Rachůnková and Tomeček *Boundary Value Problems* 2014, **2014**:172
http://www.boundaryvalueproblems.com/content/2014/1/172

Boundary Value Problems
a SpringerOpen Journal

RESEARCH ARTICLE

Open Access

Fixed point problem associated with state-dependent impulsive boundary value problems

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Abstract

The paper investigates a fixed point problem in the space $(\mathbb{W}^{1,\infty}([a,b];\mathbb{R}^n))^{p+1}$ which is connected to boundary value problems with state-dependent impulses of the form $z'(t) = f(t, z(t))$, a.e. $t \in [a,b] \subset \mathbb{R}$, $z(\tau_i+) - z(\tau_i) = J_i(\tau_i, z(\tau_i))$, $\ell(z) = c_0$. Here, the impulse instants τ_i are determined as solutions of the equations $\tau_i = \gamma_i(z(\tau_i))$, $i = 1, \ldots, p$. We assume that $n, p \in \mathbb{N}$, $c_0 \in \mathbb{R}^n$, the vector function f satisfies the Carathéodory conditions on $[a,b] \times \mathbb{R}^n$, the impulse functions J_i, $i = 1, \ldots, p$, are continuous on $[a,b] \times \mathbb{R}^n$, and the barrier functions γ_i, $i = 1, \ldots, p$, are continuous on $\mathbb{R}^n$. The operator ℓ is an arbitrary linear and bounded operator on the space of left-continuous regulated on $[a,b]$ vector valued functions and is represented by the Kurzweil-Stieltjes integral. Provided the data functions f and J_i are bounded, transversality conditions which guarantee that this fixed point problem is solvable are presented. As a result it is possible to realize the construction of a solution of the above impulsive problem.
MSC: 34B37; 34B10; 34B15

Keywords: system of ODEs of the first order; state-dependent impulses; general linear boundary conditions; transversality conditions; fixed point problem

1 Introduction

In the literature most of impulsive boundary value problems deals with impulses at fixed times. This is the case that moments, where impulses act in state variables, are known (*cf.* Section 2). The theory of these impulsive problems is widely developed and presents direct analogies with methods and results for problems without impulses. Important texts in this area are [1–6].

A different situation arises, when impulse moments satisfy a predetermined relation between state and time variables, see *e.g.* [7–12]. This case, which is represented by state-dependent impulses, is studied here, where we are interested in a system of n ($n \in \mathbb{N}$) nonlinear ordinary differential equations of the first order with state-dependent impulses and general linear boundary conditions on the interval $[a,b] \subset \mathbb{R}$. The main reason that boundary value problems with state-dependent impulses are developed significantly less than those with impulses at fixed moments is that new difficulties with an operator representation of the problem appear when examining state-dependent impulses (*cf.* Section 4). Therefore almost all existence results for boundary value problems with state-dependent impulses have been reached for periodic problems which can be transformed to fixed point problems of corresponding Poincaré maps in $\mathbb{R}^n$. Hence, the difficulties with the construction of a functional space and an operator have been cleared in the periodic case. See *e.g.* [13–16]. Other types of boundary value problems with state-dependent impulses have been studied very rarely, see [17, 18].

In this paper we construct and investigate a fixed point problem in some subset $\overline{\Omega}$ of the product space $(\mathbb{W}^{1,\infty}([a,b];\mathbb{R}^n))^{p+1}$ and we provide conditions for its solvability (*cf.* Section 4 and Theorem 14). The existence of such fixed point allows us to construct a solution of the system of differential equations

$$z'(t) = f\big(t, z(t)\big), \quad \text{a.e. } t \in [a,b] \subset \mathbb{R}, \tag{1}$$

subject to the state-dependent impulse conditions

$$z(\tau_i+) - z(\tau_i) = J_i\big(\tau_i, z(\tau_i)\big), \quad \text{where } \tau_i = \gamma_i\big(z(\tau_i)\big), i = 1, \ldots, p, \tag{2}$$

and the general linear boundary condition

$$\ell(z) = c_0. \tag{3}$$

For nonzero impulse functions J_i, $i = 1, \ldots, p$, this solution is discontinuous on $[a,b]$ and, since discontinuity points τ_i, $i = 1, \ldots, p$, are not fixed and depend on the solution through (2), such a solution has to be searched in the space $\mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n)$; see the notation below. Note that conditions which guarantee the solvability of problem (1)-(3) have not been known before. Some results for special cases of problem (1)-(3) can be found in our previous papers [19–24].

In what follows we use this notation. Let $k, m, n \in \mathbb{N}$. By $\mathbb{R}^{n\times m}$ we denote the set of all matrices of the type $n \times m$ with real valued coefficients equipped with the matrix norm

$$|A| = \max_{k\in\{1,\ldots,n\}} \sum_{j=1}^{m} |a_{kj}| \quad \text{for } A = (a_{kj})_{k,j=1}^{n,m} \in \mathbb{R}^{n\times m}.$$

Let A^T denote the transpose of $A \in \mathbb{R}^{n\times m}$. Let $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ be the set of all n-dimensional column vectors $c = (c_1, \ldots, c_n)^T$, where $c_k \in \mathbb{R}$, $k = 1, \ldots, n$, and $\mathbb{R} = \mathbb{R}^{1\times 1}$. The (vector) norm of $\mathbb{R}^n$ is a special case of the norm of $\mathbb{R}^{n\times m}$, *i.e.* it has the form

$$|x| = \max_{k\in\{1,\ldots,n\}} |x_k| \quad \text{for } x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n.$$

It is well known that

$$|Ax| \le |A||x| \quad \text{for each } A \in \mathbb{R}^{n\times n}, x \in \mathbb{R}^n.$$

By $\mathbb{C}([a,b]\times\mathbb{R}^n;\mathbb{R}^n)$, $\mathbb{C}([\alpha,\beta];\mathbb{R}^{n\times m})$ (with $-\infty < \alpha < \beta < \infty$), $\mathbb{C}(\mathbb{R}^n;\mathbb{R}^m)$ we denote the set of all mappings $x : [a,b]\times\mathbb{R}^n \to \mathbb{R}^n$, $x : [\alpha,\beta] \to \mathbb{R}^{n\times m}$, $x : \mathbb{R}^n \to \mathbb{R}^m$ with continuous components, respectively. By $\mathbb{L}^\infty([a,b];\mathbb{R}^{n\times m})$, $\mathbb{L}^1([a,b];\mathbb{R}^{n\times m})$, $\mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^{n\times m})$, $\mathbb{C}([a,b];\mathbb{R}^{n\times m})$, $\mathbb{BV}([a,b];\mathbb{R}^{n\times m})$, we denote the sets of all mappings $F : [a,b] \to \mathbb{R}^{n\times m}$ whose components are, respectively, essentially bounded functions, Lebesgue integrable functions, left-continuous regulated functions, continuous functions and functions with bounded variation on the interval $[a,b]$. Let us note that the norm in the linear space $\mathbb{L}^\infty([a,b];\mathbb{R}^{n\times m})$ is taken as

$$\|F\|_\infty = \max_{k\in\{1,\dots,n\}} \sum_{j=1}^{m} \operatorname*{ess\,sup}_{t\in[a,b]} \bigl|f_{kj}(t)\bigr| \quad \text{for } F = (f_{kj})_{k,j=1}^{n,m} \in \mathbb{L}^\infty\bigl([a,b];\mathbb{R}^{n\times m}\bigr),$$

especially, in $\mathbb{L}^\infty([a,b];\mathbb{R}^n)$

$$\|u\|_\infty = \max_{k\in\{1,\dots,n\}} \operatorname*{ess\,sup}_{t\in[a,b]} \bigl|u_k(t)\bigr| \quad \text{for } u = (u_1,\dots,u_n)^T \in \mathbb{L}^\infty\bigl([a,b];\mathbb{R}^n\bigr).$$

We will make use of the Sobolev space $\mathbb{W}^{1,\infty}([a,b];\mathbb{R}^n)$, which is the linear space of vector functions, whose components are absolutely continuous having essentially bounded first derivatives on $[a,b]$, equipped with the norm

$$\|u\|_{1,\infty} = \|u\|_\infty + \bigl\|u'\bigr\|_\infty \quad \text{for } u \in \mathbb{W}^{1,\infty}\bigl([a,b];\mathbb{R}^n\bigr).$$

By $\mathrm{Car}([a,b]\times\mathbb{R}^n;\mathbb{R}^n)$ we denote the set of all mappings $f:[a,b]\times\mathbb{R}^n\to\mathbb{R}^n$ satisfying the Carathéodory conditions on the set $[a,b]\times\mathbb{R}^n$. Finally, by χ_M we denote the characteristic function of the set $M\subset\mathbb{R}$.

Note that a mapping $u:[a,b]\to\mathbb{R}^n$ is left-continuous regulated on $[a,b]$ if for each $t\in(a,b]$ and each $s\in[a,b)$

$$u(t) = u(t-) = \lim_{\tau\to t-} u(\tau) \in \mathbb{R}^n, \qquad u(s+) = \lim_{\tau\to s+} u(\tau) \in \mathbb{R}^n.$$

$\mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^n)$ is a linear space and equipped with the sup-norm $\|\cdot\|_\infty$ it is a Banach space (see [25], Theorem 3.6). In particular, we set

$$\|u\|_\infty = \max_{k\in\{1,\dots,n\}} \Bigl(\sup_{t\in[a,b]} \bigl|u_k(t)\bigr|\Bigr) \quad \text{for } u = (u_1,\dots,u_n)^T \in \mathbb{G}_\mathrm{L}\bigl([a,b];\mathbb{R}^n\bigr).$$

A mapping $f:[a,b]\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies the Carathéodory conditions on $[a,b]\times\mathbb{R}^n$ if

- $f(\cdot,x):[a,b]\to\mathbb{R}^n$ is measurable for all $x\in\mathbb{R}^n$,
- $f(t,\cdot):\mathbb{R}^n\to\mathbb{R}^n$ is continuous for a.e. $t\in[a,b]$,
- for each compact set $K\subset\mathbb{R}^n$ there exists a function $m_K\in\mathbb{L}^1([a,b];\mathbb{R})$ such that $|f(t,x)|\le m_K(t)$ for a.e. $t\in[a,b]$ and each $x\in K$.

Throughout we assume that

$$\left.\begin{array}{l} n,p\in\mathbb{N}, \qquad f\in\mathrm{Car}([a,b]\times\mathbb{R}^n;\mathbb{R}^n),\\ c_0\in\mathbb{R}^n, \qquad J_i\in\mathbb{C}([a,b]\times\mathbb{R}^n;\mathbb{R}^n), \qquad \gamma_i\in\mathbb{C}(\mathbb{R}^n;\mathbb{R}), \quad i=1,\dots,p,\\ \ell:\mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^n)\to\mathbb{R}^n \text{ is a linear bounded operator, } i.e.\\ \ell(z) = Kz(a) + \int_a^b V(t)\,\mathrm{d}[z(t)], \quad z\in\mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^n),\\ \quad \text{where } K\in\mathbb{R}^{n\times n},\ V\in\mathbb{BV}([a,b];\mathbb{R}^{n\times n}), k=1,\dots,n. \end{array}\right\} \tag{4}$$

Now let us define a solution of problem (1)-(3).

Definition 1 A mapping $z:[a,b]\to\mathbb{R}^n$ is a solution of problem (1)-(3) if for each $i\in\{1,\dots,p\}$ there exists a unique $\tau_i\in(a,b)$ such that

$$\tau_i = \gamma_i\bigl(z(\tau_i)\bigr),$$

$a < \tau_1 < \tau_2 < \cdots < \tau_p < b$, the restrictions $z|_{[a,\tau_1]}, z|_{(\tau_1,\tau_2]}, \ldots, z|_{(\tau_p,b]}$ are absolutely continuous, z satisfies (1) for a.e. $t \in [a,b]$ and fulfills conditions (2) and (3).

2 Problem with impulses at fixed times

In this section we summarize results from the paper [23], where we investigated boundary value problems having impulses at fixed times. This is the case that the barrier functions γ_i in (2) are constant functions, *i.e.* there exist $t_1, \ldots, t_p \in \mathbb{R}$ satisfying $a < t_1 < \cdots < t_p < b$ such that

$$\gamma_i(x) = t_i \quad \text{for } i = 1, \ldots, p, x \in \mathbb{R}^n,$$

and each solution of the problem crosses ith barrier at the same time instant $\tau_i = t_i$ for $i = 1, \ldots, p$.

In [23], the following boundary value problem was investigated:

$$z'(t) = A(t)z(t) + f\big(t, z(t)\big), \quad \text{a.e. } t \in [a,b], \tag{5}$$

$$z(t_i+) - z(t_i) = J_i\big(z(t_i)\big), \quad i = 1, \ldots, p, \tag{6}$$

$$\ell(z) = c_0, \tag{7}$$

where

$$\left.\begin{array}{l} a < t_1 < \cdots < t_p < b, \qquad A \in \mathbb{L}^1([a,b];\mathbb{R}^{n\times n}), \\ f \in \mathrm{Car}([a,b]\times\mathbb{R}^n;\mathbb{R}^n), \qquad J_i \in \mathbb{C}(\mathbb{R}^n;\mathbb{R}^n), \quad i = 1, \ldots, p, \\ \ell : \mathbb{G}_{\mathrm{L}}([a,b];\mathbb{R}^n) \to \mathbb{R}^n \text{ is a linear bounded operator}, \qquad c_0 \in \mathbb{R}^n. \end{array}\right\} \tag{8}$$

In order to get an operator representation of this problem (*cf.* Theorem 4) the Green's matrix is constructed.

Definition 2 ([23], Definition 7) A mapping $G : [a,b] \times [a,b] \to \mathbb{R}^{n\times n}$ is the Green's matrix of the problem

$$z'(t) = A(t)z(t) \quad \text{for a.e. } t \in [a,b], \qquad \ell(z) = 0, \tag{9}$$

if

(a) $G(\cdot,\tau)$ is continuous on $[a,\tau]$ and on $(\tau,b]$ for each $\tau \in [a,b]$,
(b) $G(t,\cdot) \in \mathbb{BV}([a,b];\mathbb{R}^{n\times n})$ for each $t \in [a,b]$,
(c) for any $q \in \mathbb{L}^1([a,b];\mathbb{R}^n)$ the mapping

$$x(t) = \int_a^b G(t,\tau)q(\tau)\,\mathrm{d}\tau, \quad t \in [a,b]$$

is a unique solution of the problem

$$z'(t) = A(t)z(t) + q(t) \quad \text{for a.e. } t \in [a,b], \qquad \ell(z) = 0. \tag{10}$$

Lemma 3 ([23], Lemma 8) *Assume* (8). *Problem* (10) *has a unique solution if and only if*

$$\det \ell(Y) \neq 0, \tag{11}$$

where Y is a fundamental matrix of the system of differential equations in (9). *If* (11) *is valid, then there exists a Green's matrix of problem* (9), *which is in the form*

$$G(t,\tau) = Y(t)H(\tau) + \chi_{(\tau,b]}(t)Y(t)Y^{-1}(\tau), \quad t,\tau \in [a,b], \tag{12}$$

where H is defined by

$$H(\tau) = -\bigl[\ell(Y)\bigr]^{-1}\left(\int_\tau^b V(s)A(s)Y(s)\,\mathrm{d}s \cdot Y^{-1}(\tau) + V(\tau)\right), \quad \tau \in [a,b], \tag{13}$$

and it has the following properties:

(i) *G is bounded on $[a,b] \times [a,b]$*,

(ii) *$G(\cdot,\tau)$ is absolutely continuous on $[a,\tau]$ and $(\tau,b]$ for each $\tau \in [a,b]$ and its columns satisfy the differential equation from* (9) *a.e. on $[a,b]$*,

(iii) *$G(\tau+,\tau) - G(\tau,\tau) = E$ for each $\tau \in [a,b)$*,

(iv) *$G(\cdot,\tau) \in \mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^{n\times n})$ for each $\tau \in [a,b]$ and*

$$\ell\bigl(G(\cdot,\tau)\bigr) = 0 \quad \textit{for each } \tau \in [a,b).$$

Theorem 4 ([23], Theorem 11) *Let* (8) *and* (11) *be satisfied and let G be given by* (12) *with H of* (13). *Then $z \in \mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^n)$ is a fixed point of an operator $\mathcal{F} : \mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^n) \to \mathbb{G}_\mathrm{L}([a,b];\mathbb{R}^n)$ defined by*

$$(\mathcal{F}z)(t) = \int_a^b G(t,s)f\bigl(s,z(s)\bigr)\,\mathrm{d}s + \sum_{i=1}^p G(t,t_i)J_i\bigl(z(t_i)\bigr) + Y(t)\bigl[\ell(Y)\bigr]^{-1}c_0$$

for $t \in [a,b]$, if and only if z is a solution of problem (5)-(7). *Moreover, the operator $\mathcal{F}$ is completely continuous*.

Similar results can be found also in [26, Chapter 6].

Remark 5 As in [23], we denote

$$G_1(t,\tau) = Y(t)H(\tau), \qquad G_2(t,\tau) = Y(t)\bigl(H(\tau) + Y^{-1}(\tau)\bigr),$$

i.e.

$$G(t,\tau) = G_1(t,\tau)\chi_{[a,\tau]}(t) + G_2(t,\tau)\chi_{(\tau,b]}(t) = \begin{cases} G_1(t,\tau), & a \le t \le \tau \le b, \\ G_2(t,\tau), & a \le \tau < t \le b. \end{cases}$$

Remark 6 In the present paper we need the Green's matrix of problem (9) for $A \equiv 0$. Therefore $Y(t) = E$ and $\ell(Y) = K$. The existence of the Green's matrix is then equivalent with the regularity of K, *i.e.* with the assumption $\det K \neq 0$. If this is satisfied, then H from (13) is given by the formula

$$H(\tau) = -K^{-1}V(\tau), \quad \tau \in [a,b],$$

and the Green's matrix takes the form

$$G(t,\tau) = \begin{cases} -K^{-1}V(\tau), & a \le t \le \tau \le b, \\ -K^{-1}V(\tau) + E, & a \le \tau < t \le b. \end{cases}$$

In this case the matrix functions G_1, G_2 from Remark 5 are written as

$$G_1(t,\tau) = -K^{-1}V(\tau), \qquad G_2(t,\tau) = -K^{-1}V(\tau) + E, \quad t,\tau \in [a,b].$$

3 Transversality conditions

Here we formulate conditions which guarantee that each possible solution of problem (1)-(3) in some region, which will be specified later (*cf.* (21)), crosses each barrier γ_i at the unique impulse point τ_i, $i = 1,\dots,p$. Consider positive real numbers μ_j, $j = 1,\dots,n$, and denote

$$\mathcal{A} = \{(x_1,\dots,x_n)^T \in \mathbb{R}^n : |x_j| \le \mu_j, j = 1,\dots,n\}. \tag{14}$$

We assume that

$$\left.\begin{array}{l} \text{there exist disjoint subintervals } [a_i,b_i] \text{ of the interval } (a,b) \text{ such that} \\ a_1 < \cdots < a_p, a_i \le \gamma_i(x) \le b_i \text{ for } i = 1,\dots,p, x \in \mathcal{A}, \end{array}\right\} \tag{15}$$

$$\left.\begin{array}{l} \text{for each } i = 1,\dots,p, j = 1,\dots,n, \text{ there exists } \lambda_{ij} \in [0,\infty) \text{ such that} \\ \text{for each } x = (x_1,\dots,x_n)^T, y = (y_1,\dots,y_n)^T \in \mathcal{A}, \\ |\gamma_i(x) - \gamma_i(y)| \le \sum_{j=1}^n \lambda_{ij}|x_j - y_j|. \end{array}\right\} \tag{16}$$

Further we choose positive real numbers ρ_j, $j = 1,\dots,n$, and assume that

$$\sum_{j=1}^n \lambda_{ij}\rho_j < 1 \quad \text{for } i = 1,\dots,p. \tag{17}$$

Under conditions (14)-(17), which we call *transversality conditions*, we define the set

$$\mathcal{B} = \{v = (v_1,\dots,v_n)^T \in \mathbb{W}^{1,\infty}([a,b];\mathbb{R}^n) : \|v_j\|_\infty < \mu_j, \|v_j'\|_\infty < \rho_j, j = 1,\dots,n\}. \tag{18}$$

In Section 4 we define an operator $\mathcal{G}$ (*cf.* (26)) whose fixed point $(u_1,\dots,u_{p+1})$ is used for the construction of a solution z of problem (1)-(3) (*cf.* (28)). In order to get a correct definition of $\mathcal{G}$ we need to describe intersection point t of a function $v \in \overline{\mathcal{B}}$ with the barriers γ_i, $i = 1,\dots,p$. These intersection points are roots of the functions $\gamma_i(v(t)) - t$, and their uniqueness is stated in Lemma 7.

Lemma 7 *Let* $\mu_j \in \mathbb{R}$, $\mathcal{A}$ *be given by* (14), *and let* λ_{ij}, ρ_j *and* γ_i, $j = 1,\dots,n$, $i = 1,\dots,p$, *satisfy* (15), (16) *and* (17). *Finally, let* $\mathcal{B}$ *be given by* (18). *Then for each* $v \in \overline{\mathcal{B}}$ *the functions*

$$\sigma_i(t) = \gamma_i(v(t)) - t, \quad t \in [a,b], i = 1,\dots,p,$$

are continuous and decreasing on $[a,b]$ *and they have unique roots in the interval* (a,b), *i.e. for* $i \in \{1,\dots,p\}$ *there exists a unique solution of the equation*

$$t = \gamma_i(v(t)). \tag{19}$$

Proof Let $v \in \overline{\mathcal{B}}$, $i \in \{1,\ldots,p\}$. By (15),

$$\sigma_i(a) = \gamma_i\big(v(a)\big) - a > 0,$$
$$\sigma_i(b) = \gamma_i\big(v(b)\big) - b < 0$$

are valid. This together with the fact that σ is continuous on $[a,b]$ shows that σ has at least one root in (a,b). Now, we will prove that σ is decreasing, by a contradiction. Let $s_1, s_2 \in (a,b)$, $s_1 < s_2$ be such that

$$\sigma_i(s_1) = \sigma_i(s_2),$$

i.e.

$$\gamma_i\big(v(s_1)\big) - \gamma_i\big(v(s_2)\big) = s_1 - s_2.$$

From (16) and (18) we obtain

$$\begin{aligned} 0 < |s_1 - s_2| &= \big|\gamma_i\big(v(s_1)\big) - \gamma_i\big(v(s_2)\big)\big| \\ &\le \sum_{j=1}^{n} \lambda_{ij}\big|v_j(s_1) - v_j(s_2)\big| \le \sum_{j=1}^{n} \lambda_{ij}\left|\int_{s_1}^{s_2} v_j'(\xi)\,\mathrm{d}\xi\right| \\ &\le \sum_{j=1}^{n} \lambda_{ij}\big\|v_j'\big\|_\infty |s_1 - s_2| \le \sum_{j=1}^{n} \lambda_{ij}\rho_j |s_1 - s_2|. \end{aligned}$$

This contradicts (17). Therefore (19) has a unique solution. □

According to Lemma 7, for $i \in \{1,\ldots,p\}$ and $v \in \overline{\mathcal{B}}$, there exists a unique point $(\tau_i, v(\tau_i)) \in [a,b] \times [-\mu_i, \mu_i]$ which is an intersection point of the graph of v with the graph of the barrier γ_i. Therefore we define a functional $\mathcal{P}_i : \overline{\mathcal{B}} \to (a,b)$ by

$$\mathcal{P}_i v = \tau_i, \quad v \in \overline{\mathcal{B}}, i = 1,\ldots,p, \tag{20}$$

where τ_i is a solution of (19), *i.e.* a unique root of the function σ_i from Lemma 7, for $i = 1,\ldots,p$.

Since solutions are affected by impulses at the points τ_i, the functionals $\mathcal{P}_i$, $i = 1,\ldots,p$, are used in the definition of the operator $\mathcal{G}$ (*cf.* (26)), it is important to prove their properties which are presented in Lemma 8 and Corollary 9 and which are necessary for the compactness of $\mathcal{G}$ (*cf.* Lemma 13).

Lemma 8 *Let the assumptions of Lemma* 7 *be satisfied. Then for each* $i \in \{1,\ldots,p\}$ *there exists a constant* $C \ge 0$ *such that for every* $v, \tilde{v} \in \overline{\mathcal{B}}$

$$|\mathcal{P}_i v - \mathcal{P}_i \tilde{v}| \le C\|v - \tilde{v}\|_\infty.$$

Proof Let $i \in \{1,\ldots,p\}$, $v, \tilde{v} \in \overline{\mathcal{B}}$. Let us denote

$$\tau = \mathcal{P}_i v, \qquad \tilde{\tau} = \mathcal{P}_i \tilde{v}.$$

Then from (16) and (18) we get

$$\begin{aligned}|\tau-\tilde{\tau}| &= \bigl|\gamma_i\bigl(\nu(\tau)\bigr)-\gamma_i\bigl(\tilde{\nu}(\tilde{\tau})\bigr)\bigr| \le \sum_{j=1}^{n}\lambda_{ij}\bigl|\nu_j(\tau)-\tilde{\nu}_j(\tilde{\tau})\bigr| \\ &\le \sum_{j=1}^{n}\lambda_{ij}\bigl|\nu_j(\tau)-\tilde{\nu}_j(\tau)\bigr| + \sum_{j=1}^{n}\lambda_{ij}\bigl|\tilde{\nu}_j(\tau)-\tilde{\nu}_j(\tilde{\tau})\bigr| \\ &\le \sum_{j=1}^{n}\lambda_{ij}\|\nu-\tilde{\nu}\|_\infty + \sum_{j=1}^{n}\lambda_{ij}\left|\int_{\tilde{\tau}}^{\tau}\tilde{\nu}_j'(s)\,\mathrm{d}s\right| \\ &\le \sum_{j=1}^{n}\lambda_{ij}\|\nu-\tilde{\nu}\|_\infty + \sum_{j=1}^{n}\lambda_{ij}\rho_j|\tau-\tilde{\tau}|.\end{aligned}$$

Subtracting the second term from the right-hand side of the inequality we obtain

$$|\tau-\tilde{\tau}| - \sum_{j=1}^{n}\lambda_{ij}\rho_j|\tau-\tilde{\tau}| \le \sum_{j=1}^{n}\lambda_{ij}\|\nu-\tilde{\nu}\|_\infty$$

and using (17) we arrive at

$$|\tau-\tilde{\tau}| \le \frac{\sum_{j=1}^{n}\lambda_{ij}}{1-\sum_{j=1}^{n}\lambda_{ij}\rho_j}\|\nu-\tilde{\nu}\|_\infty,$$

which is the desired inequality. □

Corollary 9 *Let the assumptions of Lemma* 7 *be satisfied. Then the functionals* $\mathcal{P}_i$, $i = 1,\dots,p$, *which are given by* (20), *are continuous on* $\overline{\mathcal{B}}$ *in the norm of* $\mathbb{W}^{1,\infty}([a,b];\mathbb{R}^n)$.

4 Fixed point problem

The main result of this section is contained in Theorem 11, where we present a connection between a (discontinuous) solution z of problem (1)-(3) and a fixed point of some operator $\mathcal{G}$ which operates on ordered $(p+1)$-tuples $(u_1,\dots,u_{p+1})$ of absolutely continuous vector functions. We work with the product space

$$X = \bigl(\mathbb{W}^{1,\infty}\bigl([a,b];\mathbb{R}^n\bigr)\bigr)^{p+1},$$

where for $u \in X$ we write $u = (u_1,\dots,u_{p+1})$ and $u_k = (u_{k,1},\dots,u_{k,n})^T$, $k = 1,\dots,p+1$. The sequence of elements of X is denoted as $\{u^m\}_{m=1}^{\infty}$; and the sequence of its kth components as $\{u_k^m\}_{m=1}^{\infty}$. The space X is equipped with the norm

$$\bigl\|(u_1,\dots,u_{p+1})\bigr\|_X = \sum_{k=1}^{p+1}\|u_k\|_{1,\infty} \quad \text{for } (u_1,\dots,u_{p+1}) \in X.$$

It is well known that X is a Banach space. For the construction of a fixed point problem we need the set

$$\Omega = \mathcal{B}^{p+1} \subset X, \tag{21}$$

where $\mathcal{B}$ is defined in (18) with constants μ_j, ρ_j, $j = 1, \dots, n$, satisfying the assumptions of Lemma 7.

Now, assume that the matrix K from (4) fulfills

$$\det K \neq 0, \tag{22}$$

and consider an operator $\mathcal{F}^* : \overline{\Omega} \to (\mathbb{C}([a,b];\mathbb{R}^n))^{p+1}$ defined by

$$\begin{aligned}(\mathcal{F}^* u)_k(t) = {} & \int_a^b G(t,s) \sum_{i=1}^{p+1} \chi_{(\tau_{i-1},\tau_i)}(s) f\big(s, u_i(s)\big)\, \mathrm{d}s + \sum_{i=k}^{p} G_1(t,\tau_i) J_i\big(\tau_i, u_i(\tau_i)\big) \\ & + \sum_{i=1}^{k-1} G_2(t,\tau_i) J_i\big(\tau_i, u_i(\tau_i)\big) + Y(t)\big[\ell(Y)\big]^{-1} c_0 \end{aligned} \tag{23}$$

for $k = 1, \dots, p+1$, $t \in [a,b]$, where

$$\tau_i = \mathcal{P}_i u_i \quad \text{for } i = 1, \dots, p, \qquad \tau_0 = a, \qquad \tau_{p+1} = b, \tag{24}$$

and $\mathcal{P}_i : \overline{\mathcal{B}} \to (a,b)$, $i = 1, \dots, p$, are continuous functionals from Corollary 9. Here G_1, G_2, Y, $\ell(Y)$ take values from Remark 6. Then $(\mathcal{F}^* u)_k \in \mathbb{C}([a,b];\mathbb{R}^n)$, for $k = 1, \dots, p+1$. Assume in addition that f is essentially bounded, that is,

$$\text{there exists } \bar{f} \in \mathbb{R} \text{ such that } \big|f(t,x)\big| \leq \bar{f} \quad \text{for a.e. } t \in [a,b], \text{all } x \in \mathbb{R}^n. \tag{25}$$

Then the operator $\mathcal{F}^*$ maps $\overline{\Omega}$ to X. Unfortunately, $\mathcal{F}^*$ is not compact on $\overline{\Omega}$. We can overcome this obstacle by redefining the operator $\mathcal{F}^*$ by means of an operator $\mathcal{G} : \overline{\Omega} \to X$ given by

$$(\mathcal{G}u)_k(t) = \begin{cases} (\mathcal{F}^* u)_k(\tau_{k-1}) + \int_{\tau_{k-1}}^t f(s, u_k(s))\, \mathrm{d}s & \text{for } t < \tau_{k-1}, \\ (\mathcal{F}^* u)_k(t) & \text{for } \tau_{k-1} \leq t \leq \tau_k, \\ (\mathcal{F}^* u)_k(\tau_k) + \int_{\tau_k}^t f(s, u_k(s))\, \mathrm{d}s & \text{for } t > \tau_k, \end{cases} \tag{26}$$

where $t \in [a,b]$, $k = 1, \dots, p+1$, and τ_k are defined by (24). As we will show this will be enough for our needs (*cf.* Theorem 11).

Remark 10 The important property of the operator $\mathcal{G}$ is that for $u = (u_1, \dots, u_{p+1}) \in \overline{\Omega}$ we have

$$(\mathcal{G}u)_k'(t) = f\big(t, u_k(t)\big) \quad \text{for a.e. } t \in [a,b], k = 1, \dots, p+1.$$

Let us note that for $k \in \{1, \dots, p+1\}$ the operator $\mathcal{F}^*$ satisfies this identity only on the interval (τ_{k-1}, τ_k), because

$$\big(\mathcal{F}^* u\big)_k'(t) = \sum_{i=1}^{p+1} \chi_{(\tau_{i-1},\tau_i)}(t) f\big(t, u_i(t)\big) \quad \text{for a.e. } t \in [a,b].$$

This fact obstructs the compactness of the operator $\mathcal{F}^*$ in X.

Consider $\mathcal{A}$ from (14), and assume

$$\gamma_i\big(x + J_i(t,x)\big) \le \gamma_i(x) \quad \text{for all } (t,x) \in [a,b] \times \mathcal{A}, i = 1,\dots,p. \tag{27}$$

Then we are ready to prove the following theorem.

Theorem 11 *Let the assumptions of Lemma* 7 *and conditions* (22), (25) *and* (27) *hold. If* $u = (u_1,\dots,u_{p+1})$ *is a fixed point of the operator* $\mathcal{G}$, *then a function* z *defined by*

$$z(t) = \begin{cases} u_1(t), & t \in [a, \mathcal{P}_1 u_1], \\ u_2(t), & t \in (\mathcal{P}_1 u_1, \mathcal{P}_2 u_2], \\ \dots, & \\ u_{p+1}(t), & t \in (\mathcal{P}_p u_p, b] \end{cases} \tag{28}$$

is a solution of problem (1)-(3). *Here* $\mathcal{P}_i : \overline{\mathcal{B}} \to (a,b)$, $i = 1,\dots,p$, *are continuous functionals from Corollary* 9.

Proof Let $\mathcal{B}$ be defined by (18) and $\Omega = \mathcal{B}^{p+1}$. Further, let $u = (u_1,\dots,u_{p+1}) \in \overline{\Omega}$ be a fixed point of the operator $\mathcal{G}$. Then for each $i \in \{1,\dots,p\}$ we have $u_i \in \overline{\mathcal{B}}$ and hence, by Lemma 7, there exists a unique solution $\tau_i = \mathcal{P}_i u_i$ of the equation $t = \gamma_i(u_i(t))$. Due to (15) the inequalities

$$a = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_p < \tau_{p+1} = b$$

are valid. Let us consider z defined by (28). We will prove that z is a fixed point of the operator $\mathcal{F}$ from Theorem 4, taking

$$t_i = \tau_i \quad \text{and} \quad J_i\big(\tau_i, z(\tau_i)\big) \text{ in place of } J_i\big(z(t_i)\big), \quad i = 1,\dots,p. \tag{29}$$

Let us denote

$$\mathcal{I}_1 = [a,\tau_1], \qquad \mathcal{I}_2 = (\tau_1,\tau_2], \qquad \mathcal{I}_3 = (\tau_2,\tau_3], \qquad \dots, \qquad \mathcal{I}_{p+1} = (\tau_p, b].$$

Let us choose $k \in \{1,\dots,p+1\}$ and consider $t \in \mathcal{I}_k$. Then

$$\begin{aligned} z(t) &= u_k(t) \\ &= \sum_{i=1}^{p+1} \int_{\tau_{i-1}}^{\tau_i} G(t,s) f\big(s, u_i(s)\big)\,\mathrm{d}s + \sum_{i=k}^{p} G_1(t,\tau_i) J_i\big(\tau_i, u_i(\tau_i)\big) \\ &\quad + \sum_{i=1}^{k-1} G_2(t,\tau_i) J_i\big(\tau_i, u_i(\tau_i)\big) + Y(t)\big[\ell(Y)\big]^{-1} c_0 \\ &= \sum_{i=1}^{p+1} \int_{\tau_{i-1}}^{\tau_i} G(t,s) f\big(s, z(s)\big)\,\mathrm{d}s + \sum_{i=k}^{p} G_1(t,\tau_i) J_i\big(\tau_i, z(\tau_i)\big) \\ &\quad + \sum_{i=1}^{k-1} G_2(t,\tau_i) J_i\big(\tau_i, z(\tau_i)\big) + Y(t)\big[\ell(Y)\big]^{-1} c_0. \end{aligned}$$

Of course,

$$\sum_{i=1}^{p+1} \int_{\tau_{i-1}}^{\tau_i} G(t,s)f\bigl(s,z(s)\bigr)\,\mathrm{d}s = \int_a^b G(t,s)f\bigl(s,z(s)\bigr)\,\mathrm{d}s.$$

Let $i \in \mathbb{N}$ be such that $k \le i \le p$. Then $t \le \tau_k \le \tau_i$ and therefore Remark 5 yields

$$G_1(t,\tau_i) = G(t,\tau_i).$$

Let $i \in \mathbb{N}$ be such that $1 \le i < k$ (such i exists only if $k > 1$). Then $t > \tau_{k-1} \ge \tau_i$ and Remark 5 gives

$$G_2(t,\tau_i) = G(t,\tau_i).$$

These facts imply that

$$\sum_{i=k}^{p} G_1(t,\tau_i)J_i\bigl(\tau_i,z(\tau_i)\bigr) + \sum_{i=1}^{k-1} G_2(t,\tau_i)J_i\bigl(\tau_i,z(\tau_i)\bigr) = \sum_{i=1}^{p} G(t,\tau_i)J_i\bigl(\tau_i,z(\tau_i)\bigr).$$

Consequently, by virtue of Theorem 4, z is a solution of problem (5)-(7) with $A \equiv 0$ and (29). The function z satisfies (1) a.e. on $[a,b]$ and fulfills the boundary condition (3). In addition, since z fulfills the impulse conditions (6) with $t_i = \tau_i$, and $J_i(\tau_i, z(\tau_i))$ in place of $J_i(z(t_i))$, where $\tau_i = \gamma_i(u_i(\tau_i)) = \gamma_i(z(\tau_i))$, $i = 1,\dots,p$, we see that z fulfills (2). It remains to prove that $\tau_1,\dots,\tau_p$ are the only instants at which the function z crosses the barriers $t = \gamma_1(x),\dots,t = \gamma_p(x)$, respectively. To this aim, due to (15) and (28) it suffices to prove that

$$t \ne \gamma_i\bigl(u_{i+1}(t)\bigr) \quad \text{for all } t \in (\tau_i,b],\ i = 1,\dots,p.$$

Choose an arbitrary $i \in \{1,\dots,p\}$ and consider σ_i from Lemma 7 for $v = u_{i+1}$, *i.e.*

$$\sigma_i(t) = \gamma_i\bigl(u_{i+1}(t)\bigr) - t, \quad t \in [a,b].$$

Since z fulfills (2) we have

$$u_{i+1}(\tau_i+) = z(\tau_i+) = z(\tau_i) + J_i\bigl(\tau_i,z(\tau_i)\bigr)$$

and according to (27) we get

$$\begin{aligned}\sigma_i(\tau_i+) &= \gamma_i\bigl(u_{i+1}(\tau_i+)\bigr) - \tau_i = \gamma_i\bigl(z(\tau_i) + J_i\bigl(\tau_i,z(\tau_i)\bigr)\bigr) - \tau_i \\ &\le \gamma_i\bigl(z(\tau_i)\bigr) - \tau_i = \sigma_i(\tau_i) = 0.\end{aligned}$$

Since σ_i is decreasing on $[a,b]$ we have

$$\sigma_i(t) < \sigma_i(\tau_i+) \le 0 \quad \text{for all } t \in (\tau_i,b]. \qquad \square$$

5 Existence results

Properties of the operator $\mathcal{G}$ which is defined by (23), (24), and (26), in particular its compactness and the existence of its fixed point, will be proved in this section. Then the existence of a solution of problem (1)-(3) will follow (*cf.* Theorem 15). Besides the conditions from Section 4 we assume in addition that

$$\text{there exists } \bar{J}_i, i = 1, \dots, p, \text{such that } \left|J_i(t,x)\right| \leq \bar{J}_i \quad \text{for all } (t,x) \in [a,b] \times \mathbb{R}^n, \tag{30}$$

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x, y \in \mathcal{A}: \quad |x-y| < \delta \quad \Rightarrow \quad \left\|f(\cdot,x) - f(\cdot,y)\right\|_\infty < \varepsilon, \tag{31}$$

$$V \in \mathbb{C}\bigl([a_i,b_i];\mathbb{R}^{n\times n}\bigr), \quad i = 1,\dots,p. \tag{32}$$

Here $\mathcal{A}$ is from (14) and $[a_i,b_i]$, $i = 1,\dots,p$, are from (15).

Lemma 12 *Let the assumptions of Lemma* 7 *and conditions* (22), (25), (27), (30), (31), *and* (32) *be fulfilled. Let $\mathcal{G}$ be defined by* (23), (24), *and* (26). *Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that each $u, \tilde{u} \in \overline{\Omega}$ satisfy*

$$\sum_{i=1}^{p+1} \|\tilde{u}_i - u_i\|_\infty < \delta \quad \Rightarrow \quad \left\|(\mathcal{G}\tilde{u})_k - (\mathcal{G}u)_k\right\|_{1,\infty} < \varepsilon, \quad k = 1,\dots,p+1. \tag{33}$$

Proof Consider $\tilde{u} = (\tilde{u}_1,\dots,\tilde{u}_{p+1}), u = (u_1,\dots,u_{p+1}) \in \overline{\Omega}$ and denote

$$\begin{aligned}
\tilde{y} &= (\tilde{y}_1,\dots,\tilde{y}_{p+1}) = \bigl((\mathcal{G}\tilde{u})_1,\dots,(\mathcal{G}\tilde{u})_{p+1}\bigr),\\
y &= (y_1,\dots,y_{p+1}) = \bigl((\mathcal{G}u)_1,\dots,(\mathcal{G}u)_{p+1}\bigr),\\
\tilde{x} &= (\tilde{x}_1,\dots,\tilde{x}_{p+1}) = \bigl((\mathcal{F}^*\tilde{u})_1,\dots,(\mathcal{F}^*\tilde{u})_{p+1}\bigr),\\
x &= (x_1,\dots,x_{p+1}) = \bigl((\mathcal{F}^*u)_1,\dots,(\mathcal{F}^*u)_{p+1}\bigr),
\end{aligned}$$

where $\mathcal{F}^*$ is defined in (23). Let us choose a fixed $k \in \{1,\dots,p+1\}$.

Step 1. According to Remark 10 we have

$$\begin{aligned}
\tilde{y}_k'(t) &= (\mathcal{G}\tilde{u})_k'(t) = f\bigl(t,\tilde{u}_k(t)\bigr),\\
y_k'(t) &= (\mathcal{G}u)_k'(t) = f\bigl(t,u_k(t)\bigr) \quad \text{for a.e. } t \in [a,b].
\end{aligned} \tag{34}$$

By (31) and (34) we have

$$\forall \tilde{\varepsilon} > 0\ \exists \tilde{\delta} > 0\ \forall \tilde{u}, u \in \overline{\Omega}: \quad \|\tilde{u}_k - u_k\|_\infty < \tilde{\delta} \quad \Rightarrow \quad \left\|\tilde{y}_k' - y_k'\right\|_\infty < \tilde{\varepsilon}. \tag{35}$$

Denote (*cf.* (24))

$$\tilde{\tau}_i = \mathcal{P}_i\tilde{u}_i, \qquad \tau_i = \mathcal{P}_i u_i, \quad i = 1,\dots,p, \qquad \tilde{\tau}_0 = \tau_0 = a, \qquad \tilde{\tau}_{p+1} = \tau_{p+1} = b.$$

By Lemma 8, we have

$$\forall \tilde{\varepsilon} > 0\ \exists \tilde{\delta} > 0\ \forall \tilde{u}, u \in \overline{\Omega}: \quad \|\tilde{u}_i - u_i\|_\infty < \tilde{\delta} \quad \Rightarrow \quad |\tilde{\tau}_i - \tau_i| < \tilde{\varepsilon}, \quad i = 1,\dots,p. \tag{36}$$

Choose an arbitrary $\varepsilon > 0$. By (35), there exists $\delta_1 > 0$ such that for each $\tilde{u}, u \in \overline{\Omega}$

$$\|\tilde{u}_k - u_k\|_\infty < \delta_1 \quad \Rightarrow \quad \|\tilde{y}_k' - y_k'\|_\infty < \frac{\varepsilon}{7}. \tag{37}$$

For $t \in [a,b]$ we have

$$\tilde{y}_k(t) = \tilde{y}_k(\tilde{\tau}_k) + \int_{\tilde{\tau}_k}^t \tilde{y}_k'(s)\,\mathrm{d}s, \qquad y_k(t) = y_k(\tau_k) + \int_{\tau_k}^t y_k'(s)\,\mathrm{d}s,$$

and therefore, by (26),

$$\begin{aligned}\bigl|\tilde{y}_k(t) - y_k(t)\bigr| &\le \bigl|\tilde{y}_k(\tilde{\tau}_k) - y_k(\tau_k)\bigr| + \left|\int_{\tilde{\tau}_k}^t \tilde{y}_k'(s)\,\mathrm{d}s - \int_{\tau_k}^t y_k'(s)\,\mathrm{d}s\right| \\ &\le \bigl|\tilde{x}_k(\tilde{\tau}_k) - x_k(\tau_k)\bigr| + \left|\int_{\tau_k}^t \bigl|\tilde{y}_k'(s) - y_k'(s)\bigr|\,\mathrm{d}s\right| + \left|\int_{\tilde{\tau}_k}^{\tau_k} \bigl|\tilde{y}_k'(s)\bigr|\,\mathrm{d}s\right|.\end{aligned}$$

Then, using (25) and (34), we get

$$\|\tilde{y}_k - y_k\|_\infty \le \bigl|\tilde{x}_k(\tilde{\tau}_k) - x_k(\tau_k)\bigr| + (b-a)\|\tilde{y}_k' - y_k'\|_\infty + |\tilde{\tau}_k - \tau_k|\bar{f}.$$

Due to (35) and (36) there exists $\delta_2 \in (0, \delta_1)$ such that for each $\tilde{u}, u \in \overline{\Omega}$

$$\|\tilde{u}_k - u_k\|_\infty < \delta_2 \quad \Rightarrow \quad (b-a)\|\tilde{y}_k' - y_k'\|_\infty + |\tilde{\tau}_k - \tau_k|\bar{f} < \frac{\varepsilon}{7}. \tag{38}$$

It remains to discuss the expression $|\tilde{x}_k(\tilde{\tau}_k) - x_k(\tau_k)|$. We have

$$\begin{aligned}\tilde{x}_k(\tilde{\tau}_k) - x_k(\tau_k) &= \sum_{i=1}^{p+1}\left(\int_{\tilde{\tau}_{i-1}}^{\tilde{\tau}_i} G(\tilde{\tau}_k, s) f\bigl(s, \tilde{u}_i(s)\bigr)\,\mathrm{d}s - \int_{\tau_{i-1}}^{\tau_i} G(\tau_k, s) f\bigl(s, u_i(s)\bigr)\,\mathrm{d}s\right) \\ &\quad + \sum_{i=k}^{p}\bigl(G_1(\tilde{\tau}_k, \tilde{\tau}_i) J_i\bigl(\tilde{\tau}_i, \tilde{u}_i(\tilde{\tau}_i)\bigr) - G_1(\tau_k, \tau_i) J_i\bigl(\tau_i, u_i(\tau_i)\bigr)\bigr) \\ &\quad + \sum_{i=1}^{k-1}\bigl(G_2(\tilde{\tau}_k, \tilde{\tau}_i) J_i\bigl(\tilde{\tau}_i, \tilde{u}_i(\tilde{\tau}_i)\bigr) - G_2(\tau_k, \tau_i) J_i\bigl(\tau_i, u_i(\tau_i)\bigr)\bigr).\end{aligned} \tag{39}$$

STEP 2. Treating the first term on the right-hand side of equality (39) we have

$$\begin{aligned}&\sum_{i=1}^{p+1}\left(\int_{\tilde{\tau}_{i-1}}^{\tilde{\tau}_i} G(\tilde{\tau}_k, s) f\bigl(s, \tilde{u}_i(s)\bigr)\,\mathrm{d}s - \int_{\tau_{i-1}}^{\tau_i} G(\tau_k, s) f\bigl(s, u_i(s)\bigr)\,\mathrm{d}s\right) \\ &\quad = \sum_{i=1}^{p+1}\left(\int_{\tau_{i-1}}^{\tau_i} \bigl[G(\tilde{\tau}_k, s) f\bigl(s, \tilde{u}_i(s)\bigr) - G(\tau_k, s) f\bigl(s, u_i(s)\bigr)\bigr]\,\mathrm{d}s \right.\\ &\qquad \left. + \int_{\tilde{\tau}_{i-1}}^{\tau_{i-1}} G(\tilde{\tau}_k, s) f\bigl(s, \tilde{u}_i(s)\bigr)\,\mathrm{d}s + \int_{\tau_i}^{\tilde{\tau}_i} G(\tilde{\tau}_k, s) f\bigl(s, \tilde{u}_i(s)\bigr)\,\mathrm{d}s\right) \\ &\quad = \sum_{i=1}^{p+1}\left(\int_{\tau_{i-1}}^{\tau_i} G(\tilde{\tau}_k, s)\bigl(f\bigl(s, \tilde{u}_i(s)\bigr) - f\bigl(s, u_i(s)\bigr)\bigr)\,\mathrm{d}s\right.\end{aligned}$$

$$+\int_{\tau_{i-1}}^{\tau_i}\big(G(\tilde{\tau}_k,s)-G(\tau_k,s)\big)f\big(s,u_i(s)\big)\,\mathrm{d}s\Bigg)$$
$$+\sum_{i=1}^{p+1}\bigg(\int_{\tilde{\tau}_{i-1}}^{\tau_{i-1}}G(\tilde{\tau}_k,s)f\big(s,\tilde{u}_i(s)\big)\,\mathrm{d}s+\int_{\tau_i}^{\tilde{\tau}_i}G(\tilde{\tau}_k,s)f\big(s,\tilde{u}_i(s)\big)\,\mathrm{d}s\bigg).$$

The function G is bounded on $[a,b]\times[a,b]$; it follows from (31) that there exists $\delta_3\in(0,\delta_2)$ such that for each $\tilde{u},u\in\overline{\Omega}$

$$\sum_{i+1}^{p+1}\|\tilde{u}_i-u_i\|_\infty<\delta_3\quad\Rightarrow\quad\sum_{i=1}^{p+1}\int_{\tau_{i-1}}^{\tau_i}\big|G(\tilde{\tau}_k,s)\big(f\big(s,\tilde{u}_i(s)\big)-f\big(s,u_i(s)\big)\big)\big|\,\mathrm{d}s<\frac{\varepsilon}{7}. \tag{40}$$

In view of Remark 6

$$\int_a^b\big|G(\tilde{\tau}_k,s)-G(\tau_k,s)\big|\,\mathrm{d}s=\int_a^b\big|\chi_{[a,\tilde{\tau}_k)}(s)-\chi_{[a,\tau_k)}(s)\big|\,\mathrm{d}s=|\tilde{\tau}_k-\tau_k|,$$

and therefore, by (25) and (36), there exists $\delta_4\in(0,\delta_3)$ such that for each $\tilde{u},u\in\overline{\Omega}$

$$\sum_{i=1}^{p+1}\|\tilde{u}_i-u_i\|_\infty<\delta_4\quad\Rightarrow\quad\sum_{i=1}^{p+1}\int_{\tau_{i-1}}^{\tau_i}\big|G(\tilde{\tau}_k,s)-G(\tau_k,s)\big|\big|f\big(s,u_i(s)\big)\big|\,\mathrm{d}s<\frac{\varepsilon}{7}. \tag{41}$$

Similarly, since G is bounded on $[a,b]\times[a,b]$ and f fulfills (25), we can find $\alpha>0$ satisfying

$$\sum_{i=1}^{p+1}\bigg|\int_{\tilde{\tau}_{i-1}}^{\tau_{i-1}}G(\tilde{\tau}_i,s)f\big(s,\tilde{u}_i(s)\big)\,\mathrm{d}s+\int_{\tau_i}^{\tilde{\tau}_i}G(\tilde{\tau}_k,s)f\big(s,\tilde{u}_i(s)\big)\,\mathrm{d}s\bigg|$$
$$<\alpha\sum_{i=1}^{p+1}\big(|\tilde{\tau}_{i-1}-\tau_{i-1}|+|\tilde{\tau}_i-\tau_i|\big).$$

Consequently, by (36), there exists $\delta_5\in(0,\delta_4)$ such that for each $\tilde{u},u\in\overline{\Omega}$

$$\sum_{i=1}^{p+1}\|\tilde{u}_i-u_i\|_\infty<\delta_5$$
$$\Rightarrow\quad\sum_{i=1}^{p+1}\bigg|\int_{\tilde{\tau}_{i-1}}^{\tau_{i-1}}G(\tilde{\tau}_i,s)f\big(s,\tilde{u}_i(s)\big)\,\mathrm{d}s+\int_{\tau_i}^{\tilde{\tau}_i}G(\tilde{\tau}_k,s)f\big(s,\tilde{u}_i(s)\big)\,\mathrm{d}s\bigg|<\frac{\varepsilon}{7}. \tag{42}$$

STEP 3. Finally we discuss the second and third term on the right-hand side of equality (39). According to Remark 6, we have

$$G_1(\tilde{\tau}_k,\tilde{\tau}_i)-G_1(\tau_k,\tau_i)=-K^{-1}V(\tilde{\tau}_i)+K^{-1}V(\tau_i)=-K^{-1}\big(V(\tilde{\tau}_i)-V(\tau_i)\big),$$
$$G_2(\tilde{\tau}_k,\tilde{\tau}_i)-G_2(\tau_k,\tau_i)=-K^{-1}V(\tilde{\tau}_i)+E-\big(-K^{-1}V(\tau_i)+E\big)=-K^{-1}\big(V(\tilde{\tau}_i)-V(\tau_i)\big).$$

Therefore, due to the uniform continuity of J_i, $i=1,\dots,p$, on $[a,b]\times\mathcal{A}$ (*cf.* (4) and (14)), the uniform continuity of V on $[a_i,b_i]$, $i=1,\dots,p$ (*cf.* (32) and (15)) and by (36), there exists

$\delta \in (0, \delta_5)$ such that for each $\tilde{u}, u \in \overline{\Omega}$

$$\sum_{i=1}^{p+1} \|\tilde{u}_i - u_i\|_\infty < \delta \quad \Rightarrow \quad \sum_{i=k}^{p} \bigl| G_1(\tilde{\tau}_k, \tilde{\tau}_i) J_i\bigl(\tilde{\tau}_i, \tilde{u}_i(\tilde{\tau}_i)\bigr) - G_1(\tau_k, \tau_i) J_i\bigl(\tau_i, u_i(\tau_i)\bigr)\bigr| < \frac{\varepsilon}{7}, \quad (43)$$

$$\sum_{i=1}^{p+1} \|\tilde{u}_i - u_i\|_\infty < \delta \quad \Rightarrow \quad \sum_{i=1}^{k-1} \bigl| G_2(\tilde{\tau}_k, \tilde{\tau}_i) J_i\bigl(\tilde{\tau}_i, \tilde{u}_i(\tilde{\tau}_i)\bigr) - G_2(\tau_k, \tau_i) J_i\bigl(\tau_i, u_i(\tau_i)\bigr)\bigr| < \frac{\varepsilon}{7}. \quad (44)$$

Relations (37), (38), (40), (41), (42), (43), and (44) imply (33). □

Lemma 13 *Let the assumptions of Lemma* 12 *be fulfilled. Then the operator* $\mathcal{G}$ *defined by* (23), (24), *and* (26) *is compact on* $\overline{\Omega}$.

Proof First, we prove the continuity of $\mathcal{G}$. Choose $\varepsilon > 0$. Then there exists $\delta > 0$ such that each $u, \tilde{u} \in \overline{\Omega}$ satisfy (33). Since $\|\tilde{u}_i - u_i\|_\infty \le \|\tilde{u}_i - u_i\|_{1,\infty}$, $i = 1, \dots, p+1$, each $u, \tilde{u} \in \overline{\Omega}$ satisfy

$$\sum_{i=1}^{p+1} \|\tilde{u}_i - u_i\|_{1,\infty} < \delta \quad \Rightarrow \quad \bigl\|(\mathcal{G}\tilde{u})_k - (\mathcal{G}u)_k\bigr\|_{1,\infty} < \varepsilon, \quad k = 1, \dots, p+1.$$

Now, we prove the relative compactness of the set $\mathcal{G}(\overline{\Omega})$. Let $\{y^m\}_{m=1}^\infty$ be a sequence of elements from the set $\mathcal{G}(\overline{\Omega})$. Then there exists a sequence $\{u^m\}_{m=1}^\infty \subset \overline{\Omega}$ such that $y^m = \mathcal{G}(u^m)$ for every $m \in \mathbb{N}$. Since $u_i^m \in \overline{\mathcal{B}}$, we have (*cf.* (18))

$$\|u_i^m\|_\infty \le \mu_i, \qquad \|(u_i^m)'\|_\infty \le \rho_i$$

for each $i = 1, \dots, p+1$, $m \in \mathbb{N}$. This implies

$$\bigl|u_i^m(t_1) - u_i^m(t_2)\bigr| = \left|\int_{t_1}^{t_2} \bigl(u_i^m\bigr)'(s)\,\mathrm{d}s\right| \le \rho_i |t_1 - t_2|.$$

The Arzelà-Ascoli theorem and the diagonalization principle give the existence of a subsequence which is convergent in the $\|\cdot\|_\infty$-norm. Let us denote it as $\{u^\nu\}_{\nu=1}^\infty$. Then, by Lemma 12, for each $\varepsilon > 0$ there exist $\delta > 0$ and $\nu_0 \in \mathbb{N}$ such that for each $\nu \in \mathbb{N}$, $\nu \ge \nu_0$ the inequality $\sum_{i=1}^{p+1} \|u_i^\nu - u_i^{\nu_0}\|_\infty < \delta$ holds, and consequently, by (33),

$$\nu \ge \nu_0 \quad \Rightarrow \quad \bigl\|\bigl(\mathcal{G}u^\nu\bigr)_k - \bigl(\mathcal{G}u^{\nu_0}\bigr)_k\bigr\|_{1,\infty} < \varepsilon, \quad k = 1, \dots, p+1.$$

Therefore there exists a subsequence $\{y^\nu\}_{\nu=1}^\infty \subset \{y^m\}_{m=1}^\infty$ which is convergent in X. □

Theorem 14 *Assume that* (25) *and* (30) *hold and that numbers* μ_j, ρ_j, $j = 1, \dots, n$, *satisfy*

$$\left.\begin{aligned} \mu_j \ge{} & \bigl|K^{-1}\bigr| \sup_{s\in[a,b]} \bigl|V(s)\bigr| \bar{f}(b-a) + 2\bar{f}(b-a) \\ & + \bigl|K^{-1}\bigr| \sup_{s\in[a,b]} \bigl|V(s)\bigr| \sum_{k=1}^{p} \bar{J}_k + \sum_{k=1}^{p} \bar{J}_k + \bigl|K^{-1} c_0\bigr|, \\ \rho_j \ge{} & \bar{f}, \quad j = 1, \dots, n. \end{aligned}\right\} \quad (45)$$

Define sets $\mathcal{A}$, $\mathcal{B}$ *and* Ω *by* (14), (18), *and* (21), *respectively, and assume that conditions* (15), (16), (17), (27), (31), *and* (32) *hold. Then the operator* $\mathcal{G}$ *has a fixed point in* $\overline{\Omega}$.

Proof It suffices to show that $\mathcal{G}(\overline{\Omega}) \subset \overline{\Omega}$. Let $u \in \overline{\Omega}$ and $x = \mathcal{F}^* u$, $y = \mathcal{G}(u)$ (*cf.* (23) and (26)). That is $x = (x_1, \dots, x_{p+1})$ and $y = (y_1, \dots, y_{p+1})$, where $y_i = (y_{i,1}, \dots, y_{i,n})^T$ for $i = 1, \dots, p+1$. Choose $j \in \{1, \dots, n\}$, $i \in \{1, \dots, p+1\}$. Having in mind (24), we get by (23), (26), (45), and Remark 6

$$\begin{aligned}
|y_{i,j}(t)| &\le |y_i(t)| \\
&\le |K^{-1}| \sup_{s\in[a,b]} |V(s)| \bar{f}(b-a) + \bar{f}(b-a) \\
&\quad + |K^{-1}| \sup_{s\in[a,b]} |V(s)| \sum_{k=1}^{p} \bar{J}_k + \sum_{k=1}^{p} \bar{J}_k + |K^{-1} c_0| \\
&\le \mu_j - \bar{f}(b-a) \quad \text{for } t \in [\tau_{i-1}, \tau_i], \\
|y_{i,j}(t)| &\le |y_i(t)| \le |x_i(\tau_{i-1})| + \left| \int_{\tau_{i-1}}^{t} f(s, u_i(s))\,\mathrm{d}s \right| \\
&\le |y_i(\tau_{i-1})| + \bar{f}(b-a) \le \mu_j \quad \text{for } t < \tau_{i-1}, \\
|y_{i,j}(t)| &\le |y_i(t)| \le |x_i(\tau_i)| + \left| \int_{\tau_i}^{t} f(s, u_i(s))\,\mathrm{d}s \right| \\
&\le |y_i(\tau_i)| + \bar{f}(b-a) \le \mu_j \quad \text{for } t > \tau_i.
\end{aligned}$$

Therefore

$$\|y_{i,j}\|_\infty \le \mu_j, \quad j = 1, \dots, n, i = 1, \dots, p+1.$$

From (25) and Remark 10 we have

$$|y'_{i,j}(t)| \le |y'_i(t)| = |f(t, u_i(t))| \le \bar{f} \quad \text{for a.e. } t \in [a,b],$$

which yields, due to (45),

$$\|y'_{i,j}\|_\infty \le \rho_j, \quad j = 1, \dots, n, i = 1, \dots, p+1.$$

Consequently, by virtue of (18), $y_i \in \overline{\mathcal{B}}$ for $i = 1, \dots, p+1$, that is, $y \in \overline{\Omega}$. □

Theorems 11 and 14 give an existence result for problem (1)-(3).

Theorem 15 *Under the assumptions of Theorem* 14 *problem* (1)-(3) *has at least one solution z such that*

$$\|z\|_\infty \le \max\{\mu_1, \dots, \mu_n\}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgements
This work was supported by the grant No. 14-06958S of the Grant Agency of the Czech Republic.

Received: 27 March 2014 Accepted: 27 June 2014 Published online: 24 September 2014

References

1. Samoilenko, AM, Perestyuk, NA: Impulsive Differential Equations. World Scientific, Singapore (1995)
2. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
3. Bainov, D, Simeonov, P: Impulsive Differential Equations: Periodic Solutions and Applications. Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 66. Longman, Harlow (1993)
4. Graef, JR, Henderson, J, Ouahab, A: Impulsive Differential Inclusion. de Gruyter, Berlin (2013)
5. Yang, T: Impulsive Systems and Control: Theory and Applications. Nova Science Publishers, New York (2001)
6. Bainov, DD, Covachev, V: Impulsive Differential Equations with Small Parameter. Series on Advances in Mathematics for Applied Sciences, vol. 4. World Scientific, Singapore (1994)
7. Jiao, JJ, Cai, SH, Chen, LS: Analysis of a stage-structured predator-prey system with birth pulse and impulsive harvesting at different moments. Nonlinear Anal., Real World Appl. **12**, 2232-2244 (2011)
8. Nie, L, Teng, Z, Hu, L, Peng, J: Qualitative analysis of a modified Leslie-Gower and Holling-type II predator-prey model with state dependent impulsive effects. Nonlinear Anal., Real World Appl. **11**, 1364-1373 (2010)
9. Nie, L, Teng, Z, Torres, A: Dynamic analysis of an SIR epidemic model with state dependent pulse vaccination. Nonlinear Anal., Real World Appl. **13**, 1621-1629 (2012)
10. Tang, S, Chen, L: Density-dependent birth rate birth pulses and their population dynamic consequences. J. Math. Biol. **44**, 185-199 (2002)
11. Wang, F, Pang, G, Chen, L: Qualitative analysis and applications of a kind of state-dependent impulsive differential equations. J. Comput. Appl. Math. **216**, 279-296 (2008)
12. Cordova-Lepe, F, Pinto, M, Gonzalez-Olivares, E: A new class of differential equations with impulses at instants dependent on preceding pulses. Applications to management of renewable resources. Nonlinear Anal., Real World Appl. **13**, 2313-2322 (2012)
13. Bajo, I, Liz, E: Periodic boundary value problem for first order differential equations with impulses at variable times. J. Math. Anal. Appl. **204**, 65-73 (1996)
14. Belley, J, Virgilio, M: Periodic Duffing delay equations with state dependent impulses. J. Math. Anal. Appl. **306**, 646-662 (2005)
15. Belley, J, Virgilio, M: Periodic Liénard-type delay equations with state-dependent impulses. Nonlinear Anal., Theory Methods Appl. **64**, 568-589 (2006)
16. Frigon, M, O'Regan, D: First order impulsive initial and periodic problems with variable moments. J. Math. Anal. Appl. **233**, 730-739 (1999)
17. Benchohra, M, Graef, JR, Ntouyas, SK, Ouahab, A: Upper and lower solutions method for impulsive differential inclusions with nonlinear boundary conditions and variable times. Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal. **12**, 383-396 (2005)
18. Frigon, M, O'Regan, D: Second order Sturm-Liouville BVP's with impulses at variable times. Dyn. Contin. Discrete Impuls. Syst., Ser. A Math. Anal. **8**, 149-159 (2001)
19. Rachůnek, L, Rachůnková, L: First-order nonlinear differential equations with state-dependent impulses. Bound. Value Probl. **2013**, 195 (2013)
20. Rachůnková, L, Tomeček, J: A new approach to BVPs with state-dependent impulses. Bound. Value Probl. **2013**, 22 (2013). doi:10.1186/1687-2770-2013-22
21. Rachůnková, L, Tomeček, J: Second order BVPs with state-dependent impulses via lower and upper functions. Cent. Eur. J. Math. **12**(1), 128-140 (2014)
22. Rachůnková, L, Tomeček, J: Existence principle for BVPs with state-dependent impulses. Topol. Methods Nonlinear Anal. (to appear)
23. Rachůnková, L, Tomeček, J: Impulsive system of ODEs with general linear boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2013**, 25 (2013)
24. Rachůnková, L, Tomeček, J: Existence principle for higher order nonlinear differential equations with state-dependent impulses via fixed point theorem. Bound. Value Probl. **2014**, 118 (2014). doi:10.1186/1687-2770-2014-118
25. Hönig, CS: The Adjoint Equation of a Linear Volterra-Stieltjes Integral Equation with a Linear Constraint. Lecture Notes in Mathematics, vol. 957, pp. 118-125. Springer, Berlin (1982)
26. Azbelev, NV, Maksimov, VP, Rakhmatullina, LF: Introduction to the Theory of Functional Differential Equations. Nauka, Moscow (1991)

doi:10.1186/s13661-014-0172-9
Cite this article as: Rachůnková and Tomeček: **Fixed point problem associated with state-dependent impulsive boundary value problems.** *Boundary Value Problems* 2014 **2014**:172.