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Optimal control problem governed by a linear hyperbolic integro-differential equation and its finite element analysis

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Abstract
In this paper, the mathematical formulation for a quadratic optimal control problem governed by a linear hyperbolic integro-differential equation is established. We first show the existence and regularity for the solution of the optimal control problem. The finite element approximation is based on the optimality conditions, which are also derived. Then the *a priori* error estimates for its finite element approximation are obtained with the optimal convergence order. Furthermore some numerical tests are presented to verify the theoretical results.

Keywords: optimal control problem; linear hyperbolic integro-differential equations; optimality conditions; finite element methods; *a priori* error estimate

1 Introduction

The distributed optimal control problem has been a classic research topic in the discipline of applied mathematics. Since it is normally difficult to obtain a closed form solution, finite element approximations of optimal control problems governed by partial differential equations have been extensively studied in the literature. In particular, there have been extensive studies in convergence and *a priori* error estimates of the standard finite element approximation of optimal control problems; see for instance, [1–9], although it is impossible to give even a very brief review here.

For optimal control problems governed by classic linear PDEs such as elliptic, parabolic and hyperbolic equations, the existence and the optimality conditions are well known, see [10]. Furthermore their finite element approximation and *a priori* error estimates were established long ago, for example, see [1–7, 9]. Recently research has been carried out for the control governed by the integro-differential equations such as elliptic and parabolic integro-differential equations; see [11, 12]. However, there exists little research on the optimal control problem governed by hyperbolic integro-differential equations, in spite of the fact that such control problems are widely encountered in practical engineering applications and scientific computations. Integro-differential equations and their control of this nature appear in applications such as heat conduction in materials with memory, population dynamics, and visco-elasticity; *cf.*, *e.g.*, [13–15]. The physical backgrounds and the existence and uniqueness of the solution of the hyperbolic integro-differential equations have been studied in [15–17]. One very important characteristic of all these models is that they all express conservation of a certain quantity; mass, momentum, heat *etc.* in any moment for any subdomain.

Furthermore the finite element approximation of optimal control problem governed by hyperbolic integro-differential equations has not been studied yet, although there exists much research on the finite element approximation of hyperbolic integro-differential equations, see, *e.g.* [18, 19].

The purpose of this paper is to investigate the weak formulation of the optimal control problem governed by integro-differential equations of hyperbolic type, and then its finite element approximation. Furthermore we derive the optimality conditions and establish the *a priori* error estimates for the constrained optimal control problems. Finally we present some numerical tests to verify the theoretical results.

The outline of the paper is as follows. In Section 2, we present the weak formulation and prove the existence of the solution for the optimal control problem. In Section 3, we present the optimality conditions and the finite element approximation. In Section 4, we establish the optimal *a priori* error estimates for the finite element approximation of the control problem. Finally, we present some numerical tests, which illustrate the theoretical results.

2 Model problem and its weak formulation

Let Ω, with the Lipschitz boundary $\partial\Omega$, and Ω_U be bounded open sets in R^d, $1 \leq d \leq 3$, and $T > 0$. We introduce some Sobolev spaces. Throughout the paper, we adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,q,\Omega}$, and semi-norm $|\cdot|_{m,q,\Omega}$. Set $W_0^{m,q}(\Omega) = \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$. Also denote $W^{m,2}(\Omega)(W_0^{m,2}(\Omega))$ by $H^m(\Omega)$ $(H_0^m(\Omega))$, with norm $\|\cdot\|_{m,\Omega}$, and semi-norm $|\cdot|_{m,\Omega}$. Denote by $L^s(0,T;W^{m,q}(\Omega))$ the Banach space of all L^s integrable functions from $(0,T)$ into $W^{m,q}(\Omega)$ with norm $\|v\|_{L^s(0,T;W^{m,q}(\Omega))} = (\int_0^T \|v\|^s_{W^{m,q}(\Omega)}\,dt)^{\frac{1}{s}}$ for $s \in [1,\infty)$ and the standard modification for $s = \infty$. Similarly, one can define the spaces $H^1(0,T;W^{m,q}(\Omega))$ and $C^k(0,T;W^{m,q}(\Omega))$. The details can be found in [20]. In addition, c or C denotes a general positive constant independent of the unknowns and the mesh parameters introduced later.

To fix ideas, we will take the state space $W = L^2(0,T;V)$ with $V = H_0^1(\Omega)$ and the control space $X = L^2(0,T;U)$ with $U = L^2(\Omega_U)$. Let the observation space be $Y = L^2(0,T;H)$ with $H = L^2(\Omega)$. Let $U_{ad} \subseteq X$ be a convex subset.

We investigate the following optimal control problem governed by a hyperbolic integro-differential equation:

$$\min_{u \in U_{ad} \subset X} J\big(u, y(u)\big) = \int_0^T \big(g(y) + h(u)\big)\,dt \tag{2.1}$$

subject to

$$\begin{cases} y_{tt} + \mathbf{A}y + \int_0^t \mathbf{C}(t,\tau)y(\tau)\,d\tau = f + Bu, & \text{in } \Omega \times (0,T], \\ y = 0, & \text{on } \partial\Omega \times [0,T], \\ y|_{t=0} = y_0, \qquad y_t|_{t=0} = y_1, & \text{in } \Omega, \end{cases} \tag{2.2}$$

where u is the control, y is the state, U_{ad} is a closed convex subset with the respect to the control, f, y_0, and y^1 are some suitable functions to be specified later. $\mathbf{A}$ is a linear strongly elliptic self-adjoint partial differential operator of second order with coefficients depending smoothly on the spatial variables, and $\mathbf{C}(t,\tau)$ is an arbitrary second-order linear partial differential operator, with coefficients depending smoothly on both time and spatial variables in the closure of their respective domains; B is a suitable continuous operator. A precise formulation of this problem is given later.

Here we assume $g(\cdot)$ is a convex functional which is continuously differentiable on $L^2(\Omega)$, and $h(\cdot)$ is a strictly convex continuously differentiable functional on U. We further assume that $h(u) \longrightarrow +\infty$ as $\|u\|_U \to +\infty$ and that $g(\cdot)$ is bounded below. Details will be specified later.

In order to give the weak formulation of problem mentioned above and study the existence and regularity of the solution, we introduce the L^2-inner products

$$(f_1,f_2) = \int_\Omega f_1 f_2, \quad \forall (f_1,f_2) \in H \times H, \qquad (u,v)_U = \int_{\Omega_U} uv, \quad \forall (u,v) \in U \times U$$

and the bilinear forms

$$\begin{aligned}
&a(z,w) = (\mathbf{A}z,w),\\
&c(t,\tau;z,w) = \bigl(\mathbf{C}(t,\tau)z,w\bigr), \qquad c_t(t,\tau;z,w) = \bigl(\mathbf{C}_t(t,\tau)z,w\bigr),\\
&c_{tt}(t,\tau;z,w) = \bigl(\mathbf{C}_{tt}(t,\tau)z,w\bigr).
\end{aligned}$$

In the case that $f_1 \in V, f_2 \in V^*$, the dual pair (f_1,f_2) is understood as $\langle f_1,f_2\rangle_{V\times V^*}$.

We shall assume the convexity conditions

$$\bigl(h'(u) - h'(v), u - v\bigr) \geq c\|u - v\|^2_{0,\Omega_U}, \quad \forall u, v \in L^2(\Omega_U), \tag{2.3}$$

that is to say, $h(\cdot)$ is uniformly convex. Noting that $g(\cdot)$ is convex, it is easy to see that

$$\bigl(g'(u) - g'(v), u - v\bigr) \geq 0, \quad \forall u, v \in H^1(\Omega). \tag{2.4}$$

Also, we have

$$\bigl|(Bv,w)\bigr| \leq c\|v\|_{0,\Omega_U}\|w\|_{0,\Omega}, \quad \forall v \in L^2(\Omega_U), u \in H^1(\Omega), \tag{2.5}$$

because B is a bounded linear operator.

Then a possible weak formulation for the state equation reads

$$\begin{cases} (y_{tt},w) + a(y,w) + \int_0^t c(t,\tau;y(\tau),w)\,d\tau = (f + Bu,w), \quad \forall w \in V, t \in (0,T],\\ y|_{t=0} = y_0, \qquad y_t|_{t=0} = y_1. \end{cases} \tag{2.6}$$

From [15–17], we know that the above weak formulation has at least one solution in $y \in S(0,T) = \{y : y \in L^2(0,T;H^1_0(\Omega)), y_t \in L^2(0,T;L^2(\Omega)), y_{tt} \in L^2(0,T;H^{-1}(\Omega))\}$.

Therefore the control problem (2.1)-(2.2) can be restated as (OCP):

$$\min_{u\in U_{ad}} J\bigl(u,y(u)\bigr)$$

subject to

$$\begin{cases} (y_{tt}, w) + a(y, w) + \int_0^t c(t, \tau; y(\tau), w)\, d\tau = (f + Bu, w), \quad \forall w \in V, t \in (0, T], \\ y|_{t=0} = y_0, \qquad y_t|_{t=0} = y_1. \end{cases} \tag{2.7}$$

Next, we will analyze the existence, uniqueness, and regularity of the solution of (2.7). Assume that there are constants $c > 0$ and $C > 0$, such that for all t and τ in $[0, T]$:

$$\begin{aligned}
&\text{(a)} \quad a(z, z) \geq c\|z\|^2_{1,\Omega}, \quad \forall z \in V, \\
&\text{(b)} \quad \big|a(z, w)\big| \leq C\|z\|_{1,\Omega}\|w\|_{1,\Omega}, \quad \forall z, w \in V, \\
&\text{(c)} \quad \big|c(t, \tau; z, w)\big| \leq C\|z\|_{1,\Omega}\|w\|_{1,\Omega}, \quad \forall z, w \in V, \\
&\text{(d)} \quad \big|c_t(t, \tau; z, w)\big| \leq C\|z\|_{1,\Omega}\|w\|_{1,\Omega}, \quad \forall z, w \in V, \\
&\text{(e)} \quad \big|c_{tt}(t, \tau; z, w)\big| \leq C\|z\|_{1,\Omega}\|w\|_{1,\Omega}, \quad \forall z, w \in V.
\end{aligned} \tag{2.8}$$

In the following, we will give the existence and uniqueness of the solution of the system (2.7).

Theorem 2.1 *Assume that the above conditions* (a)-(d) *hold. There exists a unique solution* (u, y) *for the minimization problem* (2.7) *such that* $u \in L^2(0, T; L^2(\Omega_U))$, $y \in L^\infty(0, T; H_0^1(\Omega))$, $y_t \in L^\infty(0, T; L^2(\Omega))$, $y_{tt} \in L^2(0, T; H^{-1}(\Omega))$.

Proof Let $\{(u^n, y^n)\}_{n=1}^\infty$ be a minimization sequence for the system (2.7), then it is clear that $\{u^n\}_{n=1}^\infty$ are bounded in $L^2(0, T; L^2(\Omega_U))$. Thus there is a subsequence of $\{u^n\}_{n=1}^\infty$ (still denoted by $\{u^n\}_{n=1}^\infty$) such that u^n converges to u^* weakly in $L^2(0, T; L^2(\Omega_U))$. For the subsequence u^n, we have

$$\begin{aligned}
&\big(y^n_{tt}, w\big) + a\big(y^n, w\big) + \int_0^t c\big(t, \tau; y^n(\tau), w(t)\big)\, d\tau = \big(f + Bu^n, w\big), \\
&\forall w \in V, t \in (0, T].
\end{aligned} \tag{2.9}$$

Taking $w = y^n_t$ in (2.9), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\{\big\|y^n_t\big\|^2_{0,\Omega} + a\big(y^n, y^n\big)\big\} \\
&\quad = \big(f + Bu^n, y^n_t\big) - \frac{d}{dt}\int_0^t c\big(t, \tau; y^n(\tau), y^n(t)\big)\, d\tau \\
&\qquad + c\big(t, t; y^n(t), y^n(t)\big) + \int_0^t c_t\big(t, \tau; y^n(\tau), y^n(t)\big)\, d\tau, \quad t \in (0, T].
\end{aligned} \tag{2.10}$$

Integrating time from 0 to t in (2.10), we obtain

$$\begin{aligned}
\frac{1}{2}\big\|y^n_t\big\|^2_{0,\Omega} + \frac{c}{2}\big\|y^n\big\|^2_{1,\Omega} \leq{}& \frac{1}{2}\|y_1\|^2_{0,\Omega} + \frac{c}{2}\|y_0\|^2_{1,\Omega} + \int_0^t \big(f + Bu^n, y^n_t\big)\, d\tau + \varepsilon\big\|y^n\big\|^2_{1,\Omega} \\
&+ C\int_0^t \big\|y^n\big\|^2_{1,\Omega}\, d\tau + C\int_0^t \int_0^\tau \big\|y^n(s)\big\|^2_{1,\Omega}\, ds\, d\tau.
\end{aligned} \tag{2.11}$$

From (2.11) and the Gronwall lemmas, we have

$$\begin{aligned}\|y^n\|_{1,\Omega}^2 \le C\left(\|y_1\|_{0,\Omega}^2 + \|y_0\|_{1,\Omega}^2 + \int_0^t \left(f + Bu^n, y_t^n\right) d\tau\right) \\ + C\int_0^t \int_0^\tau \|y^n(s)\|_{1,\Omega}^2 \, ds\, d\tau. \end{aligned} \tag{2.12}$$

So we get

$$\begin{aligned}\|y^n\|_{1,\Omega}^2 + \int_0^t \|y^n\|_{1,\Omega}^2 \, d\tau \le C\left(\|y_1\|_{0,\Omega}^2 + \|y_0\|_{1,\Omega}^2 + \int_0^t \left(f + Bu^n, y_t^n\right) d\tau\right) \\ + C\int_0^t \left\{\|y^n(\tau)\|_{1,\Omega}^2 + \int_0^\tau \|y^n(s)\|_{1,\Omega}^2 \, ds\right\} d\tau, \end{aligned} \tag{2.13}$$

such that

$$\|y^n\|_{1,\Omega}^2 \le C\left\{\|y_1\|_{0,\Omega}^2 + \|y_0\|_{1,\Omega}^2 + \int_0^t \left(f + Bu^n, y_t^n\right) d\tau\right\}. \tag{2.14}$$

Then by (2.14) and (2.11)

$$\begin{aligned}\|y_t^n\|_{0,\Omega}^2 &\le C\left\{\|y_1\|_{0,\Omega}^2 + \|y_0\|_{1,\Omega}^2 + \int_0^t \left(f + Bu^n, y_t^n\right) d\tau\right\} \\ &\le C\left\{\|y_1\|_{0,\Omega}^2 + \|y_0\|_{1,\Omega}^2\right\} + C\int_0^t \|f + Bu^n\|_{0,\Omega} \, d\tau \cdot \sup_{0\le\tau\le t} \|y_t^n(\tau)\|_{0,\Omega}. \end{aligned} \tag{2.15}$$

Taking the supermaximum in (2.15), we obtain

$$\|y_t^n\|_{L^\infty(0,T;L^2(\Omega))}^2 \le C\left\{\|y_1\|_{0,\Omega}^2 + \|y_0\|_{1,\Omega}^2 + \|f\|_{L^2(0,T;L^2(\Omega))}^2 + \|u^n\|_{L^2(0,T;L^2(\Omega_U))}^2\right\}. \tag{2.16}$$

Then from (2.14) and (2.16), we also have

$$\|y^n\|_{L^\infty(0,T;H^1(\Omega))}^2 \le C\left\{\|y_1\|_{0,\Omega}^2 + \|y_0\|_{1,\Omega}^2 + \|f\|_{L^2(0,T;L^2(\Omega))}^2 + \|u^n\|_{L^2(0,T;L^2(\Omega_U))}^2\right\}. \tag{2.17}$$

Then we have $u^n \in L^2(0,T;L^2(\Omega_U))$, $y^n \in L^\infty(0,T;H_0^1(\Omega))$ and $y_t^n \in L^\infty(0,T;L^2(\Omega))$. Thus

$$\begin{cases} u^n \longrightarrow u \in L^2(0,T;L^2(\Omega_U)), \\ y^n \longrightarrow y \in L^\infty(0,T;H_0^1(\Omega)), \\ y^n(T) \longrightarrow y(T) \in H^1(\Omega), \\ y_t^n \longrightarrow y_t \in L^\infty(0,T;L^2(\Omega)), \\ y_t^n(T) \longrightarrow y_t(T) \in L^2(\Omega). \end{cases}$$

Integrating time from 0 to T in (2.9), we obtain

$$\begin{aligned}\left(y_t^n(T), w(T)\right) - \left(y_1, w(0)\right) - \int_0^T \left(y_t^n, w_t\right) dt + \int_0^T a\left(y^n, w\right) dt \\ + \int_0^T \int_0^t c\left(t,\tau; y^n(\tau), w\right) d\tau\, dt = \int_0^T \left(f + Bu^n, w\right) dt, \quad \forall w \in W. \end{aligned} \tag{2.18}$$

Taking the limits in (2.18) as $n \to \infty$, we have

$$\begin{aligned}&\big(y_t(T), w(T)\big) - \big(y_1, w(0)\big) - \int_0^T (y_t, w_t)\,dt + \int_0^T a(y,w)\,dt \\ &\quad + \int_0^T \int_0^t c\big(t,\tau; y(\tau), w\big)\,d\tau\,dt = \int_0^T (f + Bu, w)\,dt,\end{aligned}$$

and

$$\begin{aligned}&\int_0^T (y_{tt}, w)\,dt + \int_0^T a(y,w)\,dt + \int_0^T \int_0^t c\big(t,\tau; y(\tau), w(t)\big)\,d\tau\,dt \\ &\quad = \int_0^T (f + Bu, w)\,dt, \quad \forall w \in W.\end{aligned} \tag{2.19}$$

So we have

$$(y_{tt}, w) + a(y,w) + \int_0^t c\big(t,\tau; y(\tau), w\big)\,d\tau = (f + Bu, w), \quad \forall w \in W. \tag{2.20}$$

Further, from (2.9), we obtain

$$\begin{aligned}\|y_{tt}\|_{L^2(0,T;H^{-1}(\Omega))} &= \sup_{w \in L^2(0,T;H_0^1(\Omega))} \frac{\int_0^T (y_{tt}, w)\,dt}{\|w\|_{L^2(0,T;H_0^1(\Omega))}} \\ &\le C\big\{\|y_1\|_{0,\Omega}^2 + \|y_0\|_{1,\Omega}^2 + \|f\|_{L^2(0,T;L^2(\Omega))}^2 + \|u\|_{L^2(0,T;L^2(\Omega_U))}\big\}.\end{aligned}$$

This means $y_{tt} \in L^2(0,T;H^{-1}(\Omega))$.

Since $g(\cdot)$ is a convex function on space $L^2(0,T;L^2(\Omega))$ and $h(\cdot)$ is a strictly convex function on U, we have

$$\int_0^T \big(g(y) + h(u)\big)\,dt \le \varliminf_{n\to\infty} \int_0^T \big(g(y^n) + h(u^n)\big)\,dt.$$

So (u,y) is one solution of (2.7). Since $J(u, y(u))$ is a strictly convex function on U_{ad}, hence the solution of the minimization problem (2.7) is unique. □

The following theorem states the regularity of the solution of (2.7).

Theorem 2.2 *Assume that the above condition* (a)-(e) *holds and* $\mathbf{A}$ *is an* H^2*-regularity elliptic operator of second order and* $f, f_t, u, u_t \in C(0,T;L^2(\Omega_U))$, $y_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. *Then the solution of* (2.7) *is regular in the sense that* $y \in L^\infty(0,T;H_0^1(\Omega)) \cap L^2(0,T;H^2(\Omega))$, $y_t \in L^\infty(0,T;H^1(\Omega))$, $y_{tt} \in L^\infty(0,T;L^2(\Omega))$.

Proof Differentiating (2.2) with respect to t, we have

$$\begin{cases} y_{ttt} + \mathbf{A}y_t + \mathbf{C}(t,t)y + \int_0^t \mathbf{C}_t(t,\tau)y(\tau)\,d\tau = f_t + Bu_t, & (x,t) \in \Omega \times (0,T], \\ y = 0, & (x,t) \in \partial\Omega \times [0,T], \\ y|_{t=0} = y_0, \qquad y_t|_{t=0} = y_1, & x \in \Omega, \end{cases} \tag{2.21}$$

and we obtain

$$(y_{ttt}, w) + a(y_t, w) + c(t,t;y,w) + \int_0^t c_t\big(t,\tau;y(\tau),w\big)\,d\tau = (f_t + Bu_t, w). \tag{2.22}$$

Taking $w = y_{tt}$ in (2.22), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\{\|y_{tt}\|_{0,\Omega}^2 + a(y_t,y_t)\big\} \\
&\quad= (f_t + Bu_t, y_{tt}) - \frac{d}{dt}c(t,t;y,y_t) + c_t(t,t;y,y_t) \\
&\qquad+ c(t,t;y_t,y_t) - \frac{d}{dt}\int_0^t c_t\big(t,\tau;y(\tau),y_t\big)\,d\tau + c_t(t,t;y,y_t) \\
&\qquad+ \int_0^t c_{tt}\big(t,\tau;y(\tau),y_t\big)\,d\tau.
\end{aligned} \tag{2.23}$$

Integrating time from 0 to t in (2.23), in the same way as getting (2.16) and (2.17), we can deduce

$$\begin{aligned}
&\|y_{tt}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \|y_t\|_{L^\infty(0,T;H^1(\Omega))}^2 \\
&\quad\le C\big\{\|y_0\|_{1,\Omega}^2 + \|y_1\|_{1,\Omega}^2 + \|\mathbf{A}y_0\|_{0,\Omega}^2 + \|f\|_{L^2(0,T;L^2(\Omega))}^2 \\
&\qquad+ \|f_t\|_{L^2(0,T;L^2(\Omega))}^2 + \|u\|_{L^2(0,T;L^2(\Omega_U))}^2 + \|u_t\|_{L^2(0,T;L^2(\Omega_U))}^2\big\}.
\end{aligned}$$

Then $y_t \in L^\infty(0,T;H^1(\Omega))$ and $y_{tt} \in L^\infty(0,T;L^2(\Omega))$. Further we have

$$\begin{aligned}
&\|\mathbf{A}y\|_{L^2(0,T;L^2(\Omega))} \\
&\quad\le C\big\{\|y_{tt}\|_{L^2(0,T;L^2(\Omega))} + \|f\|_{L^2(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;L^2(\Omega_U))} + \|\mathbf{C}y\|_{L^2(0,T;L^2(\Omega))}\big\}.
\end{aligned}$$

Thus by the Gronwall lemmas, $y \in L^2(0,T;H^2(\Omega))$. This completes the proof of Theorem 2.2. □

Remark 2.3 In this paper, we suppose that $\mathbf{A}$ is independent of t. The above results also hold for the case $\mathbf{A} = \mathbf{A}(x,t)$ provided suitable smoothness of the operator $\mathbf{A}$ is assumed.

3 The optimality conditions and its finite element approximation

In this section, we study the optimality conditions and the finite element approximation for the optimal control problem governed by hyperbolic integro-differential equation.

For simplicity, we will only consider the case of quadratic objective functionals as follows:

$$J(u,y) = \int_0^T \big(g(y) + h(u)\big)\,dt = \left\{\frac{1}{2}\int_0^T \|y - z_d\|_{0,\Omega}^2\,dt + \frac{\alpha}{2}\int_0^T \|u\|_{0,\Omega_U}^2\,dt\right\}.$$

Here

$$g(y) = \frac{1}{2}\int_0^T \|y - z_d\|_{0,\Omega}^2\,dt \tag{3.1}$$

and

$$h(u) = \frac{\alpha}{2}\int_0^T \|u\|^2_{0,\Omega_U}\,dt, \tag{3.2}$$

where z_d is the observation.

3.1 The optimality conditions of model problem

The following theorem states the optimality conditions of the problem (2.7).

Theorem 3.1 *A pair $(y,u) \in S(0,T) \times X$ is the solution of the optimal control problem* (2.7), *if and only there exists a co-state $p \in S(0,T)$, such that the triple (y,p,u) satisfies the following optimality conditions*:

$$\begin{cases} (y_{tt},w) + a(y,w) + \int_0^t c(t,\tau;y(\tau),w)\,d\tau = (f + Bu, w), & \forall w \in V, t \in (0,T], \\ y|_{t=0} = y_0, \qquad y_t|_{t=0} = y_1; \end{cases} \tag{3.3}$$

$$\begin{cases} (q,p_{tt}) + a(q,p) + \int_t^T c(\tau,t;q,p(\tau))\,d\tau = (y - z_d, q), & \forall q \in V, t \in [0,T), \\ p|_{t=T} = 0, \qquad p_t|_{t=T} = 0; \end{cases} \tag{3.4}$$

$$\int_0^T \left(\alpha u + B^* p, v - u\right)_U dt \geq 0, \quad \forall v \in U_{ad}, \tag{3.5}$$

where $B : L^2(\Omega_U) \to L^2(\Omega)$ is independent with t. B^* is the adjoint operator of B.

Proof Let $J(u,y) = g(y(u)) + j(u)$, where

$$g\big(y(u)\big) = \frac{1}{2}\int_0^T \|y - z_d\|^2_{0,\Omega}\,dt, \qquad j(u) = \frac{\alpha}{2}\int_0^T \|u\|^2_{0,\Omega_U}\,dt.$$

By the standard method in [21], the optimal conditions read

$$j'(u)(v-u) + \big(g\big(y(u)\big)\big)'(v-u) \geq 0, \quad \forall v \in U_{ad}, \tag{3.6}$$

where

$$\begin{aligned} &j'(u)(v-u) \\ &\quad = \lim_{s\to 0^+} \frac{1}{s}\big(j\big(u + s(v-u)\big) - j(u)\big) \\ &\quad = \lim_{s\to 0^+} \frac{1}{s}\left(\frac{\alpha}{2}\int_0^T \left[\big\|u + s(v-u)\big\|^2_{0,\Omega_U} - \|u\|^2_{0,\Omega_U}\right] dt\right) \\ &\quad = \int_0^T (\alpha u, v-u)_U\,dt, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &\big(g\big(y(u)\big)\big)'(v-u) \\ &\quad = \lim_{s\to 0^+} \frac{1}{s}\big(g\big(y\big(u + s(v-u)\big)\big) - g\big(y(u)\big)\big) \\ &\quad = \lim_{s\to 0^+} \frac{1}{2s}\int_0^T \left[\big\|y\big(u + s(v-u)\big) - z_d\big\|^2_{0,\Omega} - \big\|y(u) - z_d\big\|^2_{0,\Omega}\right] dt \end{aligned}$$

$$
\begin{aligned}
&= \lim_{s\to 0^+} \frac{1}{2s} \int_0^T \bigl[\bigl\| y\bigl(u+s(v-u)\bigr) - y(u) \bigr\|_{0,\Omega}^2 + 2\bigl(y\bigl(u+s(v-u)\bigr) - y(u), y - z_d\bigr) \bigr]\, dt \\
&= \int_0^T \bigl(y'(u)(v-u), y - z_d\bigr)\, dt. \qquad (3.8)
\end{aligned}
$$

Next, we compute $y'(u)(v-u)$. Let us differentiate the state equation (2.7) at u in the direction v. By (2.7), we have

$$
\begin{aligned}
&\frac{1}{s}\biggl(\int_0^T \bigl(y_{tt}(u+sv) - y_{tt}(u), w\bigr)\, dt + \int_0^T a\bigl(y(u+sv) - y(u), w\bigr)\, dt \\
&\quad + \int_0^T \int_0^t c\bigl(t,\tau; y(u+sv)(\tau) - y(u)(\tau), w\bigr)\, d\tau\, dt \biggr) = \int_0^T (Bv, w)\, dt. \qquad (3.9)
\end{aligned}
$$

Taking the limits in (3.9) as $s \to 0$, we obtain

$$
\begin{aligned}
&\int_0^T \bigl(\bigl(y'(u)(v)\bigr)_{tt}, w\bigr)\, dt + \int_0^T a\bigl(y'(u)(v), w\bigr)\, dt + \int_0^T \int_0^t c\bigl(t,\tau; \bigl(y'(u)(v)\bigr)(\tau), w\bigr)\, d\tau\, dt \\
&\quad = \int_0^T (Bv, w)\, dt, \quad \forall v \in U_{ad}, w \in W, \qquad (3.10)
\end{aligned}
$$

where we used the equality that for any $z, w \in L^2(0,T;H^1(\Omega))$,

$$
\int_0^T \int_0^t c\bigl(t,\tau; z(\tau), w(t)\bigr)\, d\tau\, dt = \int_0^T \int_\tau^T c\bigl(t,\tau; z(\tau), w(t)\bigr)\, dt\, d\tau. \qquad (3.11)
$$

Then (3.10) is equivalent to

$$
\begin{aligned}
&\int_0^T \bigl(\bigl(y'(u)(v)\bigr)_{tt}, w\bigr)\, dt + \int_0^T a\bigl(y'(u)(v), w\bigr)\, dt \\
&\qquad + \int_0^T \int_t^T c\bigl(\tau, t; \bigl(y'(u)(v)\bigr)(t), w(\tau)\bigr)\, d\tau\, dt \\
&\quad = \int_0^T (Bv, w)\, dt, \quad \forall v \in U_{ad}, w \in W. \qquad (3.12)
\end{aligned}
$$

Define the co-state $p \in S(0,T)$ satisfying

$$
\begin{cases}
\int_0^T [(q_{tt}, p) + a(q,p) + \int_t^T c(\tau, t; q(t), p(\tau))\, d\tau]\, dt \\
\quad = \int_0^T (y - z_d, q)\, dt, \quad \forall q \in W, \\
p(x,T) = 0, \qquad p_t(x,T) = 0.
\end{cases} \qquad (3.13)
$$

Since $p \in S(0,T)$, (3.13) is equivalent to

$$
\begin{cases}
\int_0^T [(q, p_{tt}) + a(q,p) + \int_t^T c(\tau, t; q(t), p(\tau))\, d\tau]\, dt \\
\quad = \int_0^T (y - z_d, q)\, dt, \quad \forall q \in W, \\
p(x,T) = 0, \qquad p_t(x,T) = 0.
\end{cases} \qquad (3.14)
$$

Letting $w = p$ in (3.12), we have

$$\begin{aligned}\int_0^T \big(B(v-u),p\big)\,dt &= \int_0^T \big(v-u,B^*p\big)_U\,dt \\ &= \int_0^T \bigg[\big(y'(u)(v-u),p_{tt}\big) + a\big(y'(u)(v-u),p\big) \\ &\quad + \int_t^T c\big(\tau,t;y'(u)(v-u)(t),p(\tau)\big)\,d\tau\bigg]\,dt \\ &= \int_0^T \big(y-z_d,y'(u)(v-u)\big)\,dt, \quad \forall v\in U_{ad}. \end{aligned} \tag{3.15}$$

By (3.8) and (3.15), we have

$$\begin{aligned}\big(g\big(y(u)\big)\big)'(v-u) &= \int_0^T \big(y'(u)(v-u),y-z_d\big)\,dt \\ &= \int_0^T \big(v-u,B^*p\big)_U\,dt, \quad \forall v\in U_{ad}. \end{aligned} \tag{3.16}$$

By (3.6)-(3.8), and (3.16), the optimality conditions read

$$J'(u)(v-u) = \int_0^T \big(\alpha u + B^*p, v-u\big)_U\,dt \ge 0, \quad \forall v\in U_{ad}, \tag{3.17}$$

where p is defined in (3.14). This completes the proof of Theorem 3.1. □

3.2 Finite element approximation

In the following, we discuss the finite element approximation of the control problem (2.7). Here we only consider triangular and conforming elements.

Let Ω^h be a polygonal approximation to Ω with boundary $\partial\Omega^h$. Let T^h be a partitioning of Ω^h into disjoint regular n-simplices τ, so that $\bar{\Omega}^h = \bigcup_{\tau\in T^h}\bar{\tau}$. Each element has at most one face on $\partial\Omega^h$, and $\bar{\tau}$ and $\bar{\tau}'$ have either only one common vertex or a whole edge or face if $\bar{\tau}$ and $\bar{\tau}' \in T^h$. We further require that $P_i \in \partial\Omega^h \Longrightarrow P_i \in \partial\Omega$ where P_i $(i = 1,\dots,J)$ is the vertex set associated with the triangulation T_h. As usual, h denotes the diameter of the triangulation T^h. For simplicity, we assume that Ω is a convex polygon so that $\Omega = \Omega^h$.

Associated with T^h is a finite-dimensional subspace S^h of $C(\bar{\Omega}^h)$, such that $\chi|_\tau$ are polynomials of order m $(m \ge 1)$ for all $\chi \in S^h$ and $\tau \in T^h$. Let $V^h = \{v_h \in S_h : v_h(P_i) = 0\ (i = 1,\dots,J)\}$, $W^h = L^2(0,T;V^h)$. It is easy to see that $V^h \subset V$, $W^h \subset W$.

Let T_U^h be a partitioning of Ω_U^h into disjoint regular n-simplices τ_U, so that $\bar{\Omega}_U^h = \bigcup_{\tau_U\in T_U^h}\bar{\tau}_U$. $\bar{\tau}_U$ and $\bar{\tau}_U'$ have either only one common vertex or a whole edge or face if $\bar{\tau}_U$ and $\bar{\tau}_U' \in T_U^h$. We further require that $P_i \in \partial\Omega_U^h \Longrightarrow P_i \in \partial\Omega_U$ where P_i $(i = 1,\dots,J)$ is the vertex set associated with the triangulation T_U^h. For simplicity, we again assume that Ω_U is a convex polygon so that $\Omega_U = \Omega_U^h$.

Associated with T_U^h is another finite-dimensional subspace U^h of $L^2(\Omega_U^h)$, such that $\chi|_{\tau_U}$ are polynomials of order m $(m \ge 0)$ for all $\chi \in U^h$ and $\tau_U \in T_U^h$. Here there is no requirement of continuity. Let $X^h = L^2(0,T;U^h)$. It is easy to see that $X^h \subset X$. Let $h_\tau(h_{\tau_U})$ denote the maximum diameter of the element $\tau(\tau_U)$ in $T^h(T_U^h)$. To simplify our presentation we here only consider the piecewise constant finite element space for the approximation of the control. Let $P_0(\Omega)$ denote all the zeroth-order polynomial over Ω. Therefore we always take $X^h = \{u \in X : u(x,t)|_{x\in\tau_U} \in P_0(\tau_U), \forall t \in [0,T]\}$. U_{ad}^h is a closed convex set in X^h. For ease of exposition, in this paper we assume that $U_{ad}^h \subset (U_{ad} \cap X^h)$.

Then the finite element approximation of (OCP) is thus defined by $(OCP)^h$:

$$\min_{u_h \in U_{ad}^h} \left\{ \frac{1}{2} \int_0^T \|y_h - z_d\|_{0,\Omega}^2 \, dt + \frac{\alpha}{2} \int_0^T \|u_h\|_{0,\Omega_U}^2 \, dt \right\} \tag{3.18}$$

such that

$$\begin{cases} (\frac{\partial^2}{\partial t^2} y_h, w_h) + a(y_h, w_h) + \int_0^t c(t,\tau; y_h(\tau), w_h)\, d\tau \\ \quad = (f + Bu_h, w_h), \quad \forall w_h \in V^h, t \in (0,T], \\ y_h|_{t=0} = y_0^h, \qquad \frac{\partial}{\partial t} y_h|_{t=0} = y_1^h, \end{cases} \tag{3.19}$$

where $y_h \in W^h$, $y_0^h \in V^h$, and $y_1^h \in V^h$ are the approximations of y_0 and y_1.

Since (3.19) is a linear functional equation, and (3.18) is a strictly convex and finite dimensional optimal problem, we can prove that the problem (3.18)-(3.19) has a unique solution $(y_h, u_h) \in W^h \times U_{ad}^h$ in the same way as proving the uniqueness of the solution of (2.1)-(2.2).

It is well known that a pair $(y_h, u_h) \in W^h \times U_{ad}^h$ is a solution of (3.18)-(3.19), if and only there exists a co-state $p_h \in W^h$ such that the triple (y_h, p_h, u_h) satisfies the following optimality conditions:

$$\begin{cases} (\frac{\partial^2}{\partial t^2} y_h, w_h) + a(y_h, w_h) + \int_0^t c(t,\tau; y_h(\tau), w_h)\, d\tau = (f + Bu_h, w_h), \quad \forall w_h \in V^h, \\ y_h|_{t=0} = y_0^h, \qquad \frac{\partial}{\partial t} y_h|_{t=0} = y_1^h; \end{cases} \tag{3.20}$$

$$\begin{cases} (q_h, \frac{\partial^2}{\partial t^2} p_h) + a(q_h, p_h) + \int_t^T c(\tau, t; q_h, p_h(\tau))\, d\tau = (y_h - z_d, q_h), \quad \forall q_h \in V^h, \\ p_h|_{t=T} = 0, \qquad \frac{\partial}{\partial t} p_h|_{t=T} = 0; \end{cases} \tag{3.21}$$

$$\int_0^T \left(\alpha u_h + B^* p_h, v_h - u_h\right)_U dt \geq 0, \quad \forall v_h \in U_{ad}^h. \tag{3.22}$$

The optimality conditions in (3.20)-(3.22) are the semi-discrete approximation to the problem (3.3)-(3.5). Let π_{h_U} be the local averaging operator given by

$$(\pi_{h_U} w)|_{\tau_U} := \frac{\int_{\tau_U} w}{\int_{\tau_U} 1}, \quad \forall \tau_U \in T_U^h. \tag{3.23}$$

It is an obvious fact that $\int_{\Omega_U} w = \int_{\Omega_U} \pi_{h_U} w$ for any $w \in L^2(\Omega_U)$. By the operator π_{h_U}, (3.22) is equivalent to

$$\int_0^T \left(\alpha u_h + \pi_{h_U}(B^* p_h), v_h - u_h\right)_U dt \geq 0, \quad \forall v_h \in U_{ad}^h. \tag{3.24}$$

In the next sections, we will analyze the *a priori* error estimates of the approximation solution.

4 *A priori* error analysis

For simplicity, we consider the zero obstacle problem:

$$U_{ad} = \{v \in X; v \geq 0, \text{ a.e. } x \in \Omega_U, t \in [0,T]\}, \tag{4.1}$$

or the integration obstacle problem:

$$U_{ad} = \left\{v \in X; \int_{\Omega_U} v \geq 0, t \in [0,T]\right\}. \tag{4.2}$$

In the case of (4.1), (3.5) and (3.22) yield

$$(y,p,u) \in L^2(0,T;H^2(\Omega)) \times L^2(0,T;H^2(\Omega)) \times L^2(0,T;H^1(\Omega_U)). \tag{4.3}$$

In the case of (4.2), (3.5) and (3.22) yield

$$(y,p,u) \in L^2(0,T;H^2(\Omega)) \times L^2(0,T;H^2(\Omega)) \times L^2(0,T;H^2(\Omega_U)). \tag{4.4}$$

In the following, we will give the *a priori* error estimates in $L^\infty(0,T;H^1(\Omega))$-norm. We first present some lemmas.

Lemma 4.1 *Let U_{ad} be given by* (4.1) *or* (4.2). *Then $\pi_{h_U} w \in U_{ad}^h$ for any $w \in U_{ad}$.*

Let us introduce the auxiliary problem

$$\begin{cases} (\frac{\partial^2}{\partial t^2} y_h(u), w_h) + a(y_h(u), w_h) + \int_0^t c(t,\tau; y_h(u)(\tau), w_h)\,d\tau \\ \quad = (f + Bu, w_h), \quad \forall w_h \in V^h, \\ y_h(u)|_{t=0} = y_0^h, \qquad \frac{\partial}{\partial t} y_h(u)|_{t=0} = y_1^h; \end{cases} \tag{4.5}$$

$$\begin{cases} (q_h, \frac{\partial^2}{\partial t^2} p_h(u)) + a(q_h, p_h(u)) + \int_t^T c(\tau, t; q_h, p_h(u)(\tau))\,d\tau \\ \quad = (y - z_d, q_h), \quad \forall q_h \in V^h, \\ p_h(u)|_{t=T} = 0, \qquad \frac{\partial}{\partial t} p_h(u)|_{t=T} = 0. \end{cases} \tag{4.6}$$

Since $(y_h(u), p_h(u))$ is the standard finite element of (y,p), from [18], we get the following results.

Lemma 4.2 *Let $(y_h(u), p_h(u))$ be the solutions of the systems* (4.5)-(4.6). *Then we have the a priori error estimates*

$$\begin{aligned} &\|y - y_h(u)\|_{L^\infty(0,T;H^1(\Omega))} + \left\|\frac{\partial}{\partial t}(y - y_h(u))\right\|_{L^\infty(0,T;L^2(\Omega))} \\ &\quad + \|p - p_h(u)\|_{L^\infty(0,T;H^1(\Omega))} + \left\|\frac{\partial}{\partial t}(p - p_h(u))\right\|_{L^\infty(0,T;L^2(\Omega))} \leq Ch, \end{aligned} \tag{4.7}$$

$$\|y - y_h(u)\|_{L^2(0,T;L^2(\Omega))} + \|p - p_h(u)\|_{L^2(0,T;L^2(\Omega))} \leq Ch^2. \tag{4.8}$$

Lemma 4.3 *Let $(y_h(u), p_h(u))$ and (y_h, p_h, u_h) be the solutions of the systems* (4.5)-(4.6) *and* (3.20)-(3.22). *Then we have the a priori error estimate*

$$\begin{aligned}
&\|y_h - y_h(u)\|_{L^\infty(0,T;H^1(\Omega))} + \left\| \frac{\partial}{\partial t}\big(y_h - y_h(u)\big)\right\|_{L^\infty(0,T;L^2(\Omega))} + \|p_h - p_h(u)\|_{L^\infty(0,T;H^1(\Omega))} \\
&\quad + \left\| \frac{\partial}{\partial t}\big(p_h - p_h(u)\big)\right\|_{L^\infty(0,T;L^2(\Omega))} + \|u - u_h\|_{L^2(0,T;L^2(\Omega_U))} \le C\big(h_U + h^2\big).
\end{aligned} \tag{4.9}$$

Proof From (4.5) and (3.20), we obtain

$$\begin{cases}
(\frac{\partial^2}{\partial t^2}(y_h - y_h(u)), w_h) + a(y_h - y_h(u), w_h) + \int_0^t c(t,\tau;(y_h - y_h(u))(\tau), w_h)\,d\tau \\
\quad = (B(u_h - u), w_h), \quad \forall w_h \in V^h, \\
(y_h - y_h(u))|_{t=0} = 0, \qquad \frac{\partial}{\partial t}(y_h - y_h(u))|_{t=0} = 0.
\end{cases} \tag{4.10}$$

Similarly, from (4.6) and (3.21), we have

$$\begin{cases}
(q_h, \frac{\partial^2}{\partial t^2}(p_h - p_h(u))) + a(q_h, p_h - p_h(u)) + \int_t^T c(\tau,t;q_h,(p_h - p_h(u))(\tau))\,d\tau \\
\quad = (y_h - y, q_h), \quad \forall q_h \in V^h, \\
(p_h - p_h(u))|_{t=T} = 0, \qquad \frac{\partial}{\partial t}(p_h - p_h(u))|_{t=T} = 0.
\end{cases} \tag{4.11}$$

Taking $w_h = \frac{\partial}{\partial t}(y_h - y_h(u))$ in (4.10), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\{\big\|\big(y_h - y_h(u)\big)_t\big\|_{0,\Omega}^2 + a\big(y_h - y_h(u), y_h - y_h(u)\big)\big\} \\
&\quad = \big(B(u_h - u), \big(y_h - y_h(u)\big)_t\big) \\
&\qquad - \frac{d}{dt}\int_0^t c\big(t,\tau;\big(y_h - y_h(u)\big)(\tau), y_h - y_h(u)\big)\,d\tau + c\big(t,t;y_h - y_h(u), y_h - y_h(u)\big) \\
&\qquad + \int_0^t c_t\big(t,\tau;\big(y_h - y_h(u)\big)(\tau), y_h - y_h(u)\big)\,d\tau.
\end{aligned} \tag{4.12}$$

Integrating time from 0 to t in (4.12) and noting that $(y_h - y_h(u))|_{t=0} = 0$, $\frac{\partial}{\partial t}(y_h - y_h(u))|_{t=0} = 0$, we have

$$\begin{aligned}
&\left\|\frac{\partial}{\partial t}\big(y_h - y_h(u)\big)\right\|_{0,\Omega}^2 + \|y_h - y_h(u)\|_{1,\Omega}^2 \\
&\quad \le C\int_0^t \|u_h - u\|_{0,\Omega_U}^2\,d\tau + C\int_0^t \left\|\frac{\partial}{\partial t}\big(y_h - y_h(u)\big)\right\|_{0,\Omega}^2 d\tau + \varepsilon\|y_h - y_h(u)\|_{1,\Omega}^2 \\
&\qquad + C\int_0^t \|y_h - y_h(u)\|_{1,\Omega}^2\,d\tau + C\int_0^t\int_0^\tau \big\|\big(y_h - y_h(u)\big)(s)\big\|_{1,\Omega}^2\,ds\,d\tau.
\end{aligned} \tag{4.13}$$

Letting ε be small enough, we get

$$\begin{aligned}
&\left\|\frac{\partial}{\partial t}\big(y_h - y_h(u)\big)\right\|_{0,\Omega}^2 + \|y_h - y_h(u)\|_{1,\Omega}^2 + \int_0^t \|y_h - y_h(u)\|_{1,\Omega}^2\,d\tau \\
&\quad \le C\int_0^t \|u_h - u\|_{0,\Omega_U}^2\,d\tau + C\int_0^t \left\{\left\|\frac{\partial}{\partial t}\big(y_h - y_h(u)\big)\right\|_{0,\Omega}^2 + \|y_h - y_h(u)\|_{1,\Omega}^2 \right. \\
&\qquad \left. + \int_0^\tau \big\|\big(y_h - y_h(u)\big)(s)\big\|_{1,\Omega}^2\,ds\right\} d\tau.
\end{aligned}$$

By the Gronwall lemma, we have

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}\big(y_h-y_h(u)\big)\right\|_{L^\infty(0,T;L^2(\Omega))}+\big\|y_h-y_h(u)\big\|_{L^\infty(0,T;H^1(\Omega))}\\&\quad\le C\|u_h-u\|_{L^2(0,T;L^2(\Omega_U))}.\end{aligned}\tag{4.14}$$

Similarly letting $q_h=\frac{\partial}{\partial t}(p_h-p_h(u))$ in (4.11), we also have

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}\big(p_h-p_h(u)\big)\right\|_{L^\infty(0,T;L^2(\Omega))}+\big\|p_h-p_h(u)\big\|_{L^\infty(0,T;H^1(\Omega))}\\&\quad\le C\|y_h-y\|_{L^2(0,T;L^2(\Omega))}\\&\quad\le C\big\|y-y_h(u)\big\|_{L^2(0,T;L^2(\Omega))}+C\|u-u_h\|_{L^2(0,T;L^2(\Omega_U))}.\end{aligned}\tag{4.15}$$

From (4.14), (4.15), and Lemma 4.2, we only need to estimate $\|u-u_h\|_{L^2(0,T;L^2(\Omega_U))}$.

Since

$$\|u-u_h\|_{L^2(0,T;L^2(\Omega_U))}\le\|u-\pi_{h_U}u\|_{L^2(0,T;L^2(\Omega_U))}+\|\pi_{h_U}u-u_h\|_{L^2(0,T;L^2(\Omega_U))},$$

we need the estimate $\|\pi_{h_U}u-u_h\|_{L^2(0,T;L^2(\Omega_U))}$.

From (3.5), (3.22), we have

$$\begin{aligned}&\alpha\|\pi_{h_U}u-u_h\|^2_{L^2(0,T;L^2(\Omega_U))}\\&\quad=\alpha\int_0^T\big[(u,u-u_h)_U+(u_h,u_h-\pi_{h_U}u)_U+(u,\pi_{h_U}u-u)_U\big]\,dt\\&\quad\le\int_0^T\big[\big(B^*p,u_h-u\big)_U+\big(B^*p_h,\pi_{h_U}u-u_h\big)_U+\alpha(u,\pi_{h_U}u-u)_U\big]\,dt\\&\quad=\int_0^T\big[\big(B^*(p-p_h),u_h-\pi_{h_U}u\big)_U+\big(B^*p+\alpha u,\pi_{h_U}u-u\big)_U\big]\,dt.\end{aligned}\tag{4.16}$$

On the one hand, we take $w_h=p_h-p_h(u)$ in (4.10), and $q_h=y_h-y_h(u)$ in (4.11), and integrate time from 0 to T, to have

$$\begin{aligned}&\int_0^T\big(\big(B(u_h-u),p_h-p_h(u)\big)-\big(y_h-y,y_h-y_h(u)\big)\big)\,dt\\&\quad=\left(\frac{\partial}{\partial t}\big(y_h-y_h(u)\big),p_h-p_h(u)\right)\Big|_{t=0}^{t=T}-\left(y_h-y_h(u),\frac{\partial}{\partial t}\big(p_h-p_h(u)\big)\right)\Big|_{t=0}^{t=T}\\&\qquad+\int_0^T\int_0^t c\big(t,\tau;\big(y_h-y_h(u)\big)(\tau),\big(p_h-p_h(u)\big)(t)\big)\,d\tau\,dt\\&\qquad-\int_0^T\int_t^T c\big(\tau,t;\big(y_h-y_h(u)\big)(t),\big(p_h-p_h(u)\big)(\tau)\big)\,d\tau\,dt=0.\end{aligned}$$

Then

$$\begin{aligned}&\int_0^T\big(u_h-\pi_{h_U}u,B^*(p-p_h)\big)_U\,dt\\&\quad=\int_0^T\big(u_h-\pi_{h_U}u,B^*\big(p-p_h(u)\big)\big)_U\,dt+\int_0^T\big(y_h(u)-y_h,y_h-y\big)_U\,dt\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T \left(\pi_{h_U}u-u,B^*\left(p_h-p_h(u)\right)\right)_U\,dt\\
&\le \int_0^T \left(u_h-\pi_{h_U}u,B^*\left(p-p_h(u)\right)\right)_U\,dt+\int_0^T \left(y_h(u)-y_h,y_h(u)-y\right)dt\\
&\quad+\int_0^T \left(\pi_{h_U}u-u,B^*\left(p_h-p_h(u)\right)\right)_U\,dt\\
&\le C\left\{\left\|y-y_h(u)\right\|^2_{L^2(0,T;L^2(\Omega))}+\left\|p-p_h(u)\right\|^2_{L^2(0,T;L^2(\Omega))}+h_U^2\|u-\pi_{h_U}u\|^2_{L^2(0,T;L^2(\Omega_U))}\right\}\\
&\quad+\varepsilon\left\{\|u_h-\pi_{h_U}u\|^2_{L^2(0,T;L^2(\Omega_U))}+\left\|y_h-y_h(u)\right\|^2_{L^2(0,T;L^2(\Omega))}\right.\\
&\quad\left.+\left\|p_h-p_h(u)\right\|^2_{L^2(0,T;H^1(\Omega))}\right\}\\
&\le C\left\{h^4+h_U^2\|u-\pi_{h_U}u\|^2_{L^2(0,T;L^2(\Omega_U))}\right\}+\varepsilon\left\{\|u_h-\pi_{h_U}u\|^2_{L^2(0,T;L^2(\Omega_U))}\right.\\
&\quad\left.+\left\|y_h-y_h(u)\right\|^2_{L^2(0,T;L^2(\Omega))}+\left\|p_h-p_h(u)\right\|^2_{L^2(0,T;H^1(\Omega))}\right\}.
\end{aligned}
\tag{4.17}
$$

On the other hand

$$
\left(B^*p+\alpha u,\pi_{h_U}u-u\right)_U\le C\left(\left\|B^*p-\pi_{h_U}\left(B^*p\right)\right\|^2_{0,\Omega_U}+\|u-\pi_{h_U}u\|^2_{0,\Omega_U}\right).
\tag{4.18}
$$

Applying the above two estimates, from Lemma 4.2, we can get

$$
\|\pi_{h_U}u-u_h\|_{L^2(0,T;L^2(\Omega_U))}\le C\left(h_U+h^2\right).
\tag{4.19}
$$

Thus we complete the proof of Lemma 4.3. □

Then from Lemma 4.1, Lemma 4.2, and the triangle inequality, we have the following.

Theorem 4.4 *Let* (y,p,u) *and* (y_h,p_h,u_h) *be the solutions of the systems* (3.3)-(3.5) *and* (3.20)-(3.22). *Then we have the a priori error estimate*:

$$
\begin{aligned}
&\|y-y_h\|_{L^\infty(0,T;H^1(\Omega))}+\left\|\frac{\partial}{\partial t}(y-y_h)\right\|_{L^\infty(0,T;L^2(\Omega))}+\|p-p_h\|_{L^\infty(0,T;H^1(\Omega))}\\
&\quad+\left\|\frac{\partial}{\partial t}(p-p_h)\right\|_{L^\infty(0,T;L^2(\Omega))}+\|u-u_h\|_{L^2(0,T;L^2(\Omega_U))}\le C(h_U+h).
\end{aligned}
\tag{4.20}
$$

5 Numerical experiment

In this section, we carry out a numerical experiment to verify the *a priori* error estimates derived in Section 4. The numerical tests were done by using AFEpack software package (see [22]).

In the numerical example, we take $\Omega=\Omega_U=[0,1]^2$. We use linear finite element spaces to approximate the state and co-state, and the piecewise constant finite element spaces to approximate the control. For the time variable, a Euler backward-difference procedure is used to solve the discrete system. Here the time step size is controlled to demonstrate the relation between the error function and the spatial sizes.

The numerical example is the following control problem:

$$
\min_{u\ge 0}\frac{1}{2}\int_0^1\left\{\int_\Omega (y-z_d)^2+\int_\Omega u^2\right\}dt
\tag{5.1}
$$

Table 1 Numerical result: for adaptive time steps 50

			L^2-L^2	$L^\infty-H^1$		$L^\infty-L^2$	
# nodes	**# sides**	**# elements**	$u-u_h$	$y-y_h$	$p-p_h$	$\frac{\partial}{\partial t}(y-y_h)$	$\frac{\partial}{\partial t}(p-p_h)$
7,089	19,074	12,036	3.3e–01	2.1e–01	7.3e–01	2.9e–01	1.5e–01
26,163	74,256	48,144	1.6e–01	1.1e–01	3.5e–01	1.4e–01	7.1e–02
100,419	292,944	192,576	8e–02	5.2e–02	1.6e–01	7.1e–02	3.5e–02
393,363	1,163,616	770,304	4.9e–02	2.5e–02	8e–02	3.5e–02	1.7e–02

subject to

$$\begin{cases} y_{tt}-\Delta y-\int_0^t (t-\tau)\Delta y\,d\tau = f+u, & x\in\Omega, 0<t<1,\\ y|_{\partial\Omega}=0. \end{cases} \tag{5.2}$$

The solutions of (5.1)-(5.2) are

$$\begin{cases} p=-(T-t)^2\sin\pi x_1\sin\pi x_2, \quad T=1,\\ u=\max\{-p,0\},\\ y=t^2x_1(1-x_1)x_2(1-x_2),\\ z_d=y-p_{tt}+\Delta p+\int_t^T(t-\tau)\Delta p\,d\tau,\\ f=y_{tt}-\Delta y-\int_0^t(t-\tau)\Delta y\,d\tau-u. \end{cases} \tag{5.3}$$

The numerical results are put in Table 1. In Table 1, the errors in $L^\infty(0,T;H^1(\Omega))$ (L^2-L^2)-norm, $L^\infty(0,T;H^1(\Omega))$ (L^2-H^1)-norm and $L^\infty(0,T;L^2(\Omega))$ ($L^\infty-L^2$)-norm are listed.

From Table 1, we see that the L^2-norm convergent rate of the control variable $u-u_h$ is $O(h)$, *i.e.*, we have first-order accuracy with respect to the spatial size; the H^1-norm convergent rate of the state and co-state variables $y-y_h$ and $p-p_h$ also are $O(h)$; and the L^2-norm convergent rate of the state and co-state approximation errors $\frac{\partial}{\partial t}(y-y_h)$ and $\frac{\partial}{\partial t}(p-p_h)$ are $O(h)$, consistent with our theoretical analysis.

6 Conclusions

In this paper, a quadratic optimal control problem governed by a linear hyperbolic integro-differential equation and its finite element approximation are investigated for the first time. By selecting suitable state and control spaces, and defining the bilinear forms, the mathematical formulation is established. Then *a priori* estimates have been carried out using the standard functional analysis techniques, and the existence and regularity of the solution are provided by using these estimates. We then approximate the optimal control using the standard finite element method and study the approximation errors. Based on these studies, *a priori* error estimates with the optimal convergence rates are derived. Finally numerical results are presented. Through our investigation, it is clear the standard finite element method works well, both from the point of view of theory and practice, for the quadratic optimal control governed by a linear hyperbolic integro-differential equation when there is no convection term present. However, when there exists strong convection, it is very likely that very different finite element approximation schemes need to be used.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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