Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Global existence and blow-up for a class of nonlinear reaction diffusion problems

Juntang Ding*

*Correspondence:
djuntang@sxu.edu.cn
School of Mathematical Sciences,
Shanxi University, Taiyuan, 030006,
P.R. China

Abstract
This paper deals with the global existence and blow-up of the solution for a class of nonlinear reaction diffusion problems. The purpose of this paper is to establish conditions on the data to guarantee the blow-up of the solution at some finite time, and conditions to ensure that the solution remains global. In addition, an upper bound for the 'blow-up time', an upper estimate of the 'blow-up rate', and an upper estimate of the global solution are also specified. Finally, as applications of the obtained results, some examples are presented.

MSC: 35K57; 35K55; 35B05

Keywords: reaction diffusion problem; global existence; blow-up

1 Introduction

The global existence and blow-up for nonlinear reaction diffusion equations have been widely studied in recent years (see, for instance, [1–8]). In this paper, we consider the following problem:

$$\begin{cases} (g(x,u))_t = \nabla\cdot(a(u)b(x)\nabla u) + f(u) & \text{in } D\times(0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D\times(0,T), \\ u(x,0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases} \tag{1.1}$$

where $D\subset\mathbb{R}^N$ $(N\geq 2)$ is a bounded domain with smooth boundary ∂D, $\overline{D}$ is the closure of D, $\partial/\partial n$ is the outward normal derivative on ∂D, T is the maximal existence time of u. Set $\mathbb{R}^+ := (0,+\infty)$. We assume, throughout this paper, that $a(s)$ is a positive $C^2(\mathbb{R}^+)$ function, $b(x)$ is a positive $C^1(\overline{D})$ function, $f(s)$ is a positive $C^2(\mathbb{R}^+)$ function, $g(x,s)$ is a $C^2(\overline{D}\times\mathbb{R}^+)$ function, $g_s(x,s)>0$ for any $(x,s)\in\overline{D}\times\mathbb{R}^+$, and $u_0(x)$ is a positive $C^2(\overline{D})$ function. Under the above assumptions, it is well known from the classical parabolic equation theory [5] and maximum principle [9] that there exists a unique local positive solution for problem (1.1). Moreover, by the regularity theorem [10], $u(x,t)\in C^3(D\times(0,T))\cap C^2(\overline{D}\times[0,T))$.

Many authors discussed the global existence and blow-up for nonlinear reaction diffusion equations with Neumann boundary conditions and obtained a lot of interesting results [11–24]. Some special cases of (1.1) have been studied already. Lair and Oxley [25] investigated the following problem:

$$\begin{cases} u_t = \nabla \cdot (a(u)\nabla u) + f(u) & \text{in } D \times (0, T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \\ u(x, 0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary ∂D. The necessary and sufficient conditions characterized by functions a and f were given for the global existence and blow-up solution. Zhang [26] dealt with the following problem:

$$\begin{cases} (g(u))_t = \Delta u + f(u) & \text{in } D \times (0, T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \\ u(x, 0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary ∂D. The sufficient conditions were obtained there for the existence of global and blow-up solutions. Gao *et al.* [27] considered the following problem:

$$\begin{cases} (g(u))_t = \nabla \cdot (a(u)\nabla u) + f(u) & \text{in } D \times (0, T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0, T), \\ u(x, 0) = u_0(x) > 0 & \text{in } \overline{D}, \end{cases}$$

where $D \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with smooth boundary ∂D. The sufficient conditions were developed for the existence of global and blow-up solutions. Meanwhile, the upper estimate of the global solution, the upper bound of the 'blow-up time', and the upper estimate of the 'blow-up rate' were also given.

In this paper, we study reaction diffusion problem (1.1). Note that $f(u)$, $g(x,u)$ and $a(u)b(x)$ are nonlinear reaction, nonlinear diffusion and nonlinear convection, respectively. Since the diffusion function $g(x,u)$ depends not only on the concentration variable u but also on the space variable x, it seems that the methods of [26, 27] are not applicable for the problem (1.1). In this paper, by constructing completely different auxiliary functions from those in [26, 27] and technically using maximum principles, we obtain the conditions on the data to guarantee the blow-up of the solution at some finite time, and conditions to ensure that the solution remains global. In addition, an upper bound for the 'blow-up time', an upper estimate of the 'blow-up rate', and an upper estimate of the global solution are also given. Our results extend and supplement those obtained in [26, 27].

We proceed as follows. In Section 2 we study the blow-up solution of (1.1). Section 3 is devoted to the global solution of (1.1). A few examples are given in Section 4 to illustrate the applications of the obtained results.

2 Blow-up solution

In this section we establish sufficient conditions on the data of the problem (1.1) to produce a blow-up of the solution $u(x,t)$ at some finite time T and under these conditions we derive an explicit upper bound for T and an explicit upper estimate of the 'blow-up rate'. The main result of this section is formulated in the following theorem.

Theorem 2.1 *Let* $u(x,t)$ *be a solution of the problem* (1.1). *Assume that the data of the problem* (1.1) *satisfies the following conditions*:

(i) *for any* $(s,t) \in D \times \mathbb{R}^+$,

$$a'(s) - a(s) \le 0, \qquad \left(\frac{g_s(x,s)}{a(s)}\right)_s \le 0,$$
$$\left(\frac{a(s)f(s) - (a(s)f(s))'}{a(s)}\right)' - \frac{a(s)f(s) - (a(s)f(s))'}{a(s)} \le 0; \tag{2.1}$$

(ii) *the constant*

$$\alpha = \min_{\overline{D}} \left\{ \frac{e^{u_0}}{f(u_0)g_u(x,u_0)} \left[\nabla \cdot \big(a(u_0)b(x)\nabla u_0\big) + f(u_0)\right] \right\} > 0; \tag{2.2}$$

(iii) *the integration*

$$\int_{M_0}^{+\infty} \frac{e^s}{f(s)}\,ds < +\infty, \qquad M_0 = \max_{\overline{D}} u_0(x). \tag{2.3}$$

Then $u(x,t)$ *must blow up in a finite time* T *and*

$$T \le \frac{1}{\alpha} \int_{M_0}^{+\infty} \frac{e^s}{f(s)}\,ds \tag{2.4}$$

as well as

$$u(x,t) \le \Phi^{-1}\big(\alpha(T-t)\big), \tag{2.5}$$

where

$$\Phi(z) = \int_z^{+\infty} \frac{e^s}{f(s)}\,ds, \qquad z > 0, \tag{2.6}$$

and Φ^{-1} *is the inverse function of* Φ.

Proof Consider the auxiliary function

$$Q(x,t) = e^u u_t - \alpha f(u). \tag{2.7}$$

Now we have

$$\nabla Q = e^u u_t \nabla u + e^u \nabla u_t - \alpha f' \nabla u, \tag{2.8}$$
$$\Delta Q = e^u u_t |\nabla u|^2 + 2e^u \nabla u \cdot \nabla u_t + e^u u_t \Delta u + e^u \Delta u_t - \alpha f'' |\nabla u|^2 - \alpha f' \Delta u, \tag{2.9}$$

and

$$\begin{aligned} Q_t &= e^u (u_t)^2 + e^u (u_t)_t - \alpha f' u_t \\ &= e^u (u_t)^2 + e^u \left(\frac{ab}{g_u}\Delta u + \frac{a'b}{g_u}|\nabla u|^2 + \frac{a}{g_u}\nabla b \cdot \nabla u + \frac{f}{g_u}\right)_t - \alpha f' u_t \end{aligned}$$

$$
\begin{aligned}
&= e^u(u_t)^2 + \left(\frac{a'b}{g_u} - \frac{abg_{uu}}{g_u^2}\right)e^u u_t \Delta u + \frac{ab}{g_u}e^u \Delta u_t + \left(\frac{a''b}{g_u} - \frac{a'bg_{uu}}{g_u^2}\right)e^u u_t |\nabla u|^2 \\
&\quad + 2\frac{a'b}{g_u}e^u(\nabla u \cdot \nabla u_t) + \left(\frac{a'}{g_u} - \frac{ag_{uu}}{g_u^2}\right)e^u u_t(\nabla b \cdot \nabla u) + \frac{a}{g_u}e^u(\nabla b \cdot \nabla u_t) \\
&\quad + \left[\left(\frac{f'}{g_u} - \frac{fg_{uu}}{g_u^2}\right)e^u - \alpha f'\right]u_t. \qquad (2.10)
\end{aligned}
$$

It follows from (2.9) and (2.10) that

$$
\begin{aligned}
\frac{ab}{g_u}\Delta Q - Q_t &= \left(\frac{a'bg_{uu}}{g_u^2} - \frac{a''b}{g_u} + \frac{ab}{g_u}\right)e^u u_t|\nabla u|^2 + \left(2\frac{ab}{g_u} - 2\frac{a'b}{g_u}\right)e^u(\nabla u \cdot \nabla u_t) \\
&\quad + \left(\frac{abg_{uu}}{g_u^2} - \frac{a'b}{g_u} + \frac{ab}{g_u}\right)e^u u_t \Delta u - \alpha\frac{abf''}{g_u}|\nabla u|^2 - \alpha\frac{abf'}{g_u}\Delta u - e^u(u_t)^2 \\
&\quad + \left(\frac{ag_{uu}}{g_u^2} - \frac{a'}{g_u}\right)e^u u_t(\nabla b \cdot \nabla u) - \frac{a}{g_u}e^u(\nabla b \cdot \nabla u_t) \\
&\quad + \left[\left(\frac{fg_{uu}}{g_u^2} - \frac{f'}{g_u}\right)e^u + \alpha f'\right]u_t. \qquad (2.11)
\end{aligned}
$$

By the first equation of (1.1), we have

$$
\Delta u = \frac{g_u}{ab}u_t - \frac{a'}{a}|\nabla u|^2 - \frac{1}{b}\nabla b \cdot \nabla u - \frac{f}{ab}. \qquad (2.12)
$$

Substitute (2.12) into (2.11) to obtain

$$
\begin{aligned}
&\frac{ab}{g_u}\Delta Q - Q_t \\
&\quad = \left(\frac{(a')^2b}{ag_u} - \frac{a'b}{g_u} - \frac{a''b}{g_u} + \frac{ab}{g_u}\right)e^u u_t|\nabla u|^2 + \left(2\frac{ab}{g_u} - 2\frac{a'b}{g_u}\right)e^u(\nabla u \cdot \nabla u_t) \\
&\qquad + \frac{a}{g_u}\left(\frac{g_u}{a}\right)_u e^u(u_t)^2 - \frac{a}{g_u}e^u u_t(\nabla b \cdot \nabla u) + \alpha\frac{af'}{g_u}(\nabla b \cdot \nabla u) + \left(\frac{a'f}{ag_u} - \frac{f}{g_u} - \frac{f'}{g_u}\right)e^u u_t \\
&\qquad + \left(\alpha\frac{a'bf'}{g_u} - \alpha\frac{abf''}{g_u}\right)|\nabla u|^2 - \frac{a}{g_u}e^u(\nabla b \cdot \nabla u_t) + \alpha\frac{ff'}{g_u}. \qquad (2.13)
\end{aligned}
$$

It follows from (2.8) that

$$
\nabla u_t = e^{-u}\nabla Q - u_t\nabla u + \alpha e^{-u}f'\nabla u. \qquad (2.14)
$$

Substituting (2.14) into (2.13), we get

$$
\begin{aligned}
&\frac{ab}{g_u}\Delta Q + \left(2\frac{b}{g_u}(a' - a)\nabla u + \frac{a}{g_u}\nabla b\right) \cdot \nabla Q - Q_t \\
&\quad = \left(\frac{(a')^2b}{ag_u} + \frac{a'b}{g_u} - \frac{a''b}{g_u} - \frac{ab}{g_u}\right)e^u u_t|\nabla u|^2 + \left(2\alpha\frac{abf'}{g_u} - \alpha\frac{a'bf'}{g_u} - \alpha\frac{abf''}{g_u}\right)|\nabla u|^2 \\
&\qquad + \frac{a}{g_u}\left(\frac{g_u}{a}\right)_u e^u(u_t)^2 + \left(\frac{a'f}{ag_u} - \frac{f}{g_u} - \frac{f'}{g_u}\right)e^u u_t + \alpha\frac{ff'}{g_u}. \qquad (2.15)
\end{aligned}
$$

With (2.7), we have

$$u_t = e^{-u}Q + \alpha e^{-u}f. \tag{2.16}$$

Substitute (2.16) into (2.15) to get

$$\begin{aligned}
&\frac{ab}{g_u}\Delta Q + \left(2\frac{b}{g_u}(a'-a)\nabla u + \frac{a}{g_u}\nabla b\right)\cdot\nabla Q \\
&\quad + \left\{\frac{ab}{g_u}\left[\left(\frac{a'}{a}\right)' - \frac{a'}{a} + 1\right]|\nabla u|^2 + \frac{a}{g_u}\left[\left(\frac{f}{a}\right)' + \frac{f}{a}\right]\right\}Q - Q_t \\
&= \alpha\frac{ab}{g_u}\left[\left(\frac{af-(af)'}{a}\right)' - \frac{af-(af)'}{a}\right]|\nabla u|^2 \\
&\quad + \frac{a}{g_u}\left(\frac{g_u}{a}\right)_u e^u(u_t)^2 + \alpha\frac{f^2}{ag_u}(a'-a).
\end{aligned} \tag{2.17}$$

The assumptions (2.1) ensure that the right-hand side of (2.17) is nonpositive; that is,

$$\begin{aligned}
&\frac{ab}{g_u}\Delta Q + \left(2\frac{b}{g_u}(a'-a)\nabla u + \frac{a}{g_u}\nabla b\right)\cdot\nabla Q \\
&\quad + \left\{\frac{ab}{g_u}\left[\left(\frac{a'}{a}\right)' - \frac{a'}{a} + 1\right]|\nabla u|^2 + \frac{a}{g_u}\left[\left(\frac{f}{a}\right)' + \frac{f}{a}\right]\right\}Q - Q_t \\
&\le 0 \quad \text{in } D\times(0,T).
\end{aligned} \tag{2.18}$$

Now, by (1.1), we have

$$\frac{\partial Q}{\partial n} = e^u\frac{\partial u_t}{\partial n} + e^u u_t\frac{\partial u}{\partial n} - \alpha f'\frac{\partial u}{\partial n} = e^u\left(\frac{\partial u}{\partial n}\right)_t = 0 \quad \text{on } \partial D\times(0,T). \tag{2.19}$$

Furthermore, it follows from (2.2) that

$$\begin{aligned}
\min_{\overline{D}} Q(x,0) &= \min_{\overline{D}}\left\{\frac{e^{u_0}}{g_u(x,u_0)}\left[\nabla\cdot\left(a(u_0)b(x)\nabla u_0\right) + f(u_0)\right] - \alpha f(u_0)\right\} \\
&= \min_{\overline{D}}\left\{f(u_0)\left[\frac{e^{u_0}}{f(u_0)g_u(x,u_0)}\left[\nabla\cdot\left(a(u_0)b(x)\nabla u_0\right) + f(u_0)\right] - \alpha\right]\right\} \\
&= 0.
\end{aligned} \tag{2.20}$$

Combining (2.18)-(2.20) and applying the maximum principle [9], we find that the minimum of Q in $\overline{D}\times[0,T)$ is zero. Hence,

$$Q \ge 0 \quad \text{in } \overline{D}\times[0,T);$$

that is,

$$\frac{e^u}{f(u)}u_t \ge \alpha. \tag{2.21}$$

At the point $x_0 \in \overline{D}$, where $u_0(x_0) = M_0$, integrating (2.21) over $[0, t]$, we get

$$\int_0^t \frac{\mathrm{e}^u}{f(u)} u_t \, \mathrm{d}t = \int_{M_0}^{u(x_0,t)} \frac{\mathrm{e}^s}{f(s)} \, \mathrm{d}s \geq \alpha t.$$

By the assumption (2.3), we know that $u(x,t)$ must blow up in finite time $t = T$, moreover,

$$T \leq \frac{1}{\alpha} \int_{M_0}^{+\infty} \frac{\mathrm{e}^s}{f(s)} \, \mathrm{d}s.$$

For each fixed x, integrating the inequality (2.21) over $[t, s]$ $(0 < t < s < T)$, we obtain

$$\begin{aligned}\Phi\big(u(x,t)\big) \geq \Phi\big(u(x,t)\big) - \Phi\big(u(x,s)\big) &= \int_{u(x,t)}^{+\infty} \frac{\mathrm{e}^s}{f(s)} \, \mathrm{d}s - \int_{u(x,s)}^{+\infty} \frac{\mathrm{e}^s}{f(s)} \, \mathrm{d}s \\ &= \int_{u(x,t)}^{u(x,s)} \frac{\mathrm{e}^s}{f(s)} \, \mathrm{d}s = \int_t^s \frac{\mathrm{e}^u}{f(u)} u_t \, \mathrm{d}t \geq \alpha(s - t).\end{aligned}$$

Letting $s \to T$, we have

$$\Phi\big(u(x,t)\big) \geq \alpha(T - t),$$

which implies

$$u(x,t) \leq \Phi^{-1}\big(\alpha(T - t)\big).$$

The proof is complete. □

3 Global solution

In this section we establish sufficient conditions on the data of the problem (1.1) in order to ensure that the solution has global existence. Under these conditions, we derive an explicit upper estimate of the global solution. The main results of this section are the following theorem.

Theorem 3.1 *Let $u(x,t)$ be a solution of the problem* (1.1). *Assume that the data of the problem* (1.1) *satisfies the following conditions*:

(i) *for any* $(s,t) \in D \times \mathbb{R}^+$,

$$\begin{aligned}&a'(s) - a(s) \geq 0, \qquad \left(\frac{g_s(x,s)}{a(s)}\right)_s \geq 0, \\ &\left(\frac{a(s)f(s) - (a(s)f(s))'}{a(s)}\right)' - \frac{a(s)f(s) - (a(s)f(s))'}{a(s)} \geq 0;\end{aligned} \tag{3.1}$$

(ii) *the constant*

$$\beta = \max_{\overline{D}} \left\{ \frac{\mathrm{e}^{u_0}}{f(u_0) g_u(x,u_0)} \big[\nabla \cdot \big(a(u_0) b(x) \nabla u_0\big) + f(u_0)\big] \right\} > 0; \tag{3.2}$$

(iii) *the integration*

$$\int_{m_0}^{+\infty} \frac{\mathrm{e}^s}{f(s)}\,\mathrm{d}s < +\infty, \quad m_0 := \min_{\overline{D}} u_0(x). \tag{3.3}$$

Then $u(x,t)$ must be a global solution and

$$u(x,t) \le \Psi\bigl(\beta t + \Psi\bigl(u_0(x)\bigr)\bigr), \tag{3.4}$$

where

$$\Psi(z) = \int_{m_0}^{z} \frac{\mathrm{e}^s}{f(s)}\,\mathrm{d}s, \quad z \ge m_0, \tag{3.5}$$

and Ψ^{-1} is the inverse function of Ψ.

Proof Consider an auxiliary function

$$P(x,t) = \mathrm{e}^u u_t - \beta f(u). \tag{3.6}$$

In (2.17), by replacing Q and α by P and β, respectively, we have

$$\begin{aligned} &\frac{ab}{g_u}\Delta P + \left(2\frac{b}{g_u}\bigl(a'-a\bigr)\nabla u + \frac{a}{g_u}\nabla b\right)\cdot \nabla P \\ &\quad + \left\{\frac{ab}{g_u}\left[\left(\frac{a'}{a}\right)' - \frac{a'}{a} + 1\right]|\nabla u|^2 + \frac{a}{g_u}\left[\left(\frac{f}{a}\right)' + \frac{f}{a}\right]\right\}P - P_t \\ &\quad = \beta\frac{ab}{g_u}\left[\left(\frac{af-(af)'}{a}\right)' - \frac{af-(af)'}{a}\right]|\nabla u|^2 \\ &\quad + \frac{a}{g_u}\left(\frac{g_u}{a}\right)_u \mathrm{e}^u (u_t)^2 + \beta\frac{f^2}{ag_u}\bigl(a'-a\bigr). \end{aligned} \tag{3.7}$$

It follows from (3.1) that the right-hand side of (3.7) is nonnegative; that is,

$$\begin{aligned} &\frac{ab}{g_u}\Delta P + \left(2\frac{b}{g_u}\bigl(a'-a\bigr)\nabla u + \frac{a}{g_u}\nabla b\right)\cdot \nabla P \\ &\quad + \left\{\frac{ab}{g_u}\left[\left(\frac{a'}{a}\right)' - \frac{a'}{a} + 1\right]|\nabla u|^2 + \frac{a}{g_u}\left[\left(\frac{f}{a}\right)' + \frac{f}{a}\right]\right\}P - P_t \ge 0 \quad \text{in } D\times(0,T). \end{aligned} \tag{3.8}$$

With (1.1), we have

$$\frac{\partial P}{\partial n} = \mathrm{e}^u\frac{\partial u_t}{\partial n} + \mathrm{e}^u u_t\frac{\partial u}{\partial n} - \beta f'\frac{\partial u}{\partial n} = \mathrm{e}^u\left(\frac{\partial u}{\partial n}\right)_t = 0 \quad \text{on } \partial D\times(0,T). \tag{3.9}$$

It follows from (3.2) that

$$\begin{aligned} \max_{\overline{D}} P(x,0) &= \max_{\overline{D}}\left\{\frac{\mathrm{e}^{u_0}}{g_u(x,u_0)}\bigl[\nabla\cdot\bigl(a(u_0)b(x)\nabla u_0\bigr) + f(u_0)\bigr] - \beta f(u_0)\right\} \\ &= \max_{\overline{D}}\left\{f(u_0)\left[\frac{\mathrm{e}^{u_0}}{f(u_0)g_u(x,u_0)}\bigl[\nabla\cdot\bigl(a(u_0)b(x)\nabla u_0\bigr) + f(u_0)\bigr] - \beta\right]\right\} \\ &= 0. \end{aligned} \tag{3.10}$$

Combining (3.8)-(3.10) and applying the maximum principle, we know that the maximum of P in $\overline{D} \times [0, T)$ is zero; that is,

$$P(x,t) \le 0 \quad \text{in } \overline{D} \times [0, T). \tag{3.11}$$

From (3.11), we get

$$\frac{e^u}{f(u)} u_t \le \beta. \tag{3.12}$$

For each fixed $x \in \overline{D}$, by integrating (3.12) over $[0, t]$, we have

$$\frac{1}{\beta} \int_0^t \frac{e^u}{f(u)} u_t \, dt = \frac{1}{\beta} \int_{u_0(x)}^{u(x,t)} \frac{e^s}{f(s)} \, ds \le t. \tag{3.13}$$

It follows from (3.13) and (3.3) that $u(x,t)$ must be a global solution. Furthermore, by (3.13), we have

$$\Psi\big(u(x,t)\big) - \Psi\big(u_0(x)\big) = \int_{m_0}^{u(x,t)} \frac{e^s}{f(s)} \, ds - \int_{m_0}^{u_0(x)} \frac{e^s}{f(s)} \, ds = \int_{u_0(x)}^{u(x,t)} \frac{e^s}{f(s)} \, ds \le \beta t.$$

Hence,

$$u(x,t) \le \Psi^{-1}\big(\beta t + \Psi\big(u_0(x)\big)\big).$$

The proof is complete. □

4 Applications

When $g(x,u) \equiv g(u)$, $a(u) \equiv 1$, and $b(x) \equiv 1$ or $g(x,u) \equiv g(u)$ and $b(x) \equiv 1$, the conclusions of Theorems 2.1 and 3.1 are valid. In this sense, our results extend and supplement the results of [26, 27].

In what follows, we present several examples to demonstrate the applications of the obtained results.

Example 4.1 Let $u(x,t)$ be a solution of the following problem:

$$\begin{cases} u_t = \Delta u + |\nabla u|^2 + \frac{2}{1+|x|^2} \sum_{i=1}^3 x_i \frac{\partial u}{\partial x_i} + \frac{3e^u}{1+|x|^2} & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = 2 - (1-|x|^2)^2 & \text{in } \overline{D}, \end{cases}$$

where $D = \{x = (x_1, x_2, x_3) \mid |x|^2 = x_1^2 + x_2^2 + x_3^2 < 1\}$ is the unit ball of $\mathbb{R}^3$. The above problem may be turned into the following problem:

$$\begin{cases} (e^u(1+|x|^2))_t = \nabla \cdot (e^u(1+|x|^2)\nabla u) + 3e^{2u} & \text{in } D \times (0,T), \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D \times (0,T), \\ u(x,0) = 2 - (1-|x|^2)^2 & \text{in } \overline{D}. \end{cases}$$

Now

$$a(u) = \mathrm{e}^u, \qquad b(x) = 1 + |x|^2, \qquad f(u) = 3\mathrm{e}^{2u},$$
$$g(x,u) = \mathrm{e}^u\left(1 + |x|^2\right), \qquad u_0(x) = 2 - \left(1 - |x|^2\right)^2.$$

By setting

$$s = |x|^2,$$

we have $0 \le s \le 1$ and

$$\begin{aligned}\alpha &= \min_{\overline{D}}\left\{\frac{\mathrm{e}^{u_0}}{f(u_0)g_u(x,u_0)}\left[\nabla\cdot\left(a(u_0)b(x)\nabla u_0\right) + f(u_0)\right]\right\}\\ &= \min_{\overline{D}}\left\{\frac{12 + 16|x|^2 - 44|x|^4 - 16|x|^6 + 16|x|^8 + 3\exp[2 - (1-|x|^2)^2]}{3(1+|x|^2)\exp[2-(1-|x|^2)^2]}\right\}\\ &= \min_{0\le s\le 1}\left\{\frac{12 + 16s - 44s^2 - 16s^3 + 16s^4 + 3\exp[2-(1-s)^2]}{3(1+s)\exp[2-(1-s)^2]}\right\}\\ &= 0.1391.\end{aligned}$$

It is easy to check that (2.1)-(2.3) hold. By Theorem 2.1, $u(x,t)$ must blow up in a finite time T and

$$T \le \frac{1}{\alpha}\int_{M_0}^{+\infty}\frac{\mathrm{e}^s}{f(s)}\,\mathrm{d}s = \frac{1}{0.1391}\int_2^{+\infty}\frac{1}{3\mathrm{e}^s}\,\mathrm{d}s = 0.3243$$

as well as

$$u(x,t) \le \Phi^{-1}\left(\alpha(T-t)\right) = \ln\frac{1}{0.4173(T-t)}.$$

Example 4.2 Let $u(x,t)$ be a solution of the following problem:

$$\begin{cases} u_t = \Delta u + 2|\nabla u|^2 + \frac{2}{1+|x|^2}\sum_{i=1}^3 x_i\frac{\partial u}{\partial x_i} + \frac{\mathrm{e}^{-\frac{2}{3}u}}{2(1+|x|^2)} & \text{in } D\times(0,T),\\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D\times(0,T),\\ u(x,0) = 1 + (1-|x|^2)^2 & \text{in } \overline{D},\end{cases}$$

where $D = \{x = (x_1,x_2,x_3) \mid |x|^2 = x_1^2 + x_2^2 + x_3^2 < 1\}$ is the unit ball of $\mathbb{R}^3$. The above problem can be transformed into the following problem:

$$\begin{cases} (\mathrm{e}^{2u}(1+|x|^2))_t = \nabla\cdot(2\mathrm{e}^{2u}(1+|x|^2)\nabla u) + \mathrm{e}^{\frac{u}{2}} & \text{in } D\times(0,T),\\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial D\times(0,T),\\ u(x,0) = 1 + (1-|x|^2)^2 & \text{in } \overline{D}.\end{cases}$$

Now we have

$$a(u) = 2\mathrm{e}^{2u}, \qquad b(x) = 1 + |x|^2, \qquad f(u) = \mathrm{e}^{\frac{u}{2}},$$
$$g(x,u) = \mathrm{e}^{2u}\left(1 + |x|^2\right), \qquad u_0(x) = 1 + \left(1 - |x|^2\right)^2.$$

In order to determine the constant β, we assume

$$s = |x|^2,$$

then $0 \le s \le 1$ and

$$\begin{aligned}
\beta &= \max_{\overline{D}} \left\{ \frac{e^{u_0}}{f(u_0) g_u(x, u_0)} \left[\nabla \cdot \big(a(u_0) b(x) \nabla u_0 \big) + f(u_0) \right] \right\} \\
&= \max_{\overline{D}} \left\{ \left(\exp\left[\frac{1}{2} + \frac{1}{2} \big(1 - |x|^2\big)^2 \right] \right. \right. \\
&\quad \times \left[-24 + 64|x|^2 - 8|x|^4 - 64|x|^6 + 64|x|^8 + \exp\left(-\frac{3}{2} - \frac{3}{2} \big(1 - |x|^2\big)^2 \right) \right] \Bigg) \\
&\quad \left. \Big/ \big(1 + |x|^2\big) \right\} \\
&= \max_{0 \le s \le 1} \left\{ \frac{\exp[\frac{1}{2} + \frac{1}{2}(1-s)^2][-24 + 64s - 8s^2 - 64s^3 + 64s^8 + \exp(-\frac{3}{2} - \frac{3}{2}(1-s)^2)]}{1+s} \right\} \\
&= 26.5635.
\end{aligned}$$

Again, it is easy to check that (3.1)-(3.3) hold. By Theorem 3.1, $u(x,t)$ must be a global solution and

$$\begin{aligned}
u(x,t) &\le \Psi^{-1}\big(\beta t + \Psi\big(u_0(x)\big)\big) = 2 \ln\left[13.2818t + \exp\left(\frac{1}{2} u_0(x) \right) \right] \\
&= 2 \ln\left[13.2818t + \exp\left(\frac{1}{2} + \frac{1}{2} \big(1 - |x|^2\big)^2 \right) \right].
\end{aligned}$$

Competing interests
The author declares that there is no conflict of interests regarding the publication of this paper.

Author's contributions
All results belong to Juntang Ding.

Acknowledgements
This work was supported by the National Natural Science Foundation of China (Nos. 61074048 and 61174082), the Research Project Supported by Shanxi Scholarship Council of China (Nos. 2011-011 and 2012-011), and the Higher School '131' Leading Talent Project of Shanxi Province.

Received: 8 April 2014 Accepted: 25 June 2014 Published online: 25 September 2014

References

1. Quittner, P, Souplet, P: Superlinear Parabolic Problems: Blow-Up, Global Existence and Steady States. Birkhäuser Advanced Texts. Birkhäuser, Basel (2007)
2. Galaktionov, VA, Vázquez, JL: The problem of blow-up in nonlinear parabolic equations. Discrete Contin. Dyn. Syst. **8**, 399-433 (2002)
3. Deng, K, Levine, HA: The role of critical exponents in blow-up theorems: the sequel. J. Math. Anal. Appl. **243**, 85-126 (2000)
4. Bandle, C, Brunner, H: Blow-up in diffusion equations: a survey. J. Comput. Appl. Math. **97**, 3-22 (1998)
5. Samarskii, AA, Galaktionov, VA, Kurdyumov, SP, Mikhailov, AP: Blow-Up in Problems for Quasilinear Parabolic Equations. Nauka, Moscow (1987) (in Russian); English translation, Walter de Gruyter, Berlin, 1995
6. Levine, HA: The role of critical exponents in blow-up theorems. SIAM Rev. **32**, 262-288 (1990)
7. Ding, JT: Blow-up solutions and global existence for quasilinear parabolic problems with Robin boundary conditions. Abstr. Appl. Anal. **2014**, Article ID 324857 (2014)
8. Zhang, LL: Blow-up of solutions for a class of nonlinear parabolic equations. Z. Anal. Anwend. **25**, 479-486 (2006)
9. Protter, MH, Weinberger, HF: Maximum Principles in Differential Equations. Prentice-Hall, Englewood Cliffs (1967)

10. Friedman, A: Partial Differential Equation of Parabolic Type. Prentice-Hall, Englewood Cliffs (1964)
11. Qu, CY, Bai, XL, Zheng, SN: Blow-up versus extinction in a nonlocal *p*-Laplace equation with Neumann boundary conditions. J. Math. Anal. Appl. **412**, 326-333 (2014)
12. Li, FS, Li, JL: Global existence and blow-up phenomena for nonlinear divergence form parabolic equations with inhomogeneous Neumann boundary conditions. J. Math. Anal. Appl. **385**, 1005-1014 (2012)
13. Gao, WJ, Han, YZ: Blow-up of a nonlocal semilinear parabolic equation with positive initial energy. Appl. Math. Lett. **24**, 784-788 (2011)
14. Song, JC: Lower bounds for the blow-up time in a non-local reaction-diffusion problem. Appl. Math. Lett. **24**, 793-796 (2011)
15. Ding, JT, Li, SJ: Blow-up and global solutions for nonlinear reaction-diffusion equations with Neumann boundary conditions. Nonlinear Anal. TMA **68**, 507-514 (2008)
16. Jazar, M, Kiwan, R: Blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **25**, 215-218 (2008)
17. Soufi, AE, Jazar, M, Monneau, R: A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **24**, 17-39 (2007)
18. Payne, LE, Schaefer, PW: Lower bounds for blow-up time in parabolic problems under Neumann conditions. Appl. Anal. **85**, 1301-1311 (2006)
19. Ishige, K, Yagisita, H: Blow-up problems for a semilinear heat equations with large diffusion. J. Differ. Equ. **212**, 114-128 (2005)
20. Ishige, K, Mizoguchi, N: Blow-up behavior for semilinear heat equations with boundary conditions. Differ. Integral Equ. **16**, 663-690 (2003)
21. Mizoguchi, N: Blow-up rate of solutions for a semilinear heat equation with Neumann boundary condition. J. Differ. Equ. **193**, 212-238 (2003)
22. Mizoguchi, N, Yanagida, E: Blow-up of solutions with sign changes for a smilinear diffusion equation. J. Math. Anal. Appl. **204**, 283-290 (1996)
23. Deng, K: Blow-up behavior of the heat equations with Neumann boundary conditions. J. Math. Anal. Appl. **188**, 641-650 (1994)
24. Chen, XY, Matano, H: Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations. J. Differ. Equ. **78**, 160-190 (1989)
25. Lair, AV, Oxley, ME: A necessary and sufficient condition for global existence for degenerate parabolic boundary value problem. J. Math. Anal. Appl. **221**, 338-348 (1998)
26. Zhang, HL: Blow-up solutions and global solutions for nonlinear parabolic problems. Nonlinear Anal. TMA **69**, 4567-4574 (2008)
27. Gao, XY, Ding, JT, Guo, BZ: Blow-up and global solutions for quasilinear parabolic equations with Neumann boundary conditions. Appl. Anal. **88**, 183-191 (2009)

doi:10.1186/s13661-014-0168-5
Cite this article as: Ding: Global existence and blow-up for a class of nonlinear reaction diffusion problems. *Boundary Value Problems* 2014 **2014**:168.