eta(0,x) = \eta_0(x), & x \in \mathbf{R}, \\ m(t,x+1) = m(t,x), & t > 0, x \in \mathbf{R}, \\ \eta(t,x+1) = \eta(t,x), & t > 0, x \in \mathbf{R}, \end{cases} \tag{1}$$

where $m = (1 - 2\partial_x^2 + \partial_x^4)u = (1 - \partial_x^2)^2 u$ and $\eta = (1 - \partial_x^2)(\bar{\eta} - \bar{\eta}_0)$ are periodic on the $x$-variable and $\bar{\eta}_0$ is taken as a constant and $\mathbf{R}$ is the set of real numbers. In fact, system (1) is a generalization of two components for the following equation (if $\eta = 0$ in system (1)):

$$m_t + um_x + 2u_x m = 0, \qquad m = (1 - \partial_x^2)^2 u. \tag{2}$$

Equation (2) is firstly derived as the Euler-Poincaré differential equation on the Bott-Virasoro group with respect to the $H^2$ metric [1], and it is known as a modified Camassa-Holm equation and also viewed as a geodesic equation on some diffeomorphism group

[1]. It is shown in [1] that the dynamics of Eq. (2) on the unit circle S is significant different from those of Camassa-Holm equation. For example, Eq. (2) does not conform with a blow-up solution in finite time.

If $m = (1 - \partial_x^2)u$ and $\eta = \bar{\eta} = \rho$ in system (1), system (1) becomes the famous two-component Camassa-Holm system,

$$\begin{cases} (1-\partial_x^2)u_t + u(1-\partial_x^2)u_x + 2u_x(1-\partial_x^2)u + \rho\rho_x = 0, & t > 0, x \in \mathbf{R}, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in \mathbf{R}, \end{cases} \tag{3}$$

where the variable $u(t,x)$ represents the horizontal velocity of the fluid, and $\rho(t,x)$ is related to the free surface elevation from equilibrium with the boundary assumptions, $u \to 0$ and $\rho \to 1$ as $|x| \to \infty$. System (3) was found originally in [2], but it was firstly derived rigorously by Constantin and Ivanov [3]. The system has bi-Hamiltonian structure and is complete integrability. Since the birth of the system, a lot of literature was devoted to the investigation of the two-component Camassa-Holm system, for example, Chen *et al.* [4] established a reciprocal transformation between the two-component Camassa-Holm system and the first negative flow of the AKNS hierarchy. Escher *et al.* [5] used Kato theory to establish local well-posedness for the two-component system and presented some precise blow-up scenarios for strong solutions of the system. Gui and Liu [6, 7] established the local well-posedness for the two-component Camassa-Holm system in the Besov spaces and derived the wave-breaking mechanism and the exact blow-up rate. The dynamics in the periodic case for system (3) was considered in [8]. The other results related to the system can be found in [9–21].

If $m = (1 - \partial_x^2)u$ in system (1), system (1) becomes a modified version of the two-component Camassa-Holm system,

$$\begin{cases} (1-\partial_x^2)u_t + u(1-\partial_x^2)u_x + 2u_x(1-\partial_x^2)u + \eta\bar{\eta}_x = 0, & t > 0, x \in \mathbf{R}, \\ \eta_t + (u\eta)_x = 0, & t > 0, x \in \mathbf{R}, \end{cases} \tag{4}$$

where $u$ denotes the velocity field, $\bar{\eta}$ and $\eta$ represent the average density (or depth) and pointwise density (or depth). System (4) is introduced by Holm *et al.* in [22] and is viewed as geodesic motion on the semidirect product Lie group with respect to a certain metric [22]. System (4) admits peaked solutions in the velocity and average density [22], but it is not integrable unlike system (3). For some other recent work one is referred to Refs. [23–26] for details.

The motivations of the present paper is to find whether or not system (1) has some different dynamics from system (4) mathematically, such as wave breaking and a global solution. Comparing with the modified two-component Camasssa-Holm equation [23], we investigate the local well-posedness, global existence, and a wave-breaking criterion in the Sobolev space. One of the difficulties is the acquisition of the priori estimates $\|u_{xxx}\|_{L^\infty(\mathrm{S})}$. The difficulty has been overcome by Lemma 4.8. We use the technique of [7, 27] to derive a wave-breaking criterion for strong solutions of the system (1) in the low Sobolev spaces $H^s(\mathrm{S}) \times H^{s-1}(\mathrm{S})$ with $s > \frac{3}{2}$. It needs to point out that in the Sobolev spaces $H^s(\mathrm{R}) \times H^{s-1}(\mathrm{R})$ with $s > \frac{3}{2}$ the wave-breaking of the solution of system (4) only depends on the slope of the component $u$ of solution [7]. However, the wave-breaking of the solution for system

(1) is determined only by the slope of the component $\rho$ of solution definitely. It implies that there are differences between system (1) and system (4). On the other hand, we derive a sufficient condition for global solution in the Sobolev space $H^s(S)$ with $s > 3$, which can be done because $\|u_{xx}\|_{L^\infty(S)}$ and $\|\rho_x\|_{L^\infty(S)}$ can be controlled by $\|u\|_{H^s(S)}$ and $\|\rho\|_{H^{s-1}(S)}$ separately if $s > 3$.

The rest of this paper is organized as follows. Section 2 states the main results of present paper. Section 3 is devoted to the study of the local existence and uniqueness of a solution for system (1) by using the Kato theorem. In Section 4, we employ the transport equation theory to prove a wave-breaking criterion in the low Sobolev space $H^s(S) \times H^{s-1}(S)$ with $s > \frac{3}{2}$. The global existence result for system (1) is proved in Section 5.

## 2 The main results

We denote by $*$ the convolution and let $[A,B] = AB - BA$ denote the commutator between $A$ and $B$. Note that if $g(x) := 1 + 2\sum_{n=1}^{\infty} \frac{1}{1+2n^2+n^4}\cos(nx)$, then $(1-\partial_x^2)^{-2}f = g * f$ for all $f \in L^2(R)$ and $g * m = u$ (see [1]). We let $C$ denote all of different positive constants which depend only on initial data. To investigate dynamics of system (1) for the Cauchy problem on the circle, we rewrite system (1) by taking $\rho = \bar{\eta} - \bar{\eta}_0$ and $\eta = \rho - \rho_{xx}$:

$$\begin{cases} u_t + uu_x = -\partial_x g * [u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2 - \frac{1}{2}\rho_x^2], & t > 0, x \in \mathbf{R}, \\ \rho_t + u\rho_x = -\partial_x(1-\partial_x^2)^{-1}(u_x\rho_x) - (1-\partial_x^2)^{-1}(u_x\rho), & t > 0, x \in \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbf{R}, \\ u(t,x+1) = u(t,x), & t > 0, x \in \mathbf{R}, \\ \rho(t,x+1) = \rho(t,x), & t > 0, x \in \mathbf{R}. \end{cases} \tag{5}$$

The main results of the present paper are listed as follows.

**Theorem 2.1** *Given $z_0 = (u_0, \rho_0) \in H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$ $(s > \frac{7}{2})$, there exist a maximal $T = T(\|z_0\|_{H^s(\mathbf{S})\times H^{s-1}(\mathbf{S})})$ and a unique solution $z = (u,\rho)$ to problem* (5), *such that*

$$z = z(\cdot, z_0) \in C\big([0,T); H^s \times H^{s-1}\big) \cap C^1\big([0,T); H^{s-1} \times H^{s-2}\big).$$

*Moreover, the solution depends continuously on the initial data, the mapping*

$$z_0 \to z(\cdot, z_0) : H^s \times H^{s-1} \to C\big([0,T); H^s \times H^{s-1}\big) \cap C^1\big([0,T); H^{s-1} \times H^{s-2}\big)$$

*is continuous.*

The following wave-breaking criterion shows the wave breaking is only determined by the slope of $\rho$ but not the slope of $u$.

**Theorem 2.2** *Let $z_0 = (u_0, \rho_0) \in H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$, $s > \frac{3}{2}$ and $T$ be the maximal existence time of the solution $z = (u,\rho)$ to system* (5). *Assume $m_0 \in L^2(\mathbf{S})$ and $T < \infty$. Then*

$$\int_0^T \big\|\partial_x \rho(\tau)\big\|_{L^\infty}\,d\tau = \infty.$$

A sufficient condition of global existence is given in the following.

**Theorem 2.3** *Let $z_0 = (u_0, \rho_0) \in H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$, $s > 3$. Then system (5) admits a unique solution*

$$z = (u, \rho) \in C\big([0,\infty); H^s \times H^{s-1}\big) \cap C^1\big([0,\infty); H^{s-1} \times H^{s-2}\big).$$

## 3 Local well-posedness

In this section, we establish the local well-posedness by using Kato theory [28].

Set $Y = H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$, $X = H^{s-1}(\mathbf{S}) \times H^{s-2}(\mathbf{S})$, $\Lambda = (1-\partial_x^2)^{\frac{1}{2}}$, $Q = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$ and $f(z) = \begin{pmatrix} -\partial_x(1-\partial_x^2)^{-2}(u^2+u_x^2-\frac{7}{2}u_{xx}^2-3u_xu_{xxx}+\frac{1}{2}\rho^2-\frac{1}{2}\rho_x^2) \\ -\partial_x(1-\partial_x^2)^{-1}(u_x\rho_x)-(1-\partial_x^2)^{-1}(u_x\rho) \end{pmatrix}$.

In order to verify Theorem 2.1, we need the following lemmas in which $\mu_1$, $\mu_2$, $\mu_3$, and $\mu_4$ are constants depending only on $\max\{\|z\|_Y, \|y\|_Y\}$.

**Lemma 3.1** *The operator $A(z) = \begin{pmatrix} u\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix}$ belongs to $G(H^{s-1}(\mathbf{S}) \times H^{s-2}(\mathbf{S}), 1, \beta)$.*

**Lemma 3.2** *Let $A(z) = \begin{pmatrix} u\partial_x & 0 \\ 0 & u\partial_x \end{pmatrix}$, then $A(z) \in L(H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S}), H^{s-1}(\mathbf{S}) \times H^{s-2}(\mathbf{S}))$. Moreover, for all $z, y, w \in H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$,*

$$\big\|\big(A(z)-A(y)\big)w\big\|_{H^{s-1}\times H^{s-2}} \le \mu_1 \|z-y\|_{H^s\times H^{s-1}} \|w\|_{H^s\times H^{s-1}}.$$

**Lemma 3.3** *For $s > \frac{5}{2}$, $z, y \in H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$ and $w \in H^{s-1}(\mathbf{S}) \times H^{s-2}(\mathbf{S})$, we have $B(z) = QA(z)Q^{-1} - A(z) \in L(H^{s-1} \times H^{s-2})$ and*

$$\big\|\big(B(z)-B(y)\big)w\big\|_{H^{s-1}\times H^{s-2}} \le \mu_2 \|z-y\|_{H^s\times H^{s-1}} \|w\|_{H^{s-1}\times H^{s-2}}.$$

The proofs of Lemmas 3.1-3.3 can be found in [5].

**Lemma 3.4** ([28]) *Let $r$, $t$ be real numbers such that $-r < t \le r$. Then*

$$\|fg\|_{H^t} \le C\|f\|_{H^r}\|g\|_{H^t}, \quad \text{if } r > \frac{1}{2},$$
$$\|fg\|_{H^{t+r-\frac{1}{2}}} \le C\|f\|_{H^r}\|g\|_{H^t}, \quad \text{if } r < \frac{1}{2},$$

*where $C$ is a positive constant depending on $r$, $t$.*

**Lemma 3.5** *Let*

$$f(z) = \begin{pmatrix} -\partial_x(1-\partial_x^2)^{-2}(u^2+u_x^2-\frac{7}{2}u_{xx}^2-3u_xu_{xxx}+\frac{1}{2}\rho^2-\frac{1}{2}\rho_x^2) \\ -\partial_x(1-\partial_x^2)^{-1}(u_x\rho_x)-(1-\partial_x^2)^{-1}(u_x\rho) \end{pmatrix}.$$

*Then $f(z)$ is bounded on bounded sets in $H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$ with $s > \frac{7}{2}$ and satisfies the following*:

(a) $\|f(z)-f(y)\|_{H^s\times H^{s-1}} \le \mu_3\|z-y\|_{H^s\times H^{s-1}}$, $z, y \in H^s \times H^{s-1}$;

(b) $\|f(z)-f(y)\|_{H^{s-1}\times H^{s-2}} \le \mu_4\|z-y\|_{H^{s-1}\times H^{s-2}}$, $z, y \in H^{s-1} \times H^{s-2}$.

*Proof* (a) Let $y = (v, \sigma)$, we have

$$\begin{aligned}
&\|f(z) - f(y)\|_{H^s \times H^{s-1}} \\
&\quad \le \left\| \partial_x (1 - \partial_x^2)^{-2} \left( u^2 + u_x^2 - \frac{7}{2} u_{xx}^2 - 3u_x u_{xxx} - v^2 - v_x^2 + \frac{7}{2} v_{xx}^2 + 3 v_x v_{xxx} \right) \right\|_{H^s} \\
&\qquad + \left\| \partial_x (1 - \partial_x^2)^{-2} \left( \frac{1}{2}\rho^2 - \frac{1}{2}\sigma^2 \right) \right\|_{H^s} \\
&\qquad + \left\| \partial_x (1 - \partial_x^2)^{-2} \left( \frac{1}{2}\rho_x^2 - \frac{1}{2}\sigma_x^2 \right) \right\|_{H^s} + \left\| (1 - \partial_x^2)^{-1} (u_x \rho - v_x \sigma) \right\|_{H^{s-1}} \\
&\qquad + \left\| \partial_x (1 - \partial_x^2)^{-1} (u_x \rho_x - v_x \sigma_x) \right\|_{H^{s-1}} \\
&\quad \le \|u^2 - v^2\|_{H^{s-3}} + \|u_x^2 - v_x^2\|_{H^{s-3}} + \|u_{xx}^2 - v_{xx}^2\|_{H^{s-3}} \\
&\qquad + \|u_x u_{xxx} - v_x v_{xxx}\|_{H^{s-3}} + \|\rho^2 - \sigma^2\|_{H^{s-3}} + \|\rho_x^2 - \sigma_x^2\|_{H^{s-3}} \\
&\qquad + \|u_x \rho_x - v_x \sigma_x\|_{H^{s-2}} + \|u_x \rho - v_x \sigma\|_{H^{s-3}}.
\end{aligned} \tag{6}$$

Noting that $s > \frac{7}{2}$, we have

$$\|u^2 - v^2\|_{H^{s-3}} \le C\|u + v\|_{H^{s-3}} \|u - v\|_{H^{s-3}} \le C\|u - v\|_{H^s}, \tag{7}$$

$$\|u_x^2 - v_x^2\|_{H^{s-3}} \le C\|u - v\|_{H^s}, \tag{8}$$

$$\|u_{xx}^2 - v_{xx}^2\|_{H^{s-3}} \le C\|u - v\|_{H^s}, \tag{9}$$

$$\begin{aligned}
&\|u_x u_{xxx} - v_x v_{xxx}\|_{H^{s-3}} \\
&\quad \le \|u_x u_{xxx} - u_x v_{xxx} + u_x v_{xxx} - v_x v_{xxx}\|_{H^{s-3}} \\
&\quad \le \|u_x u_{xxx} - u_x v_{xxx}\|_{H^{s-3}} + \|u_x v_{xxx} - v_x v_{xxx}\|_{H^{s-3}} \\
&\quad \le C\|u\|_{H^{s-2}} \|u - v\|_{H^s} + C\|u - v\|_{H^{s-1}} \|v\|_{H^s} \\
&\quad \le C\|u - v\|_{H^s},
\end{aligned} \tag{10}$$

$$\|\rho^2 - \sigma^2\|_{H^{s-3}} \le C\|\rho - \sigma\|_{H^{s-1}}, \tag{11}$$

and

$$\|\rho_x^2 - \sigma_x^2\|_{H^{s-3}} \le C\|\rho - \sigma\|_{H^{s-1}}. \tag{12}$$

Similarly, for the last two terms on the right-hand side of Eq. (6), we get

$$\begin{aligned}
\|u_x \rho_x - v_x \sigma_x\|_{H^{s-2}} &\le \|u_x \rho_x - u_x \sigma_x\|_{H^{s-2}} + \|u_x \sigma_x - v_x \sigma_x\|_{H^{s-2}} \\
&\le C\|u\|_{H^s} \|\rho - \sigma\|_{H^{s-1}} + C\|\sigma\|_{H^{s-1}} \|u - v\|_{H^s}
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\|u_x \rho - v_x \sigma\|_{H^{s-3}} &\le \|u_x \rho - u_x \sigma\|_{H^{s-3}} + \|u_x \sigma - v_x \sigma\|_{H^{s-3}} \\
&\le C\|u\|_{H^{s-2}} \|\rho - \sigma\|_{H^{s-1}} + C\|\sigma\|_{H^{s-3}} \|u - v\|_{H^s}.
\end{aligned} \tag{14}$$

Therefore, from Eqs. (7)-(14), we obtain

$$\begin{aligned}\|f(z)-f(y)\|_{H^s\times H^{s-1}} &\le C\|u-v\|_{H^s}+C\|\rho-\sigma\|_{H^{s-1}}\\ &= \mu_3\|z-y\|_{H^s\times H^{s-1}},\end{aligned}\tag{15}$$

from which we know (a) holds.

Now, we prove (b). We have

$$\begin{aligned}&\|f(z)-f(y)\|_{H^{s-1}\times H^{s-2}}\\ &\quad\le \left\|\partial_x\left(1-\partial_x^2\right)^{-2}\left(u^2+u_x^2-\frac{7}{2}u_{xx}^2-3u_xu_{xxx}-v^2-v_x^2+\frac{7}{2}v_{xx}^2+3v_xv_{xxx}\right)\right\|_{H^{s-1}}\\ &\qquad+\left\|\partial_x\left(1-\partial_x^2\right)^{-2}\left(\frac{1}{2}\rho^2-\frac{1}{2}\sigma^2\right)\right\|_{H^{s-1}}\\ &\qquad+\left\|\partial_x\left(1-\partial_x^2\right)^{-2}\left(\frac{1}{2}\rho^2-\frac{1}{2}\sigma^2\right)\right\|_{H^{s-1}}+\left\|\left(1-\partial_x^2\right)^{-1}(u_x\rho-v_x\sigma)\right\|_{H^{s-2}}\\ &\qquad+\left\|\partial_x\left(1-\partial_x^2\right)^{-1}(u_x\rho_x-v_x\sigma_x)\right\|_{H^{s-2}}\\ &\quad\le \left\|u^2-v^2\right\|_{H^{s-4}}+\left\|u_x^2-v_x^2\right\|_{H^{s-4}}+\left\|u_{xx}^2-v_{xx}^2\right\|_{H^{s-4}}\\ &\qquad+\|u_xu_{xxx}-v_xv_{xxx}\|_{H^{s-4}}+\left\|\rho_x^2-\sigma_x^2\right\|_{H^{s-4}}+\left\|\rho^2-\sigma^2\right\|_{H^{s-4}}\\ &\qquad+\|u_x\rho-v_x\sigma\|_{H^{s-4}}+\|u_x\rho_x-v_x\sigma_x\|_{H^{s-3}}.\end{aligned}\tag{16}$$

Note that $s>\frac{7}{2}$. Using Lemma 3.4 with $t=s-4$ and $r=s-3$ gives rise to

$$\begin{aligned}\left\|u^2-v^2\right\|_{H^{s-4}} &\le \left\|(u+v)(u-v)\right\|_{H^{s-4}}\\ &\le C\|u+v\|_{H^{s-4}}\|u-v\|_{H^{s-3}}\le C\|u-v\|_{H^{s-1}}.\end{aligned}\tag{17}$$

In an analogous way to Eq. (17), we have

$$\left\|u_x^2-v_x^2\right\|_{H^{s-4}}\le C\|u-v\|_{H^{s-1}},\tag{18}$$

$$\left\|u_{xx}^2-v_{xx}^2\right\|_{H^{s-4}}\le C\|u-v\|_{H^{s-1}},\tag{19}$$

$$\left\|\rho^2-\sigma^2\right\|_{H^{s-4}}\le C\|\rho-\sigma\|_{H^{s-2}},\tag{20}$$

and

$$\left\|\rho_x^2-\sigma_x^2\right\|_{H^{s-4}}\le C\|\rho-\sigma\|_{H^{s-2}}.\tag{21}$$

For the fourth term on the right-hand side of Eq. (16), one has

$$\begin{aligned}\|u_xu_{xxx}-v_xv_{xxx}\|_{H^{s-4}} &\le \|u_xu_{xxx}-u_xv_{xxx}+u_xv_{xxx}-v_xv_{xxx}\|_{H^{s-4}}\\ &\le \|u_xu_{xxx}-u_xv_{xxx}\|_{H^{s-4}}+\|u_xv_{xxx}-v_xv_{xxx}\|_{H^{s-4}}\\ &\le C\|u_{xxx}-v_{xxx}\|_{H^{s-4}}\|u_x\|_{H^{s-2}}+\|u_x-v_x\|_{H^{s-2}}\|v_{xxx}\|_{H^{s-4}}\\ &\le C\|u-v\|_{H^{s-1}},\end{aligned}\tag{22}$$

where we used Lemma 3.4.

In an analogous way to Eq. (22), we can estimate the last two terms on the right-hand side of Eq. (16):

$$\begin{aligned}\|u_x\rho_x - v_x\sigma_x\|_{H^{s-3}} &\le \|u_x\rho_x - u_x\sigma_x\|_{H^{s-3}} + \|u_x\sigma_x - v_x\sigma_x\|_{H^{s-3}} \\ &\le C\|\rho - \sigma\|_{H^{s-2}} + C\|u - v\|_{H^{s-1}}\end{aligned} \tag{23}$$

and

$$\begin{aligned}\|u_x\rho - v_x\sigma\|_{H^{s-4}} &\le \|u_x\rho - u_x\sigma\|_{H^{s-4}} + \|u_x\sigma - v_x\sigma\|_{H^{s-4}} \\ &\le C\|\rho - \sigma\|_{H^{s-2}} + C\|u - v\|_{H^{s-1}}.\end{aligned} \tag{24}$$

Therefore, from Eqs. (16)-(24), we deduce

$$\begin{aligned}\|f(z) - f(y)\|_{H^{s-1}\times H^{s-2}} &\le C\|u - v\|_{H^{s-1}} + C\|\rho - \sigma\|_{H^{s-2}} \\ &= \mu_4\|z - y\|_{H^{s-1}\times H^{s-2}}.\end{aligned} \tag{25}$$

This completes the proof of Lemma 3.5. □

*Proof of Theorem* 2.1 Applying the Kato theorem for abstract quasi-linear evolution equations of hyperbolic type [28], Lemmas 3.1-3.3 and 3.5, we obtain the local well-posedness of system (5) in $H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$, $s > \frac{7}{2}$, and

$$z = z(\cdot, z_0) \in C\big([0,T); H^s \times H^{s-1}\big) \cap C^1\big([0,T); H^{s-1} \times H^{s-2}\big).$$ □

## 4 Wave-breaking criterion

In order to prove Theorem 2.2, the following lemmas are crucial.

**Lemma 4.1** ([1, 7, 29]) *The following estimates hold*:

(i) *For* $s \ge 0$,

$$\|fg\|_{H^s} \le C\big(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}\big). \tag{26}$$

(ii) *For* $s > 0$,

$$\|f\partial_x g\|_{H^s} \le C\big(\|f\|_{H^{s+1}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|\partial_x g\|_{H^s}\big). \tag{27}$$

(iii) *For* $s \le 0$,

$$\|fg\|_{H^s} \le C\|f\|_{L^\infty}\|g\|_{H^s}, \tag{28}$$

*where C is a constant independent of f and g*.

**Lemma 4.2** ([29, 30]) *Suppose that* $s > -\frac{d}{2}$. *Let v be a vector field such that* $\nabla v$ *belongs to* $L^1([0,T]; H^{s-1})$ *if* $s > 1 + \frac{d}{2}$ *or to* $L^1([0,T]; H^{\frac{d}{2}} \cap L^\infty)$ *otherwise. Suppose also that* $f_0 \in H^s$,

*$F \in L^1([0,T];H^s)$ and that $f \in L^\infty([0,T];H^s) \cap C([0,T];S')$ solves the d-dimensional linear transport equations*

$$\begin{cases} f_t + v \cdot \nabla f = F, \\ f|_{t=0} = f_0. \end{cases} \tag{29}$$

*Then $f \in C([0,T];H^s)$. More precisely, there exists a constant $C$ depending only $s$, $p$, and $d$, and such that the following statements hold*:

(1) *If $s \neq 1 + \frac{d}{2}$,*

$$\|f\|_{H^s} \leq \|f_0\|_{H^s} + C\int_0^t \|F(\tau)\|_{H^s}\,d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{H^s}\,d\tau, \tag{30}$$

*or hence*

$$\|f\|_{H^s} \leq e^{CV(t)}\left(\|f_0\|_{H^s} + \int_0^t e^{-CV(t)}\|F(\tau)\|_{H^s}\,d\tau\right) \tag{31}$$

*with $V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{\frac{d}{2}} \cap L^\infty}\,d\tau$ if $s < 1 + \frac{d}{2}$ and $V(t) = \int_0^t \|\nabla v(\tau)\|_{H^{s-1}}\,d\tau$ else.*

(2) *If $f = v$, then for all $s > 0$, the estimates* (30) *and* (31) *hold with $V(t) = \int_0^t \|\partial_x u(\tau)\|_{L^\infty}\,d\tau$.*

**Lemma 4.3** ([7]) *Let $0 < \sigma < 1$. Suppose that $f_0 \in H^\sigma$, $g \in L^1([0,T];H^\sigma)$, $v$, $\partial_x v \in L^1([0,T];L^\infty)$ and $f \in L^\infty([0,T];H^\sigma) \cap C([0,T];S')$ solves the 1-dimensional linear transport equation*

$$\begin{cases} f_t + v\partial_x f = g, \\ f|_{t=0} = f_0. \end{cases} \tag{32}$$

*Then $f \in C([0,T];H^\sigma)$. More precisely, there exists a constant $C$ depending only on $\sigma$, such that the following statement holds*:

$$\|f\|_{H^\sigma} \leq \|f_0\|_{H^\sigma} + C\int_0^t \|g(\tau)\|_{H^\sigma}\,d\tau + C\int_0^t V'(\tau)\|f(\tau)\|_{H^\sigma}\,d\tau, \tag{33}$$

*or hence*

$$\|f\|_{H^\sigma} \leq e^{CV(t)}\left(\|f_0\|_{H^\sigma} + \int_0^t C\|g(\tau)\|_{H^\sigma}\,d\tau\right) \tag{34}$$

*with $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|\partial_x v(\tau)\|_{L^\infty})\,d\tau$.*

**Lemma 4.4** *For all $x \in \mathbf{R}$, the following statements hold*:

$$\text{(i)} \quad \|\partial_x^3 g\|_{L^\infty} \leq 2 + \ln 2 + \pi \tag{35}$$

*and*

$$\text{(ii)} \quad \|u_{xx}\|_{L^\infty} \leq C\|u_{xxx}\|_{L^\infty}. \tag{36}$$

*Proof* Let $g(x)$ be the Green's function for the operator $(1-\partial_x^2)^2$. Then from

$$\left(1-2\partial_x^2+\partial_x^4\right)g(x)=\delta(x)=\sum_{n=-\infty}^{\infty}e^{inx},$$

we get

$$g(x)=\sum_{n=-\infty}^{\infty}\frac{1}{1+2n^2+n^4}e^{inx}=1+2\sum_{n=1}^{\infty}\frac{1}{1+2n^2+n^4}\cos(nx).$$

Hence,

$$g_{xxx}(x)=2\sum_{n=1}^{\infty}\frac{n^3}{1+2n^2+n^4}\sin(nx).$$

From the Fourier series, we know

$$h(x)=\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}=\frac{\pi}{2}\left(1-\frac{x}{\pi}\right)\quad\text{for } 0<x<2\pi,$$

from which we get

$$\begin{aligned}\left|2h(x)-g_{xxx}\right| &= \left|2\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{n^3}{1+2n^2+n^4}\right)\sin(nx)\right|\\ &\leq 2\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{n^3}{1+2n^2+n^4}\right)\\ &= 2\sum_{n=1}^{\infty}\left(\frac{1}{n(1+n^2)}+\frac{n}{(1+n^2)^2}\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\sum_{n=2}^{\infty}\left(\frac{1}{n(1+n^2)}+\frac{n}{(1+n^2)^2}\right) &\leq \lim_{n\to\infty}\int_1^n\left(\frac{1}{x}-\frac{x}{1+x^2}+\frac{x}{(1+x^2)^2}\right)dx\\ &= \frac{1}{2}\ln 2+\frac{1}{4}.\end{aligned}$$

Hence, we have

$$\left\|\partial_x^3 g\right\|_{L^\infty}\leq 2+\ln 2+\pi.$$

This completes the proof of (i).

Now, we prove (ii). Let $x_0\in\mathbf{S}$ satisfy $u_{xxx}(x_0)=0$. Then, for all $x\in\mathbf{S}$, we have

$$u_{xx}=\int_{x_0}^{x}u_{xxx}\,dx,$$

from which one finds

$$\|u_{xx}\|_{L^\infty}\leq C\|u_{xxx}\|_{L^\infty}.$$

□

**Lemma 4.5** *Let $z_0 = (u_0, \rho_0) \in H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$ with $s \geq 2$. Suppose that $T$ is the maximal existence time of solution $z = (u, \rho)$ of system* (5) *with the initial data $z_0$. Then, for all $t \in [0, T)$, the following conservation law holds*:

$$H = \left(\int_S \left(u^2 + 2u_x^2 + u_{xx}^2 + \rho^2 + \rho_x^2\right) dx\right)^{\frac{1}{2}} = \left(\int_S \left(u_0^2 + 2u_{0x}^2 + u_{0xx}^2 + \rho_0^2 + \rho_{0x}^2\right) dx\right)^{\frac{1}{2}}. \tag{37}$$

*Proof* Multiplying the first equation of system (1) by $u$ and integrating by parts, we reach

$$\frac{1}{2}\frac{d}{dt}\int_S \left(u^2 + 2u_x^2 + u_{xx}^2\right) dx + \int_S u\rho\rho_x\, dx - \int_S u\rho_x\rho_{xx}\, dx = 0. \tag{38}$$

Multiplying the second equation of system (1) by $\rho$ and integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int_S \left(\rho^2 + \rho_x^2\right) dx + \int_S u\rho\rho_x\, dx - \int_S u\rho\rho_{xxx}\, dx + \int_S u_x\rho^2\, dx - \int_S u_x\rho\rho_{xx}\, dx = 0, \tag{39}$$

which together with Eq. (38) yields

$$\frac{1}{2}\frac{d}{dt}\int_S \left(u^2 + 2u_x^2 + u_{xx}^2 + \rho^2 + \rho_x^2\right) dx = 0, \tag{40}$$

which implies Eq. (37).

Let us consider the following differential equation.

$$\begin{cases} q_t = u(t,q), & t \in [0,T), \\ q(0,x) = x, & x \in R, \end{cases} \tag{41}$$

where $u$ denotes the first component of solution $z$ to system (5). □

**Lemma 4.6** (See [25]) *Let $u \in C([0,T); H^s(\mathbf{S})) \cup C^1([0,T); H^{s-1}(\mathbf{S}))$, $s \geq 2$. Then Eq.* (41) *has a unique solution $q \in C^1([0,T) \times \mathbf{S}; \mathbf{S})$. Moreover, the map $q(t,\cdot)$ is an increasing diffeomorphism of $R$ with*

$$q_x(t,x) = \exp\left(\int_0^t u_x\left(s, q(s,x)\right) ds\right), \quad \forall (t,x) \in [0,T) \times R.$$

**Lemma 4.7** *Let $z_0 = (u_0, \rho_0) \in H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$ with $s \geq 2$ and $T > 0$ be the maximal existence time of the corresponding solution $z = (u, \rho)$ to system* (5). *Then we have*

$$(\rho_{xx} - \rho)\left(t, q(t,x)\right)q_x(t,x) = \rho_{0xx}(x) - \rho_0(x), \quad \forall (t,x) \in [0,T) \times \mathbf{S}. \tag{42}$$

*Moreover, for all $(t,x) \in [0,T) \times \mathbf{S}$, we have*

$$\|\rho_{xx}\|_{L^\infty} \leq H + |\rho_{0xx} - \rho_0| e^{TH} := \mu. \tag{43}$$

*Proof* Differentiating the left-hand side of Eq. (42) with respect to $t$ and making use of system (5), we get

$$\begin{aligned} &\frac{d}{dt}\big[\big(\rho_{xx}\big(t,q(t,x)\big)-\rho\big(t,q(t,x)\big)\big)q_x(t,x)\big] \\ &\quad = (\rho_{txx}+\rho_{xxx}q_t-\rho_t-\rho_x q_t)q_x+(\rho_{xx}-\rho)q_{tx} \\ &\quad = (\rho_{txx}+\rho_{xxx}u-\rho_t-\rho_x u+\rho_{xx}u_x-\rho u_x)q_x \\ &\quad = 0. \end{aligned}$$

This proves Eq. (42). From Eq. (42), we obtain for all $t \in [0,T)$

$$|\rho_{xx}|-|\rho| \le |\rho_{xx}-\rho| = |\rho_{0xx}-\rho_0|e^{-\int_0^t u_x(s,q(s,x))\,ds},$$

which results in

$$\begin{aligned} \|\rho_{xx}\|_{L^\infty} &\le \big(\|u_0\|_{H^2}^2+\|\rho_0\|_{H^1}^2\big)^{\frac{1}{2}}+|\rho_{0xx}-\rho_0|e^{t(\|u_0\|_{H^2}^2+\|\rho_0\|_{H^1}^2)^{\frac{1}{2}}} \\ &\le H+|\rho_{0xx}-\rho_0|e^{TH}, \end{aligned} \tag{44}$$

where Lemma 4.5 is used. This completes the proof of Lemma 4.7. □

**Lemma 4.8** *Let $z_0=(u_0,\rho_0)\in H^s(\mathbf{S})\times H^{s-1}(\mathbf{S})$ with $s\ge 2$. Suppose that $m_0\in L^2(\mathbf{S})$ and $T$ is the maximal existence time of solution $z=(u,\rho)$ of system* (5) *with the initial data $z_0$.*

$$\|u_{xxx}\|_{L^\infty} \le C\sqrt{\|m_0\|_{L^2}^2+\big(H^2+H\mu\big)T}\,e^{\frac{1}{2}(3H+H^2+H\mu)T} := M. \tag{45}$$

*Proof* Multiplying the first equation of system (1) by $m$ and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_S m^2\,dx = -\int_S umm_x\,dx-2\int_S u_x m^2\,dx-\int_S m\rho\rho_x\,dx+\int_S m\rho_x\rho_{xx}\,dx, \tag{46}$$

which results in

$$\frac{d}{dt}\int_S m^2\,dx = -3\int_S u_x m^2\,dx-2\int_S m\rho\rho_x\,dx+2\int_S m\rho_x\rho_{xx}\,dx. \tag{47}$$

By the Hölder inequality, Eq. (47) ensures that

$$\begin{aligned} \frac{d}{dt}\|m\|_{L^2}^2 &\le 3\|u_x\|_{L^\infty}\|m\|_{L^2}^2+2\|m\|_{L^2}\|\rho\|_{L^2}\|\rho_x\|_{L^\infty} \\ &\quad +2\|m\|_{L^2}\|\rho_x\|_{L^2}\|\rho_{xx}\|_{L^\infty} \\ &\le 3\|u_x\|_{L^\infty}\|m\|_{L^2}^2+\big(1+\|m\|_{L^2}^2\big) \\ &\quad \times\big(\|\rho\|_{L^2}\|\rho_x\|_{L^\infty}+\|\rho_x\|_{L^2}\|\rho_{xx}\|_{L^\infty}\big) \\ &\le \|m\|_{L^2}^2\big(3\|u_x\|_{L^\infty}+\|\rho\|_{L^2}\|\rho_x\|_{L^\infty}+\|\rho_x\|_{L^2}\|\rho_{xx}\|_{L^\infty}\big) \\ &\quad +\big(\|\rho\|_{L^2}\|\rho_x\|_{L^\infty}+\|\rho_x\|_{L^2}\|\rho_{xx}\|_{L^\infty}\big). \end{aligned}$$

Applying the Gronwall inequality, we get

$$\|m\|_{L^2}^2 \le \left(\|m_0\|_{L^2}^2 + \int_0^t \big(\|\rho\|_{L^2}\|\rho_x\|_{L^\infty} + \|\rho_x\|_{L^2}\|\rho_{xx}\|_{L^\infty}\big)\,d\tau\right) \times e^{\int_0^t (3\|u_x\|_{L^\infty} + \|\rho\|_{L^2}\|\rho_x\|_{L^\infty} + \|\rho_x\|_{L^2}\|\rho_{xx}\|_{L^\infty})\,d\tau},$$

which, together with Eqs. (37) and (43), yields

$$\|m\|_{L^2}^2 \le \big(\|m_0\|_{L^2}^2 + \big(H^2 + H\mu\big)T\big)e^{(3H+H^2+H\mu)T}. \tag{48}$$

On the other hand, from Lemma 4.4, we deduce

$$\|u_{xxx}\|_{L^\infty} = \|g_{xxx} * m\|_{L^\infty} \le \|g_{xxx}\|_{L^\infty}\|m\|_{L^1} \le C\|m\|_{L^2}. \tag{49}$$

Therefore, from Eq. (48) we deduce that Eq. (45) holds. This completes the proof of Lemma 4.8. □

Next, we give the proof of Theorem 2.2.

*Proof of Theorem* 2.2 We split the proof of Theorem 2.2 into five steps.

*Step* 1. For $s \in (\frac{3}{2}, 2)$, applying Lemma 4.3 to the second equation, we have

$$\begin{aligned}\|\rho\|_{H^{s-1}(\mathbf{S})} &\le \|\rho_0\|_{H^{s-1}} + C\int_0^t \|\rho\|_{H^{s-1}}\big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\big)\,d\tau \\ &\quad + C\int_0^t \big\|\partial_x\big(1-\partial_x^2\big)^{-1}(u_x\rho_x) + \big(1-\partial_x^2\big)^{-1}(u_x\rho)\big\|_{H^{s-1}}\,d\tau.\end{aligned}$$

From Lemma 4.1(iii), we get

$$\big\|\partial_x\big(1-\partial_x^2\big)^{-1}(u_x\rho_x)\big\|_{H^{s-1}(\mathbf{S})} \le C\|\rho\|_{H^{s-1}}\|u_x\|_{L^\infty} \tag{50}$$

and

$$\big\|\big(1-\partial_x^2\big)^{-1}(u_x\rho)\big\|_{H^{s-1}(\mathbf{S})} \le C\|\rho\|_{H^{s-1}}\|u_x\|_{L^\infty}. \tag{51}$$

From Eqs. (50) and (51), we obtain

$$\|\rho\|_{H^{s-1}(\mathbf{S})} \le \|\rho_0\|_{H^{s-1}} + C\int_0^t \|\rho\|_{H^{s-1}}\big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\big)\,d\tau. \tag{52}$$

On the other hand, using Lemma 4.2, we get from the first equation of system (5)

$$\begin{aligned}&\big\|u(t)\big\|_{H^s(\mathbf{S})} \\ &\quad \le C\int_0^t \left\|\partial_x g * \left[u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2 - \frac{1}{2}\rho_x^2\right]\right\|_{H^s} d\tau \\ &\qquad + \|u_0\|_{H^s} + C\int_0^t \big\|u(t)\big\|_{H^s}\big\|\partial_x u(\tau)\big\|_{L^\infty}\,d\tau.\end{aligned}$$

Thanks to Lemma 4.1(iii), one has

$$\begin{aligned}
&\left\|\partial_x g * \left[u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2 - \frac{1}{2}\rho_x^2\right]\right\|_{H^s} \\
&\quad \le C\left\|u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2 - \frac{1}{2}\rho_x^2\right\|_{H^{s-3}} \\
&\quad \le C\big(\|u\|_{H^{s-3}}\|u\|_{L^\infty} + \|u_x\|_{H^{s-3}}\|u_x\|_{L^\infty} + \|u_{xx}\|_{H^{s-3}}\|u_{xx}\|_{L^\infty} \\
&\qquad + \|u_{xxx}\|_{H^{s-3}}\|u_x\|_{L^\infty} + \|\rho\|_{H^{s-3}}\|\rho\|_{L^\infty} + \|\rho_x\|_{H^{s-3}}\|\rho_x\|_{L^\infty}\big).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\left\|u(t)\right\|_{H^s(\mathbf{S})} \\
&\quad \le \|u_0\|_{H^s} + C\int_0^t \|u\|_{H^s}\big(\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty}\big)\,d\tau \\
&\qquad + C\int_0^t \left\|\rho(\tau)\right\|_{H^{s-1}}\big(\left\|\rho(\tau)\right\|_{L^\infty} + \left\|\rho_x(\tau)\right\|_{L^\infty}\big)\,d\tau,
\end{aligned} \tag{53}$$

which, together with Eq. (52), ensures that

$$\begin{aligned}
&\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}} \\
&\quad \le \|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} + C\int_0^t \big(\|u\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}}\big) \\
&\qquad \times \big(\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty} + \|\rho\|_{L^\infty} + \|\rho_x\|_{L^\infty}\big)\,d\tau.
\end{aligned} \tag{54}$$

Using the Gronwall inequality and Lemma 4.4, we have

$$\begin{aligned}
&\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}} \\
&\quad \le \big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\big) \\
&\qquad \times e^{C\int_0^t(\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xxx}\|_{L^\infty} + \|\rho\|_{L^\infty} + \|\rho_x\|_{L^\infty})\,d\tau}.
\end{aligned} \tag{55}$$

From Lemmas 4.5 and 4.8, we get

$$\begin{aligned}
&\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}} \\
&\quad \le \big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\big)e^{(C(M+H)T + C\int_0^t \|\rho_x\|_{L^\infty}\,d\tau)}.
\end{aligned} \tag{56}$$

Therefore, if the maximal existence time $T < \infty$ satisfies $\int_0^t \|\rho_x\|_{L^\infty}\,d\tau < \infty$, we get from Eq. (56)

$$\limsup_{t\to T}\big(\left\|u(t)\right\|_{H^s} + \left\|\rho(t)\right\|_{H^{s-1}}\big) < \infty, \tag{57}$$

which contradicts the assumption on the maximal time $T < \infty$. This completes the proof of Theorem 2.2 for $s \in (\frac{3}{2}, 2)$.

*Step* 2. For $s \in [2, \frac{5}{2})$, applying Lemma 4.2 to the second equation of system (5), we get

$$\begin{aligned}\|\rho\|_{H^{s-1}(\mathbf{S})} &\leq \|\rho_0\|_{H^{s-1}} + C\int_0^t \|\rho\|_{H^{s-1}}\|\partial_x u\|_{L^\infty\cap H^{\frac{1}{2}}}\,d\tau \\ &\quad + C\int_0^t \big\|\partial_x\big(1-\partial_x^2\big)^{-1}(u_x\rho_x) + \big(1-\partial_x^2\big)^{-1}(u_x\rho)\big\|_{H^{s-1}}\,d\tau.\end{aligned}$$

Using Eqs. (50) and (51) gives rise to

$$\|\rho\|_{H^{s-1}(\mathbf{S})} \leq \|\rho_0\|_{H^{s-1}} + C\int_0^t \|\rho\|_{H^{s-1}}\|\partial_x u\|_{L^\infty\cap H^{\frac{1}{2}}}\,d\tau,$$

which, together with Eq. (53), yields

$$\begin{aligned}&\big\|u(t)\big\|_{H^s} + \big\|\rho(t)\big\|_{H^{s-1}} \\ &\quad\leq \|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}} + C\int_0^t \big(\|u\|_{H^s} + \big\|\rho(t)\big\|_{H^{s-1}}\big) \\ &\qquad\times \big(\|u\|_{L^\infty} + \|u\|_{H^{\frac{3}{2}+\varepsilon}} + \|u_{xx}\|_{L^\infty} + \|\rho\|_{L^\infty} + \|\rho_x\|_{L^\infty}\big)\,d\tau,\end{aligned} \tag{58}$$

where $\varepsilon \in (0, \frac{1}{2})$ and we used the fact that $H^{\frac{1}{2}+\varepsilon} \hookrightarrow L^\infty \cap H^{\frac{1}{2}}$.

Using the Gronwall inequality and Lemma 4.4, we have

$$\begin{aligned}&\big\|u(t)\big\|_{H^s} + \big\|\rho(t)\big\|_{H^{s-1}} \\ &\quad\leq \big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\big) \\ &\qquad\times e^{C\int_0^t(\|u\|_{L^\infty} + \|u\|_{H^{\frac{3}{2}+\varepsilon}} + \|u_{xxx}\|_{L^\infty} + \|\rho\|_{L^\infty} + \|\rho_x\|_{L^\infty})\,d\tau}.\end{aligned} \tag{59}$$

From Lemmas 4.5 and 4.8, we get

$$\begin{aligned}&\big\|u(t)\big\|_{H^s} + \big\|\rho(t)\big\|_{H^{s-1}} \\ &\quad\leq \big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\big)e^{(C(M+H)T + C\int_0^t \|\rho_x\|_{L^\infty}\,d\tau)}.\end{aligned} \tag{60}$$

Using the argument as in Step 1 one completes Theorem 2.2 for $s \in [2, \frac{5}{2})$.

*Step* 3. For $s \in (2, 3)$, differentiating once the second equation of system (5) with respect to $x$, we have

$$\partial_t\rho_x + u\partial_x\rho_x - \big(1-\partial_x^2\big)^{-1}(u_x\rho_x) - \partial_x\big(1-\partial_x^2\big)^{-1}(u_x\rho) = 0. \tag{61}$$

Using Lemma 4.3, we get

$$\|\rho_x\|_{H^{s-2}(\mathbf{S})} \leq \|\rho_{0x}\|_{H^{s-2}} + C\int_0^t \|\rho\|_{H^{s-1}}\big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\big)\,d\tau, \tag{62}$$

where we used the following estimates:

$$\big\|\big(1-\partial_x^2\big)^{-1}(u_x\rho_x)\big\|_{H^{s-2}} \leq C\|u_x\rho_x\|_{H^{s-4}} \leq C\|\rho\|_{H^{s-1}}\|u_x\|_{L^\infty}$$

and

$$\left\|\partial_x\left(1-\partial_x^2\right)^{-1}(u_x\rho)\right\|_{H^{s-2}} \le C\|u_x\rho\|_{H^{s-3}} \le C\|\rho\|_{H^{s-1}}\|u_x\|_{L^\infty},$$

where Lemma 4.1(iii) was used.

Using Eqs. (62), (53), and (52) (where $s-1$ is replaced by $s-2$) yields

$$\begin{aligned}&\left\|u(t)\right\|_{H^s}+\left\|\rho(t)\right\|_{H^{s-1}}\\&\quad\le \|u_0\|_{H^s}+\|\rho_0\|_{H^{s-1}}+C\int_0^t\left(\|u\|_{H^s}+\left\|\rho(t)\right\|_{H^{s-1}}\right)\\&\qquad\times\left(\|u\|_{L^\infty}+\|u_x\|_{L^\infty}+\|u_{xx}\|_{L^\infty}+\|\rho\|_{L^\infty}+\|\rho_x\|_{L^\infty}\right)d\tau.\end{aligned}\tag{63}$$

Using the Gronwall inequality again, we have

$$\begin{aligned}&\left\|u(t)\right\|_{H^s}+\left\|\rho(t)\right\|_{H^{s-1}}\\&\quad\le\left(\|u_0\|_{H^s}+\|\rho_0\|_{H^{s-1}}\right)\\&\qquad\times e^{C\int_0^t(\|u\|_{L^\infty}+\|u_x\|_{L^\infty}+\|u_{xx}\|_{L^\infty}+\|\rho\|_{L^\infty}+\|\rho_x\|_{L^\infty})\,d\tau}.\end{aligned}\tag{64}$$

From Lemmas 4.4, 4.5, and 4.8, we get

$$\left\|u(t)\right\|_{H^s}+\left\|\rho(t)\right\|_{H^{s-1}}\le\left(\|u_0\|_{H^s}+\|\rho_0\|_{H^{s-1}}\right)e^{(C(M+H)T+C\int_0^t\|\rho_x\|_{L^\infty}\,d\tau)}.\tag{65}$$

Using the argument as in Step 1 one completes Theorem 2.2 for $s\in(2,3)$.

*Step* 4. For $s=k\in\mathbf{N}$, $k\ge 3$, differentiating $k-2$ times the second equation of system (5) with respect to $x$, we obtain

$$\begin{aligned}&(\partial_t+u\partial_x)\partial_x^{k-2}\rho+\sum_{l_1+l_2=k-3,l_1,l_2\ge0}C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho\\&\quad+\partial_x^{k-1}\left(1-\partial_x^2\right)^{-1}(u_x\rho_x)+\partial_x^{k-2}\left(1-\partial_x^2\right)^{-1}(u_x\rho)=0.\end{aligned}\tag{66}$$

Using Lemma 4.2, we get from Eq. (66)

$$\begin{aligned}\left\|\partial_x^{k-2}\rho\right\|_{H^1}\le{}&\left\|\partial_x^{k-2}\rho_0\right\|_{H^1}+C\int_0^t\left\|\partial_x^{k-2}\rho\right\|_{H^1}\|\partial_xu\|_{H^{\frac12}\cap L^\infty}\,d\tau\\&+C\int_0^t\left\|\sum_{l_1+l_2=k-3,l_1,l_2\ge0}C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho\right.\\&\left.+\partial_x^{k-1}\left(1-\partial_x^2\right)^{-1}(u_x\rho_x)+\partial_x^{k-2}\left(1-\partial_x^2\right)^{-1}(u_x\rho)\right\|_{H^1}d\tau.\end{aligned}\tag{67}$$

From Lemma 3.4 and Lemma 4.1, we have

$$\begin{aligned}&\left\|\partial_x^{k-1}\left(1-\partial_x^2\right)^{-1}(u_x\rho_x)\right\|_{H^1}\le C\|u_x\rho_x\|_{H^{k-2}}\le C\|\rho\|_{H^{s-1}}\|u\|_{H^{s-1}},\\&\left\|\partial_x^{k-2}\left(1-\partial_x^2\right)^{-1}(u_x\rho)\right\|_{H^1}\le C\|u_x\rho\|_{H^{k-3}}\\&\qquad\qquad\le C\left(\|u\|_{H^{s-2}}\|\rho\|_{L^\infty}+\|\rho\|_{H^{s-3}}\|u_x\|_{L^\infty}\right),\end{aligned}$$

and

$$\Big\| \sum_{l_1+l_2=k-3,l_1,l_2\geq 0} C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho \Big\|_{H^1} \leq C\|\rho\|_{H^{s-1}}\|u\|_{H^{s-1}}.$$

Therefore, we deduce that

$$\|\partial_x^{k-2}\rho\|_{H^1} \leq \|\partial_x^{k-2}\rho_0\|_{H^1} + C\int_0^t \big(\|u\|_{H^s} + \|\rho\|_{H^{s-1}}\big)\big(\|u\|_{H^{s-1}} + \|\rho\|_{L^\infty}\big)\,d\tau. \tag{68}$$

From the Gagliardo-Nirenberg inequality, we have for $\sigma \in (0,1)$

$$\|\rho\|_{H^{s-1}} \leq C\big(\|\rho\|_{H^\sigma} + \|\partial_x^{k-2}\rho\|_{H^1}\big).$$

On the other hand, Eq. (52) implies that

$$\|\rho\|_{H^\sigma(\mathbf{S})} \leq \|\rho_0\|_{H^\sigma} + C\int_0^t \|\rho\|_{H^\sigma}\big(\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\big)\,d\tau, \tag{69}$$

which, together with Eq. (68), yields

$$\|\rho\|_{H^{s-1}} \leq C\|\rho_0\|_{H^{s-1}} + C\int_0^t \big(\|u\|_{H^s} + \|\rho\|_{H^{s-1}}\big)\big(\|u\|_{H^{s-1}} + \|\rho\|_{L^\infty}\big)\,d\tau. \tag{70}$$

Note that $s-3 \geq 0$. Using Lemma 4.1 and 4.2, we get

$$\begin{aligned} &\|u(t)\|_{H^s(\mathbf{S})} \\ &\quad\leq C\int_0^t \|u\|_{H^s}\big(\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty} + \|u_{xxx}\|_{L^\infty}\big)\,d\tau \\ &\qquad + \|u_0\|_{H^s} + C\int_0^t \|\rho(\tau)\|_{H^{s-1}}\big(\|\rho(\tau)\|_{L^\infty} + \|\rho_x(\tau)\|_{L^\infty}\big)\,d\tau, \end{aligned} \tag{71}$$

which, together with Eq. (70), results in

$$\begin{aligned} &\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}} \\ &\quad\leq C\big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\big) + C\int_0^t \big(\|u\|_{H^s} + \|\rho(t)\|_{H^{s-1}}\big) \\ &\qquad \times \big(\|u\|_{H^{s-1}} + \|\rho\|_{L^\infty} + \|\rho_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty} + \|u_{xxx}\|_{L^\infty}\big)\,d\tau. \end{aligned} \tag{72}$$

Using the Gronwall inequality, Lemma 4.4, we get

$$\begin{aligned} &\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}} \\ &\quad\leq C\big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\big) \\ &\qquad \times e^{C\int_0^t(\|u\|_{H^{s-1}} + \|\rho\|_{L^\infty} + \|\rho_x\|_{L^\infty} + \|u_{xxx}\|_{L^\infty})\,d\tau}. \end{aligned} \tag{73}$$

If $T < \infty$ satisfies $\int_0^T \|\rho_x\|_{L^\infty}\,d\tau < \infty$, applying Step 2 and induction assumption, we obtain from Lemma 4.5 and Lemma 4.8 that $\|u\|_{H^{s-1}} + \|\rho\|_{L^\infty} + \|u_{xxx}\|_{L^\infty}$ is uniformly bounded.

From Eq. (73), we get

$$\limsup_{t\to T}\left(\left\|u(t)\right\|_{H^s}+\left\|\rho(t)\right\|_{H^{s-1}}\right)<\infty,$$

which contradicts the assumption that $T<\infty$ is the maximal existence time. This completes the proof of Theorem 2.2 for $s=k\in N$ and $k\geq 3$.

*Step* 5. For $s\in(k,k+1)$, $k\in N$ and $k\geq 3$, differentiating $k-1$ times the second equation of system (5) with respect to $x$, we obtain

$$\begin{aligned}(\partial_t+u\partial_x)\partial_x^{k-1}\rho&+\sum_{l_1+l_2=k-2,l_1,l_2\geq 0}C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho\\&+\partial_x^k\left(1-\partial_x^2\right)^{-1}(u_x\rho_x)+\partial_x^{k-1}\left(1-\partial_x^2\right)^{-1}(u_x\rho)=0.\end{aligned}\tag{74}$$

Using Lemma 4.3 with $s-k\in(0,1)$, we get from Eq. (74)

$$\begin{aligned}&\left\|\partial_x^{k-1}\rho\right\|_{H^{s-k}}\\&\quad\leq C\int_0^t\left\|\partial_x^{k-1}\rho\right\|_{H^{s-k}}\left(\|u\|_{L^\infty}+\|\partial_x u\|_{L^\infty}\right)d\tau\\&\qquad+\left\|\partial_x^{k-1}\rho_0\right\|_{H^{s-k}}+C\int_0^t\left\|\sum_{l_1+l_2=k-2,l_1,l_2\geq 0}C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho\right.\\&\qquad\left.+\partial_x^k\left(1-\partial_x^2\right)^{-1}(u_x\rho_x)+\partial_x^{k-1}\left(1-\partial_x^2\right)^{-1}(u_x\rho)\right\|_{H^{s-k}}d\tau.\end{aligned}\tag{75}$$

For each $\varepsilon\in(0,\frac{1}{2})$, using Lemmas 3.4 and 4.1, and the fact that $H^{\frac{1}{2}+\varepsilon}\hookrightarrow L^\infty$, we have

$$\left\|\partial_x^k\left(1-\partial_x^2\right)^{-1}(u_x\rho_x)\right\|_{H^{s-k}}\leq C\|u_x\rho_x\|_{H^{s-2}}\leq C\|\rho\|_{H^{s-1}}\|u\|_{H^{s-1}},\tag{76}$$

$$\begin{aligned}&\left\|\partial_x^{k-1}\left(1-\partial_x^2\right)^{-1}(u_x\rho)\right\|_{H^{s-k}}\\&\quad\leq C\|u_x\rho\|_{H^{s-3}}\leq C\left(\|\rho\|_{H^{s-2}}\|u\|_{L^\infty}+\|\rho\|_{L^\infty}\|u_x\|_{H^{s-3}}\right),\end{aligned}\tag{77}$$

and

$$\begin{aligned}&\left\|\sum_{l_1+l_2=k-2,l_1,l_2\geq 0}C_{l_1,l_2}\partial_x^{l_1+1}u\partial_x^{l_2+1}\rho\right\|_{H^{s-k}}\\&\quad\leq C\sum_{l_1+l_2=k-2,l_1,l_2\geq 0}C_{l_1,l_2}\left(\left\|\partial_x^{l_1+1}u\right\|_{H^{s-k+1}}\left\|\partial_x^{l_2}\rho\right\|_{L^\infty}\right.\\&\qquad\left.+\left\|\partial_x^{l_1+1}u\right\|_{L^\infty}\left\|\partial_x^{l_2+1}\rho\right\|_{H^{s-k}}\right)\\&\quad\leq C\left(\|u\|_{H^s}\|\rho\|_{H^{k-\frac{3}{2}+\varepsilon}}+\|u\|_{H^{k-\frac{1}{2}+\varepsilon}}\|\rho\|_{H^{s-1}}\right).\end{aligned}\tag{78}$$

Therefore, from Eqs. (75), (77), and (78), we get

$$\begin{aligned}\left\|\partial_x^{k-1}\rho\right\|_{H^{s-k}}\leq\left\|\partial_x^{k-1}\rho_0\right\|_{H^{s-k}}&+C\int_0^t\left(\|u\|_{H^s}+\|\rho\|_{H^{s-1}}\right)\\&\times\left(\|u\|_{H^{k-\frac{1}{2}+\varepsilon}}+\|\rho\|_{H^{k-\frac{3}{2}+\varepsilon}}\right)d\tau.\end{aligned}\tag{79}$$

Using Lemma 4.2 in the first equation of system (5) for $s \in (k, k+1)$ with $k \geq 3$, we obtain

$$\begin{aligned}
&\|u(t)\|_{H^s(\mathbf{S})} \\
&\quad \leq \|u_0\|_{H^s} + C\int_0^t \|u\|_{H^s}\big(\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|u_{xx}\|_{L^\infty}\big)\,d\tau \\
&\qquad + C\int_0^t \|\rho(\tau)\|_{H^{s-1}}\big(\|\rho(\tau)\|_{L^\infty} + \|\rho_x(\tau)\|_{L^\infty}\big)\,d\tau,
\end{aligned} \tag{80}$$

which, together with Eqs. (79) and (52) (where $s-1$ is replaced by $s-k$), shows that

$$\begin{aligned}
&\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}} \\
&\quad \leq C\big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\big) + C\int_0^t \big(\|u\|_{H^s} + \|\rho(t)\|_{H^{s-1}}\big) \\
&\qquad \times \big(\|u\|_{H^{k-\frac{1}{2}+\varepsilon}} + \|\rho\|_{H^{k-\frac{3}{2}+\varepsilon}}\big)\,d\tau.
\end{aligned} \tag{81}$$

Using the Gronwall inequality again, we get

$$\begin{aligned}
&\|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}} \\
&\quad \leq C\big(\|u_0\|_{H^s} + \|\rho_0\|_{H^{s-1}}\big)e^{C\int_0^t (\|u\|_{H^{k-\frac{1}{2}+\varepsilon}} + \|\rho\|_{H^{k-\frac{3}{2}+\varepsilon}})\,d\tau}.
\end{aligned} \tag{82}$$

Noting that $k - \frac{1}{2} + \varepsilon < k$, $k - \frac{3}{2} + \varepsilon < k - 1$ and $k \geq 3$, and applying Step 4, we obtain $\|u\|_{H^{k-\frac{1}{2}+\varepsilon}} + \|\rho\|_{H^{k-\frac{3}{2}+\varepsilon}}$ is uniformly bounded. Thus, we complete the proof of Theorem 2.2 for $s \in (k, k+1)$, $k \in N$ and $k \geq 3$.

Therefore, from Step 1 to Step 5, we finish the proof of Theorem 2.2. □

## 5 Global solution

To prove Theorem 2.3, we need the following lemmas.

**Lemma 5.1** ([31]) *Let* $r > 0$. *If* $u \in H^r \cap W^{1,\infty}$ *and* $v \in H^{r-1} \cup L^\infty$, *then*

$$\big\|\big[\Lambda^r, u\big]v\big\|_{L^2} \leq C\big(\|u_x\|_{L^\infty}\big\|\Lambda^{r-1}\big\|_{L^2} + \big\|\Lambda^r u\big\|_{L^2}\|v\|_{L^\infty}\big).$$

**Lemma 5.2** *Let* $z_0 = (u_0, \rho_0) \in H^s(\mathbf{S}) \times H^{s-1}(\mathbf{S})$, $s > 3$. *Then* $\|z\|_{H^s \times H^{s-1}} = \|(u,\rho)\|_{H^s \times H^{s-1}}$ *is finite for* $0 < t < \infty$.

*Proof* Applying $\Lambda^s$ to $u_t = -uu_x - f(u)$, where $f(u) = \partial_x \Lambda^{-4}(u^2 + u_x^2 - \frac{7}{2}u_{xx}^2 - 3u_x u_{xxx} + \frac{1}{2}\rho^2 - \frac{1}{2}\rho_x^2)$, and multiplying by $\Lambda^s u$ and the integrating over $\mathbf{S}$, we have

$$\frac{d}{dt}\int_S \big(\Lambda^s u\big)^2\,dx = -2\int_S \Lambda^s u \Lambda^s u u_x\,dx - 2\int_S \Lambda^s u \Lambda^s f(u)\,dx. \tag{83}$$

From Lemma 5.1 and the Cauchy inequality, we obtain

$$\begin{aligned}
\int_S \Lambda^s u \Lambda^s u u_x\,dx &\leq \int_S \Lambda^s u\big(\Lambda^s u u_x - u\Lambda^s u_x\big)\,dx + \int_S \big(\Lambda^s u\big)u\Lambda^s u_x\,dx \\
&\leq C\|u_x\|_{L^\infty}\|u\|_{H^s}^2.
\end{aligned} \tag{84}$$

The Cauchy inequality ensures

$$\int_S \Lambda^s u \Lambda^s f(u)\,dx \le \|u\|_{H^s}\big\|f(u)\big\|_{H^s} \tag{85}$$

and

$$\begin{aligned}\big\|f(u)\big\|_{H^s} &\le C\left\|u^2+u_x^2-\frac{7}{2}u_{xx}^2-3u_xu_{xxx}+\frac{1}{2}\rho^2-\frac{1}{2}\rho_x^2\right\|_{H^{s-3}}\\ &\le C\big(\big\|u^2\big\|_{H^{s-3}}+\big\|u_x^2\big\|_{H^{s-3}}+\big\|u_{xx}^2\big\|_{H^{s-3}}+\|u_xu_{xxx}\|_{H^{s-3}}\\ &\quad+\big\|\rho^2\big\|_{H^{s-3}}+\big\|\rho_x^2\big\|_{H^{s-3}}\big)\\ &\le C\big(\|u\|_{L^\infty}\|u\|_{H^{s-3}}+\|u_x\|_{L^\infty}\|u_x\|_{H^{s-3}}\\ &\quad+\|u_{xx}\|_{L^\infty}\|u_{xx}\|_{H^{s-3}}+\|u_x\|_{H^{s-2}}\|u_{xx}\|_{L^\infty}\\ &\quad+\|u_x\|_{L^\infty}\|u_{xxx}\|_{H^{s-3}}+\|\rho\|_{L^\infty}\|\rho\|_{H^{s-3}}+\|\rho_x\|_{L^\infty}\|\rho_x\|_{H^{s-3}}\big),\end{aligned} \tag{86}$$

where we have used Lemma 4.1.

Hence,

$$\frac{d}{dt}\|u\|_{H^s}^2 \le C_1\big(\|u\|_{H^s}^2+\|u\|_{H^s}^3+\|u\|_{H^s}\|\rho\|_{H^{s-1}}^2\big), \tag{87}$$

where $C_1 = C_1(\|z_0\|_{H^s\times H^{s-1}})$.

Applying $\Lambda^{s-1}$ to $\rho_t = -u\rho_x - \partial_x(1-\partial_x^2)^{-1}(u_x\rho_x) - (1-\partial_x^2)^{-1}(u_x\rho)$, and multiplying by $\Lambda^{s-1}\rho$ and the integrating over $\mathbf{S}$, we have

$$\begin{aligned}&\frac{d}{dt}\int_S\big(\Lambda^{s-1}\rho\big)^2\,dx\\ &\quad=-2\int_S\Lambda^{s-1}\rho\Lambda^{s-1}(u\rho_x)\,dx-2\int_S\Lambda^{s-1}\rho\Lambda^{s-1}\big(1-\partial_x^2\big)^{-1}(u_x\rho)\,dx\\ &\qquad-2\int_S\Lambda^{s-1}\rho\Lambda^{s-1}\partial_x\big(1-\partial_x^2\big)^{-1}(u_x\rho_x)\,dx.\end{aligned} \tag{88}$$

We will estimate each of the terms on the right-hand side of Eq. (88). Note that $\int_S \Lambda^{s-1}\rho\Lambda^{s-3}(u_x\rho)\,dx = \int_S \Lambda^{s-2}\rho\Lambda^{s-2}(u_x\rho)\,dx$. Using Lemmas 4.5 and 5.1 and the Cauchy inequality, we have

$$\begin{aligned}&\int_S\Lambda^{s-1}\rho\Lambda^{s-1}\big(1-\partial_x^2\big)^{-1}(u_x\rho)\,dx\\ &\quad=\int_S\Lambda^{s-2}\rho\Lambda^{s-2}(u_x\rho)\,dx\\ &\quad=\int_S\Lambda^{s-2}\rho\big[\Lambda^{s-2},u_x\big]\rho\,dx+\int_S\Lambda^{s-2}\rho u_x\Lambda^{s-2}\rho\,dx\\ &\quad\le C\big(\|u\|_{H^s}\|\rho\|_{H^{s-1}}^2+\|\rho\|_{H^{s-1}}^2\big),\end{aligned} \tag{89}$$

$$\begin{aligned}&\int_S\Lambda^{s-1}\rho\Lambda^{s-1}\partial_x\big(1-\partial_x^2\big)^{-1}(u_x\rho_x)\,dx\\ &\quad=-\int_S\Lambda^{s-2}\rho_x\Lambda^{s-2}(u_x\rho_x)\,dx\le C\big(\|u\|_{H^s}\|\rho\|_{H^{s-1}}^2+\|\rho\|_{H^{s-1}}^2\big),\end{aligned} \tag{90}$$

and

$$\int_S \Lambda^{s-1}\rho\Lambda^{s-1}(u\rho_x)\,dx \le C\big(\|u\|_{H^s}\|\rho\|_{H^{s-1}}^2 + \|\rho\|_{H^{s-1}}^2\big). \tag{91}$$

It follows from Eqs. (89)-(91) that

$$\frac{d}{dt}\|\rho\|_{H^{s-1}}^2 \le C_1\big(\|u\|_{H^s}\|\rho\|_{H^{s-1}}^2 + \|\rho\|_{H^{s-1}}^2\big), \tag{92}$$

which together with Eq. (87) yields

$$\begin{aligned}
&\frac{d}{dt}\big(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2\big)\\
&\quad\le C\big(\|u\|_{H^s}^2 + \|u\|_{H^s}^3 + \|u\|_{H^s}\|\rho\|_{H^{s-1}}^2 + \|\rho\|_{H^{s-1}}^2\big)\\
&\quad\le C\big(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2\big)\big(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1\big),
\end{aligned} \tag{93}$$

which implies

$$\frac{\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2}{\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1} \le \frac{\|u_0\|_{H^s}^2 + \|\rho_0\|_{H^{s-1}}^2}{\|u_0\|_{H^s}^2 + \|\rho_0\|_{H^{s-1}}^2 + 1}e^{Ct}. \tag{94}$$

Note that $0 \le t < \infty$, and we get from Eq. (94)

$$\frac{\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2}{\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 + 1} \le \frac{\|u_0\|_{H^s}^2 + \|\rho_0\|_{H^{s-1}}^2}{\|u_0\|_{H^s}^2 + \|\rho_0\|_{H^{s-1}}^2 + 1},$$

which results in

$$\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 \le \|u_0\|_{H^s}^2 + \|\rho_0\|_{H^{s-1}}^2. \tag{95}$$

This completes the proof of Lemma 5.2. □

*Proof of Theorem* 2.3 Theorem 2.3 is a direct consequence of Theorem 2.1 and Lemma 5.2. □

**Remark** We have discussed some dynamics of system (1) in the periodic case. In fact, the above results hold true with $m = (1-\partial_x^2)^k$, $k \ge 2$ in the periodic case. We have

$$\begin{cases} m_t + um_x + 2u_x m + \eta\bar{\eta}_x = 0, & t > 0, x \in \mathbf{R},\\ \eta_t + (u\eta)_x = 0, & t > 0, x \in \mathbf{R}. \end{cases} \tag{96}$$

More precisely, the local well-posedness Theorem 2.1 and the global existence result Theorem 2.3 hold true in the Sobolev space $H^s(\mathbf{S}) \times H^{s-k+1}(\mathbf{S})$ with $s > 2k - \frac{1}{2}$, the wave-breaking criterion Theorem 2.2 is shown to be true under the condition $m_0 \in L^2(\mathbf{S})$.

**Competing interests**
The authors declare that they have no competing interests.

### Authors' contributions

Two authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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