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Spreading-vanishing dichotomy in a degenerate logistic model with general logistic nonlinear term

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Abstract

In this paper, we study the degenerate logistic equation with a free boundary and general logistic term in higher space dimensions and heterogeneous environment, which is used to describe the spreading of a new or invasive species. We first prove the existence and uniqueness of the local solution for the free boundary problem by the contraction mapping theorem, then we show that the solution can be expanded to all time using suitable estimates. Finally, we prove the spreading-vanishing dichotomy.

Keywords: logistic equation; free boundary problem; spreading-vanishing dichotomy; invasion ecology

1 Introduction

It is an important problem to study the spreading of the invasive species in invasion ecology, which is an interesting branch of ecology. Using differential equations to study ecology becomes a main approach in ecological research. Most of the ecological phenomena such as species extinction can be explained by the nature of the differential equations. In the research of the spreading of the muskrat in Europe, Skellam observed the well-known phenomenon that many animal species spread to a new environment in a linear speed, which means the spreading radius eventually shows a linear growth speed against times [1]. Firstly, he calculated the square root of the area of the muskrat range from a map, which gives the spreading radius. Then he plotted it against times and observed the data points lay on a straight line. Several mathematical models have been proposed to discuss this phenomenon (see [2]).

The most successful mathematical model to describe the problem is the following logistic equation over the entire space $\mathbb{R}^n$:

$$u_t - d\Delta = u(a - bu), \quad t > 0, x \in \mathbb{R}^n, \tag{1}$$

where $u = u(t,x)$ stands for the population density of a spreading species, d is the diffusion rate, a is the intrinsic growth rate, a/b means the habitat carrying capacity. Fisher [3] and Kolomogorov *et al.* [4] made a pioneering contribution on this problem. They proved the problem admits traveling wave solutions of the problem (1) for space dimension $n = 1$: for any $c \geq c^* := 2\sqrt{ad}$, there is a solution $u(t,x) := W(x - ct)$ satisfying

$$W'(y) < 0 \quad \text{for } y \in \mathbb{R}, \qquad W(-\infty) = a/b, \qquad W(+\infty) = 0;$$

or there is no such solution if $c < c^*$. The constant c^* is regarded as the minimal speed of the traveling waves. Fisher claimed that the constant c^* is the spreading speed for the advantageous gene and proved it by a probabilistic argument. Then Aronson and Weinberger gave a clearer description and a rigorous proof for this phenomenon (see [5]).

Although the approach predicts the successful spreading and the establishment of a new species with a nontrivial initial population u_0, it has the obvious shortcoming that it is regardless of its initial population and the initial area, which is in sharp contrast with the real-world observations. It is a well-known conclusion that the large time behavior of a species determined by its initial size. In the real-world environment, an animal species either profits from the same species, or it will be hurt by the competition with the same species. This phenomenon is called 'Allee effect', namely there exists a critical population density such that the species can establish itself when the density is greater than the critical value, or it will die out on the other hand [6]. To include the 'Allee effect', we usually replace the logistic term in (1) by a bistable function $f_0(u)$ as follows:

$$f_0(u) = au(1-u)(u-\theta), \quad \theta \in \left(0, \frac{1}{2}\right).$$

In 2010, Du used a free boundary problem to describe the spreading of species in [7, 8]. First, Du studied the problem in one space dimension and with a homogeneous environment in [7] and then extended the conclusions to higher space dimensions and a heterogeneous environment in [8]. Considering the following free boundary problem with logistic term in the same way as the problem (1), Du proved both spreading and vanishing can happen depending on the initial size:

$$\begin{cases} u_t - d\Delta u = u(\alpha(r) - \beta(r)u), & t > 0, 0 < r < h(t), \\ u_r(t,0) = 0, \quad u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_r(t,h(t)), & t > 0, \\ h(0) = h_0, \quad u(0,r) = u_0(r), & 0 \leq r \leq h_0, \end{cases} \tag{2}$$

where $u(t,r)$, $r = |x|$, $x \in \mathbb{R}^n$ $(n \geq 2)$, $\Delta u = u_{rr} + \frac{N-1}{r}u_r$, and $r = h(t)$ is the moving boundary to be determined, and the initial function $u_0(r)$ satisfies

$$u_0 \in C^2([0,h_0]), \qquad u_0'(0) = u_0(h_0) = 0, \qquad u_0 > 0 \quad \text{in } [0,h_0]. \tag{3}$$

Problem (2) describes spreading of a new species over an n-dimensional habitat with an initial population density u_0, which occupies an initial region B_{h_0}. (Here B_R stands for the ball with the center at 0 and radius R.) The free boundary $|x| = h(t)$ stands for the spreading front, which is the boundary of the ball $B_{h(t)}$. The radius of the free boundary increases with a speed that is proportional to the population gradient at the front: $h'(t) = -\mu u_r(t,h(t))$. In the same way as (1), the coefficient function $\alpha(|x|)$ means an intrinsic growth, $\beta(|x|)$ represents an intra-specific competition, and d is the diffusion rate.

The free boundary is governed by the equation $h'(t) = -\mu u_x(t,h(t))$, which is a special case of the well-known Stefan condition. The condition has been applied in a number of problems. For example, it was used to describe the melting of ice in contact with water [9], in the modeling of oxygen in the muscle [10], and in wound healing [11].

Du has proved that the problem (2) admits a unique solution for all the $t > 0$ with $u(t,r) > 0$ and $h'(t) > 0$. Moreover, compared with the traditional logistic equation, the solution of the free boundary problem (2) is typical of the spreading-vanishing dichotomy. All this means that as $t \to \infty$ the species either successfully spreads to the entire new environment and stabilizes at a positive equilibrium (called spreading), in the case that $h(t) \to \infty$ and $u(t,x) \to a/b$, or it fails to establish itself and dies out in the long run (called vanishing), in the sense that $h(t) \to h_\infty \leq \frac{\pi}{2}\sqrt{\frac{d}{a}}$ and $u(t,x) \to 0$. The criteria for spreading or vanishing are as follows: If the radius of the initial region is greater than a critical size R^*, namely $h_0 \geq R^*$, then the spreading always occurs for all the initial function u_0 satisfying (3). On the other hand, if $h_0 < R^*$, whether spreading or vanishing happens is determined by the initial population u_0 and the coefficient μ in the Stefan condition.

Compared with the free boundary problem (2) and the problem (1), (2) is more similar to the spreading process in real world. At first, compared with the persistent spreading in the model (1), both spreading and vanishing can occur in the model (2) depending on the initial size. Next, for any finite $t > 0$, the solution $u(t,x)$ of the problem (2) is supported on a finite domain of x, which expands with the increase of t. However, in the problem (1), the solution is always positive for all the $x \in \mathbb{R}^n$ as $t > 0$.

The logistic term of the form $u(a - bu)$ has been thoroughly discussed by Du in [7, 8]. In ecology, this logistic term is too simple to describe the phenomenon in the real world. Thus, we will study a more complex logistic term as follows:

$$\begin{cases} u_t - d\Delta u = \alpha(r)u - \beta(r)f(u), & t > 0, 0 < r < h(t), \\ u_r(t,0) = 0, \quad u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_r(t,h(t)), & t > 0, \\ h(0) = h_0, \quad u(0,r) = u_0(r), & 0 \leq r \leq h_0. \end{cases} \tag{4}$$

Here, the main condition is the same as the problem (2), where $u(t,r)$, $r = |x|$, $x \in \mathbb{R}^n$ ($n \geq 2$), $\Delta u = u_{rr} + \frac{N-1}{r}u_r$, $r = h(t)$ is a moving boundary to be determined, h_0, μ and d are given positive constants, $\alpha, \beta \in C^{\nu_0}([0,\infty))$, $\nu_0 \in (0,1)$, and there exist positive constants $\kappa_1 \leq \kappa_2$ such that

$$\kappa_1 \leq \alpha(r) \leq \kappa_2, \qquad \kappa_1 \leq \beta(r) \leq \kappa_2 \quad \text{for } r \in [0,\infty). \tag{5}$$

The initial function $u_0(r)$ satisfies

$$u_0 \in C^2\big([0,h_0]\big), \qquad u_0'(0) = u_0(h_0) = 0, \qquad u_0 > 0 \quad \text{in } [0,h_0]. \tag{6}$$

Moreover, the logistic nonlinear term $f(u) \in C^1((0,+\infty])$ satisfy the conditions (A1) and (A2) listed below:

(A1) $f(s) > 0$ and $\frac{f(s)}{s}$ is increasing on $[0,+\infty)$;

(A2) $\int_1^\infty F(t)^{-1/2}\,dt < \infty$, where $F(s) = \int_0^t f(s)\,ds$.

Keller [12] and Osserman [13] proposed these conditions in 1957. These conditions have been used widely to study those functions which behave like u^q ($q > 1$). We can easily obtain $\lim_{s\to 0} f(s)/s = 0$, $\lim_{s\to +\infty} f(s)/s = +\infty$ from condition (A2). Clearly, u^q is a special case.

In Section 2, we first prove the existence and uniqueness of the local solution for the free boundary problem (4) (Theorem 2.1) by the contraction mapping theorem, then we show that the solution can be expanded to all $t > 0$ using suitable estimates (Theorem 2.3). Finally, we prove the spreading-vanishing dichotomy in Section 3.

2 Existence and uniqueness for the free boundary problem

In this section, we will prove the existence and uniqueness for the problem (4). The approaches were introduced in [7] and some changes on it are needed.

Theorem 2.1 *For any given* u_0 *satisfying* (5) *and any constant* $\nu \in (0,1)$, *there is a* $T > 0$ *such that problem* (4) *has a unique solution*

$$(u,h) \in C^{(1+\nu)/2,1+\nu}(D_T) \times C^{1+\nu/2}\big([0,T]\big);$$

moreover,

$$\|u\|_{C^{(1+\nu)/2,1+\nu}(D_T)} + \|h\|_{C^{1+\nu/2}([0,T])} \leq C, \tag{7}$$

where $D_T = \{(t,r) \in \mathbb{R}^2 : y \in [0,T], r \in [0,h(t)]\}$, *C and T only depend on* h_0, ν *and* $\|u_0\|_{C^2([0,h_0])}$.

Proof At first, we follow [8] and [11] to straighten the free boundary. Then the problem (4) becomes

$$\begin{cases} w_t - Adw_{ss} - (Bd + h'C + Dd)w_s = \widetilde{\alpha} w - \widetilde{\beta} f(w), & t > 0, 0 < s < h_0, \\ w = 0, \qquad h'(t) = -\mu w_s, & t > 0, s = h_0, \\ w_s(t,0) = 0, & t > 0, \\ h(0) = h_0, \qquad w(0,s)u_0(s), & 0 \leq s \leq h_0, \end{cases} \tag{8}$$

where $A = A(h(t),s)$, $B = B(h(t),s)$, $C = C(h(t),s)$, $D = D(h(t),s)$, $\widetilde{\alpha} = \widetilde{\alpha}(h(t),s)$, $\widetilde{\beta} = \widetilde{\beta}(h(t),s)$.

We denote $\widetilde{h}_0 = -\mu u_0'(h_0)$ and $\Delta_T = [0,T] \times [0,h_0]$ for $0 < T \leq \frac{h_0}{8(1+h_0)}$,

$$\mathcal{D}_{1T} = \big\{w \in C(\Delta_T) : w(0,s) = u_0(s), \|w - u_0\|_{C(\Delta_T)} \leq 1\big\}$$

and

$$\mathcal{D}_{2T} = \big\{h \in C^1\big([0,T]\big) : h(0) = h_0, h'(0) = \widetilde{h}_0, \big\|h' - \widetilde{h}_0\big\|_{C([0,T])} \leq 1\big\}.$$

It is easily seen that $\mathcal{D} := \mathcal{D}_{1T} \times \mathcal{D}_{2T}$ is a complete metric space with the following metric:

$$d\big((w_1,h_1),(w_2,h_2)\big) = \|w_1 - w_2\|_{C(\Delta_T)} + \big\|h_1' - h_2'\big\|_{C[0,T]}.$$

For $h_1, h_2 \in \mathcal{D}_{2T}$, due to $h_1(0) = h_2(0) = h_0$, we have

$$\|h_1 - h_2\|_{C([0,T])} \le T\|h_1' - h_2'\|_{C([0,T])}. \tag{9}$$

Next, we use the contraction mapping theorem to prove the existence and uniqueness of the local solution. Firstly, for any given $(w, h) \in \mathcal{D}_{1T} \times \mathcal{D}_{2T}$, we have

$$|h(t) - h_0| \le T(1 + \widetilde{h}_0) \le \frac{h_0}{8}.$$

Thus, for $0 \le s \le h_0/2$, we have $\zeta(s) \equiv 0$ and for such s,

$$A \equiv 1, \qquad B \equiv C \equiv 0, \qquad D \equiv (N-1)/s;$$

therefore

$$-Adw_s s - (Bd + h'C + Dd)w_s = -d\Delta w \quad \text{in the ball } |y| \le \frac{h_0}{2}.$$

So, although $D = D(h(t), s)$ is singular at $s = 0$,

$$Adw_{ss} + (Bd + h'C + Dd)w_s$$

still represents an elliptic operator acting on $w = w(t, y)$ $(= w(t, |y|))$ over the ball $|y| \le h_0$, whose coefficients are continuous in (t, y) when $h \in \mathcal{D}_{2T}$.

Applying L^p theory and the Sobolev imbedding theorem [14], we find that the following initial boundary value problem:

$$\begin{cases} \overline{w}_t - Ad\overline{w}_{ss} - (Bd + h'C + Dd)\overline{w}_s = \widetilde{\alpha}w - \widetilde{\beta}f(w), & t > 0, 0 \le s < h_0, \\ \overline{w}_s(t, 0) = 0, \qquad \overline{w}(t, h_0) = 0, & t > 0, \\ \overline{w}(0, s) = u_0(s), & 0 \le s \le h_0 \end{cases} \tag{10}$$

has a unique solution $\overline{w} \in C^{(1+\nu)/2, 1+\nu}(\Delta_T)$ for any $(w, h) \in \mathcal{D}$ and

$$\|\overline{w}\|_{C^{(1+\nu)/2,1+\nu}(\Delta_T)} \le C_1, \tag{11}$$

where C_1 is a constant dependent on h_0, ν and $\|u_0\|_{C^2[0,h_0]}$.

Let

$$\overline{h}(t) := h_0 - \int_0^t \mu\overline{w}_s(\tau, h_0)\, d\tau, \tag{12}$$

we have

$$\overline{h}'(t) = -\mu\overline{w}_s(t, h_0), \qquad \overline{h}(0) = h_0, \qquad \overline{h}'(0) = -\mu\overline{w}_s(0, h_0) = \widetilde{h}_0,$$

and $\overline{h}' \in C^{\nu/2}([0, T])$ with

$$\|\overline{h}'\|_{C^{\nu/2}([0,T])} \le C_2 := \mu C_1. \tag{13}$$

Now, we define $\mathcal{F}:\mathcal{D}\to C(\Delta_T)\times C_1([0,T])$,

$$\mathcal{F}(w,h)=(\overline{w},\overline{h}).$$

Clearly $(w,h)\in\mathcal{D}$ is a solution of (8) if and only if it is the fixed point of $\mathcal{F}$.

Due to (11) and (13), we have

$$\begin{aligned}&\|\overline{h}'-\widetilde{h}_0\|_{C([0,T])}\le\|\overline{h}'\|_{C^{\nu/2}([0,T])}T^{\nu/2}\le\mu C_1T^{\nu/2},\\&\|\overline{w}-u_0\|_{C(\Delta_T)}\le\|\overline{w}-u_0\|_{C^{(1+\nu)/2,0}(\Delta_T)}T^{(1+\nu)/2}\le C_1T^{(1+\nu)/2}.\end{aligned}$$

Thus, if we let $T\le\min\{(\mu C_1)^{-2/\nu},C_1^{-2/(1+\nu)}\}$, $\mathcal{F}$ maps $\mathcal{D}$ into itself.

Next, we will prove that if $T>0$ is sufficiently small, $\mathcal{F}$ is a contraction mapping on $\mathcal{D}$. In fact, if we let $(w_i,h_i)\in\mathcal{D}(1,2)$ and denote $(\overline{w}_i,\overline{h}_i)=\mathcal{F}(w_i,h_i)$, we obtain

$$\|\overline{w}_i\|_{C^{(1+\nu)/2,1+\nu}(\Delta_T)}\le C_1,\qquad\|\overline{h}_i'(t)\|_{C^{\nu/2}([0,T])}\le C_2$$

by (11) and (13).

Setting $W=\overline{w}_1-\overline{w}_2$, then $W(t,s)$ satisfies

$$\begin{aligned}&W_t-A(h_2,s)dW_{ss}-\big(B(h_2,s)d+h'C(h_2,s)+D(h_2,s)d\big)W_s\\&\quad=\big(A(h_1,s)-A(h_2,s)\big)d\overline{w}_{1,ss}+\big(B(h_1,s)-B(h_2,s)+D(h_1,s)-D(h_2,s)\big)d\overline{w}_{1,s}\\&\qquad+\big(h_1'C(h_1,s)-h_2'C(h_2,s)\big)\overline{w}_{1,s}+(w_1-w_2)\left(\widetilde{\alpha}(h_1,s)-\widetilde{\beta}(h_1,s)\left(\frac{f(w_1)-f(w_2)}{w_1-w_2}\right)\right)\\&\qquad+w_2\left(\big(\widetilde{\alpha}(h_1,s)-\widetilde{\alpha}(h_2,s)\big)-\big(\widetilde{\beta}(h_1,s)-\widetilde{\beta}(h_2,s)\big)\frac{f(w_2)}{w_2}\right),\quad t>0,0\le s<h_0,\\&W_s(t,0)=0,\qquad W(t,h_0),\quad t>0,\\&W(0,s)=0,\quad 0\le s\le h_0.\end{aligned}$$

Using the L^p estimates and the Sobolev imbedding theorem, we have

$$\|\overline{w}_1-\overline{w}_2\|_{C^{(1+\nu)/2,1+\nu}(\Delta_T)}\le C_3\big(\|w_1-w_2\|_{C(\Delta_T)}+\|h_1-h_2\|_{C^1([0,T])}\big),\tag{14}$$

where C_3 depends on C_1, C_2 and A, B, C, D. Taking the difference of the equations for $\overline{h}_1$, $\overline{h}_2$ results in

$$\|\overline{h}_1'-\overline{h}_2'\|_{C^{\nu/2}([0,T])}\le\mu\big(\|\overline{w}_{1,s}-\overline{w}_{2,s}\|_{C^{\nu/2,0}(\Delta_T)}\big).\tag{15}$$

Combining (9), (14), and (15), and assuming $T\le 1$, we get

$$\|\overline{w}_1-\overline{w}_2\|_{C^{(1+\nu)/2,1+\nu}(\Delta_T)}+\|\overline{h}_1'-\overline{h}_2'\|_{C^{\nu/2}([0,T])}\le C_4\big(\|w_1-w_2\|_{C(\Delta_T)}+\|h_1'-h_2'\|_{C^1([0,T])}\big),$$

where C_4 depends on C_3 and μ. Therefore for

$$T:=\min\left\{1,\left(\frac{1}{2C_4}\right)^{2/\nu},(\mu C_1)^{-2/\nu},C_1^{-2/(1+\nu)},\frac{h_0}{8(1+\widetilde{h}_0)}\right\},$$

we have

$$\begin{aligned}
&\|\overline{w}_1 - \overline{w}_2\|_{C(\Delta_T)} + \|\overline{h}_1' - \overline{h}_2'\|_{C([0,T])} \\
&\quad \le T^{(1+\nu)/2}\|\overline{w}_1 - \overline{w}_2\|_{C^{(1+\nu)/2,1+\nu}(\Delta_T)} + T^{\nu/2}\|\overline{h}_1' - \overline{h}_2'\|_{C^{\nu/2}([0,T])} \\
&\quad \le C_4 T^{\nu/2}\big(\|w_1 - w_2\|_{C(\Delta_T)} + \|h_1' - h_2'\|_{C([0,T])}\big) \\
&\quad \le \frac{1}{2}\big(\|w_1 - w_2\|_{C(\Delta_T)} + \|h_1' - h_2'\|_{C([0,T])}\big).
\end{aligned}$$

This means that $\mathcal{F}$ is a contraction mapping on $\mathcal{D}$. By the contraction mapping theorem, we find that $\mathcal{F}$ has a unique fixed point (w,h) in $\mathcal{D}$. Moreover, it follows that we have the Schauder estimates $h \in C^{1+\nu/2}(0,T]$ and $w \in C^{1+\nu/2,2+\nu}((0,T] \times [0,h_0])$. Moreover, we have (11) and (13). This shows that $(w(t,s),h(t))$ is a unique local classical solution of the problem (8). □

Next, we will use some suitable estimates to show that the solution can be extended to all $t > 0$.

Lemma 2.2 *Let* (u,h) *be the solution to the problem* (2) *defined on* $t \in (0,T_0]$ *for some* $T_0 \in (0,+\infty]$. *Then there exist constants* C_1 *and* C_2 *independent of* T_0 *such that*

$$0 < u(t,r) \le C_1, \qquad 0 < h'(t) \le C_2 \quad \textit{for } 0 \le r < h(t), t \in (0,T_0).$$

Proof By the strong maximum principle, we have

$$u(t,r) > 0, \qquad u_r\big(t,h(t)\big) < 0 \quad \text{for } 0 < t < T_0, 0 \le r < h(t).$$

Thus $t \in (0,T_0)$ for $h'(t) > 0$.

Due to (5), using the comparison principle, we have $u(t,r) \le \overline{u}(t)$ for $t \in (0,T_0)$, $r \in [0,h(t)]$, where $\overline{u}(t)$ is the solution of following problem:

$$\frac{d\overline{u}}{dt} = \kappa_2\overline{u} - \kappa_1 f(\overline{u}), \quad t > 0; \qquad \overline{u}(0) = \|u_0\|_\infty. \tag{16}$$

By the condition (A2), we easily obtain $\lim_{s\to 0} f(s)/s = 0$, $\lim_{s\to+\infty} f(s)/s = +\infty$. Thus, there exists an s^* such that $f(s^*)/s^* = \kappa_2/\kappa_1$. Clearly, the s^* is the supremum of the problem (16). Therefore, we have

$$u(t,r) \le C_1 := \sup_{t\ge 0}\overline{u}(t).$$

Next, using the approach in [8], it is easy to prove that $h'(t) \le C_2$ for $t \in (0,T_0)$, where C_2 is independent of T_0. Then the proof is complete. □

Theorem 2.3 *The solution of the problem* (4) *exists and is unique for all* $t \in (0,\infty)$. *Moreover, the unique solution* (u,h) *depends continuously on* u_0 *and the parameters appearing in* (4).

The proof is the same as Theorem 4.3 in [8]. So we omit the details.

3 Spreading-vanishing dichotomy

By Lemma 2.2, we see that $r = h(t)$ is monotonic and therefore there exists $h_\infty \in (0, +\infty]$ such that $\lim_{t\to+\infty} h(t) = h_\infty$. Let $\lambda_1(d, \alpha, R)$ be the principal eigenvalue of the problem

$$\begin{cases} -d\Delta\phi = \lambda\alpha(|x|)\phi & \text{in } B_R, \\ \phi = 0 & \text{on } \partial B_R. \end{cases} \tag{17}$$

It is well known that $\lambda(d, \alpha, \cdot)$ is a strictly decreasing continuous function and

$$\lim_{R\to 0^+} \lambda_1(d, \alpha, R) = +\infty, \qquad \lim_{R\to+\infty} \lambda_1(d, \alpha, R) = 0.$$

Thus, for fixed $d > 0$ and $\alpha \in C^{\nu_0}([0, \infty))$, there is a unique $R^* := R^*(d, \alpha)$ such that

$$\lambda_1\bigl(d, \alpha, R^*\bigr) = 1 \tag{18}$$

and

$$1 > \lambda_1(d, \alpha, R) \quad \text{for } R > R^*; \qquad 1 < \lambda_1(d, \alpha, R) \quad \text{for } R < R^*.$$

By the following two lemmas, we can obtain the spreading-vanishing dichotomy.

Lemma 3.1 *If $h_\infty < +\infty$, then $h_\infty \le R^*$, and* $\lim_{t\to+\infty} \|u(t, \cdot)\|_{C([0,h(t)])} = 0$.

Proof We first prove $h_\infty \le R^*$. Arguing by contradiction, we suppose $h_\infty > R^*$ and there is a $T > 0$ such that $h(t) > R^*$ for all the $t \ge T$. Therefore, for all $t \ge T$, we have

$$1 > \lambda_1\bigl(d, \alpha, h(t)\bigr).$$

Moreover, for any sufficiently small $\varepsilon > 0$, there exists a $T_0 := T_0(\varepsilon) > T$ such that

$$R^* < h_\infty - \varepsilon < h(t) < h_\infty \quad \text{for } t \ge T_0.$$

Consider the following problem:

$$\begin{cases} w_t - d\Delta w = \alpha(r)w - \beta(r)f(w), & t \ge T_0, r \in [0, h_\infty - \varepsilon], \\ w_r(t, 0) = 0, \qquad w(t, h_\infty - \varepsilon) = 0, & t \ge T_0, \\ w(T_0, r) = u(T_0, r), & r \in [0, h_\infty - \varepsilon], \end{cases} \tag{19}$$

which is a logistic problem $\lambda_1(d, \alpha, h_\infty - \varepsilon) \le 1$. Clearly, the problem (19) has a unique positive solution $\underline{w} = \underline{w}_\varepsilon(t, r)$ (see Proposition 3.3 in [10]). We have

$$\underline{w}(t, \cdot) \to V_{h_\infty-\varepsilon} \quad \text{in } C^2\bigl([0, h_\infty - \varepsilon]\bigr) \text{ as } t \to \infty, \tag{20}$$

where $V_{h_\infty-\varepsilon}(r)$ is the unique positive solution of the following problem:

$$\begin{cases} -d\Delta V = \alpha(r)V - \beta(r)f(V) & \text{in } B_{h_\infty-\varepsilon}, \\ V = 0 & \text{on } \partial B_{h_\infty-\varepsilon}. \end{cases} \tag{21}$$

Using the comparison principle

$$u(t,r) \geq \underline{w}(r,t) \quad \text{for } t > T_0, r \in [0, h_\infty - \varepsilon], \tag{22}$$

which implies that

$$\underline{\lim}_{t\to+\infty} u(t,r) \geq V_{h_\infty-\varepsilon}(r), \quad r \in [0, h_\infty - \varepsilon]. \tag{23}$$

On the other hand, consider the problem

$$\begin{cases} w_t - d\Delta w = \alpha(r)w - \beta(r)f(w), & t \geq T_0, r \in [0, h_\infty], \\ w_r(t,0) = 0, \qquad w(t,h_\infty) = 0, & t \geq T_0, \\ w(T_0,r) = \widetilde{u}(T_0,r), & r \in [0, h_\infty], \end{cases} \tag{24}$$

where

$$\widetilde{u}(T_0,r) = \begin{cases} u(T_0,r) & \text{for } r \in [0, h(T_0)], \\ 0 & \text{for } r \in (h(T_0), h_\infty]. \end{cases}$$

Clearly, the problem (24) also has a unique positive solution

$$\overline{w}(t,\cdot) \to V_{h_\infty} \quad \text{in } C^2\big([0,h_\infty]\big) \text{ as } t \to +\infty, \tag{25}$$

where V_{h_∞} is a unique positive solution of the following problem:

$$\begin{cases} -d\Delta V = \alpha(r)V - \beta(r)f(V) & \text{in } B_{h_\infty}, \\ V = 0 & \text{on } \partial B_{h_\infty}. \end{cases} \tag{26}$$

In the same way, the comparison principle implies that

$$u(t,r) \leq \widetilde{w}(t,r) \quad \text{for } r \in \big[0, h(t)\big] \tag{27}$$

and

$$\overline{\lim}_{t\to+\infty} \leq V_{h_\infty}(r) \quad \text{for } r \in [0, h_\infty]. \tag{28}$$

Using a compactness and uniqueness argument, we can easily obtain

$$V_{h_\infty} \to V_{h_\infty} \quad \text{in } C^2_{\text{loc}}\big([0,h_\infty]\big) \text{ as } \varepsilon \to 0^+.$$

Therefore, it follows from (23), (28) and the arbitrariness of ε that

$$\lim_{t\to\infty} u(t,r) = V_{h_\infty}(r) \quad \text{for } r \in [0, h_\infty). \tag{29}$$

By the argument of Lemma 2.2 in [8], we obtain

$$\big\| u(t,\cdot) - V_{h_\infty} \big\|_{C^2([0,h(t)])} \to 0 \quad \text{as } t \to \infty.$$

Thus

$$u_r(t,h(t)) \to V'_{h_\infty}(h_\infty) < 0 \quad \text{for } t \to \infty,$$

which implies that

$$h'(t) = -\mu u_r(t,h(t)) \to -\mu V'_{h_\infty}(h_\infty) > 0 \quad \text{as } t \to \infty.$$

Hence $h_\infty = \infty$, which contradicts our assumption $h_\infty < \infty$. So we have $h_\infty \leq R^*$.

Next, we will prove that $\|u(t,\cdot)\|_{C([0,h(t)])} \to 0$ as $t \to \infty$. Let $\overline{u}(t,r)$ be the unique positive solution of the following problem:

$$\begin{cases} \overline{u}_t - d\Delta\overline{u} = \alpha(r)\overline{u} - \beta(r)f(\overline{u}), & t > 0, 0 < r < h_\infty, \\ \overline{u}_r(t,0) = 0, \qquad \overline{u}(t,h_\infty) = 0, & t > 0, \\ \overline{u}(0,r) = \widetilde{u}_0(r), & 0 < r < h_\infty, \end{cases} \tag{30}$$

where

$$\widetilde{u}_0(r) = \begin{cases} u_0(r), & 0 \leq r \leq h_0, \\ 0, & r \geq h_0. \end{cases}$$

The comparison principle implies $0 \leq u(t,r) \leq \overline{u}(t,r)$ for $t > 0$ and $r \in [0,h(t)]$. Due to $h_\infty < R^*$, we have $1 \leq \lambda_1(d,\alpha,h_\infty)$ and it follows from a well-known conclusion about logistic equation that $\overline{u}(t,r) \to 0$ uniformly for $r \in [0,h_\infty]$ as $t \to +\infty$ (see [10]). Therefore, we get $\lim_{t\to+\infty}\|u(t,\cdot)\|_{C([0,h(t)])} = 0$. □

Lemma 3.2 *If* $h_\infty = +\infty$, *then*

$$\lim_{t\to+\infty} u(t,r) = \widehat{U}(r) \quad \textit{locally uniformly for } r \in [0,+\infty), \tag{31}$$

where $\widehat{U}(|x|)$ *is the unique positive solution of the following problem*:

$$-d\Delta u = \alpha(|x|)u - \beta(|x|)f(u), \quad x \in \mathbb{R}^n. \tag{32}$$

Proof By [15], we find that the problem (32) has a unique positive solution. Moreover, the solution must be radially symmetric since (32) is invariant under rotations around the origin of $\mathbb{R}^n$.

To prove (31), we use a squeezing argument in [16]. Consider the Dirichlet problem

$$-d\Delta v = \alpha(r)v - \beta(r)f(v), \qquad v(R) = 0,$$

and the boundary blow-up problem

$$-d\Delta w = \alpha(r)w - \beta(r)f(w), \qquad w(R) = +\infty.$$

Clearly, these problems have positive radial solutions v_R and w_R for large R. It follows from the comparison principle in [15] that v_R increases to the unique positive solution $\widehat{U}$ of (32) as $R \to +\infty$ and w_R decreases to $\widehat{U}$.

Choose an increasing sequence R_k such that $R_k \to +\infty$ as $k \to \infty$, and $1 > \lambda_1(d, \alpha, R_k)$ for all k. Then both v_{R_k} and w_{R_k} converge to $\widehat{U}$ as $k \to \infty$. For each k, we can find a $T_k > 0$ such that $h(t) \geq R_k$ for $t \geq T_k$. Note that the following problem:

$$\begin{cases} w_t - d\Delta w = \alpha(r)w - \beta(r)f(w), & t \geq T_k, r \in [0, R_k], \\ w_r(t,0) = 0, \qquad w(t, R_k) = 0, & t \geq T_k, \\ w(T_k, r) = u(T_k, r), & r \in [0, R_k] \end{cases} \tag{33}$$

has a unique positive solution $w_k(t,r)$, and

$$w_k(t,r) \to v_{R_k}(r) \quad \text{uniformly for } r \in [0, R_k] \text{ as } t \to +\infty. \tag{34}$$

Using the comparison principle, we have

$$w_k(t,r) \leq u(t,r) \quad \text{for } t \geq T_k \text{ and } r \in [0, R_k].$$

Thus

$$\varliminf_{t \to +\infty} u(t,r) \geq v_{R_k}(r) \quad \text{uniformly in } r \in [0, R_k].$$

Let $k \to \infty$, we have

$$\varliminf_{t \to +\infty} u(t,r) \geq \widehat{U}(r) \quad \text{locally uniformly for } r \in [0, +\infty]. \tag{35}$$

Similarly, by the proof of Theorem 4.1 in [16], we obtain

$$\varlimsup_{t \to +\infty} u(t,r) \leq w_{R_k}(r) \quad \text{uniformly for } r \in [0, R_k].$$

Let $k \to \infty$, we have

$$\varlimsup_{t \to +\infty} u(t,r) \leq \widehat{U}(r) \quad \text{locally uniformly for } r \in [0, +\infty]. \tag{36}$$

Therefore, (31) follows from (35) and (36). □

Combining Lemmas 3.1 and 3.2, we can easily obtain the spreading-vanishing dichotomy as follows.

Theorem 3.3 *Let* $(u(t,r), h(r))$ *be the solution of the free boundary problem* (2). *Then the following alternative holds*:

1. *Spreading*: $h_\infty = +\infty$ *and*

$$\lim_{t \to \infty} u(t,r) = \widehat{U}(r) \quad \textit{locally uniformly for } r \in [0, \infty).$$

2. *Vanishing*: $h_\infty \leq R^*$ *and* $\lim_{t \to \infty} \|u(t, \cdot)\|_{C([0,h(t)])} = 0$.

Next, we will discuss when the two alternatives occur exactly. We divide the argument into two cases:

$$\text{(a)} \quad h_0 \geq R^*, \qquad \text{(b)} \quad h_0 < R^*.$$

In case (a), we can easily obtain $h_\infty > R^*$ since $h'(t) > 0$ for all $t > 0$. Therefore, the following conclusion follows from Lemma 3.1.

Theorem 3.4 *If $h_0 \geq R^*$, then $h_\infty = +\infty$.*

In the same way as in the discussion in [8], we need a comparison principle which can be used to estimate both $u(t,r)$ and the free boundary $r = h(t)$ to study case (b).

Lemma 3.5 (Comparison principle) *Suppose $T \in (0,\infty)$, $\overline{h} \in C^1([0,T])$, $\overline{u} \in C^{1,2}(D_T^*)$ with $D_T^* = \{(t,r) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq r \leq \overline{h}(t)\}$, and*

$$\begin{cases} \overline{u}_t - d\Delta\overline{u} \geq \alpha(r)\overline{u} - \beta(r)f(\overline{u}), & 0 < t \leq T, 0 < r < \overline{h}(t), \\ \overline{u} = 0, \qquad \overline{h}'(t) \geq -\mu\overline{u}_r, & 0 < t \leq T, r = \overline{h}(t), \\ \overline{u}_r(t,0) \leq 0, & 0 < t \leq T. \end{cases}$$

If we have

$$h_0 \leq \overline{h}_t \in (0,T], \qquad u_0(r) \leq \overline{u}(0,r) \in [0,h_0],$$

then the solution (u,h) of the free boundary problem (4) *satisfies*

$$h(t) \leq \overline{h}(t) \in (0,T], \qquad u(r,t) \leq \overline{u}(r,t) \quad \text{for } t \in (0,T] \text{ and } r \in \big(0,h(t)\big).$$

Proof For small $\varepsilon > 0$, in the problem (4), let $h_0^\varepsilon := h_0(1-\varepsilon)$ replace $h_0 := h(0)$, $\mu_\varepsilon := \mu(1-\varepsilon)$ replace μ, and $u_0^\varepsilon(r)$ replace u_0, where $u_0^\varepsilon \in C^2([0,h_0^\varepsilon])$ satisfying

$$0 < u_0^\varepsilon(r) \leq u(0,r) \quad \text{in } \big[0,h_0^\varepsilon\big], \qquad u_0^\varepsilon\big(h_0^\varepsilon\big) = 0$$

and

$$u_0^\varepsilon\left(\frac{h_0}{h_0^\varepsilon}r\right) \to u(0,r) \quad \text{in } C^2\big([0,h_0]\big) \text{ as } \varepsilon \to 0.$$

We denote by $(u_\varepsilon,h_\varepsilon)$ the unique solution of the above problem.

We claim that $h_\varepsilon(t) < \overline{h}(t)$ for all $t \in (0,T]$. Clearly, it is true for small $t > 0$. If our claim is wrong, then we will find a first $t^* \leq T$ such that $h_\varepsilon(t) < \overline{h}(t)$ for $t \in (0,t^*)$ and $h_\varepsilon(t^*) = \overline{h}(t^*)$, which implies

$$h_\varepsilon'\big(t^*\big) \geq \overline{h}'\big(t^*\big). \tag{37}$$

Now, we compare u_ε and $\overline{u}$ over the region $\Omega_{t^*} := \{(t,r) \in \mathbb{R}^2; 0 < t \leq t^*, 0 \leq r < h_\varepsilon(t)\}$. Using the strong maximum principle in Ω_{t^*}, we have $u_\varepsilon(t,r) < \overline{u}(t,r)$. Thus $w(t,r) := \overline{u}(t,r) - u_\varepsilon(t,r) > 0$ with $w(t^*,h_\varepsilon(t^*)) = 0$. It follows that $w_r(t^*,h_\varepsilon(t^*)) \leq 0$. Then we obtain $(u_\varepsilon)_r(t^*,h(t^*))$. Due to $\mu_\varepsilon < \mu$, it follows that $h_\varepsilon'(t^*) < h'(t^*)$, which contradicts (37). This shows our claim is correct. Then applying the usual comparison principle for Ω_T, we obtain $u_\varepsilon < \overline{u}$.

Due to the unique solution $(u_\varepsilon,h_\varepsilon)$ depending continuously on the parameters in (4), as $\varepsilon \to 0$, $(u_\varepsilon,h_\varepsilon)$ converges to the unique solution (u,h) of the problem (4). Setting $\varepsilon \to 0$, we can get the conclusion. □

Now, we consider case (b). As in [8], we first examine the case that μ is large, then we investigate the case $\mu > 0$ is small. Finally, we use Lemma 3.5 to show that there exists a critical μ^* such that spreading occurs when $\mu > \mu^*$ and vanishing happens when $\mu \in (0, \mu^*]$.

Lemma 3.6 *Suppose $h_0 \leq R^*$, then there exists $\mu^0 > 0$ depending on u_0 such that spreading occurs when $\mu > \mu^0$.*

Proof We prove it by contradiction. Suppose that there is an increasing sequence μ_k satisfying $\mu_k \to +\infty$ as $k \to \infty$ such that the unique solution (u^k, h^k) of the problem (4) with $\mu = \mu_k$ satisfies $h^k_\infty := \lim_{t\to\infty} h^k(t) < +\infty$ for all the k. Thus, by Lemma 3.1, we have $h^k_\infty \leq R^*$ and hence

$$u^k(t,r) \leq w^*(t,r) \quad \text{for } t > 0, r \in \bigl[0, h^k(t)\bigr], \tag{38}$$

where $w^*(t,r)$ is the unique solution of following problem:

$$\begin{cases} w_t - d\Delta w = \alpha(r)w - \beta(r)f(w), & t > 0, r \in [0, R^*], \\ w_r(t,0) = 0, \qquad w(t,R^*) = 0, & t > 0, \\ w(0,r) = \widehat{u}_0(t), & r \in [0, R^*], \end{cases}$$

where

$$\widehat{u}_0(t) = \begin{cases} u_0(t), & r \in [0, h_0], \\ 0, & r \in (h_0, R^*]. \end{cases}$$

Due to the fact that $1 = \lambda_1(d, \alpha, R^*)$, we have

$$\lim_{t\to+\infty} \bigl\| w^*(t,\cdot) \bigr\|_{C([0,R^*])} \to 0. \tag{39}$$

Combining (38) and (39), we obtain $\lim_{t\to\infty} u^k(t,\cdot) \to 0$ uniformly for k. Therefore, it follows from conditions (A1) and (A2) that there exists a $T > 0$ independent of k such that

$$\frac{f(u^k)}{u^k} \leq \frac{\kappa_1}{\kappa_2} \quad \text{for } t > T \text{ and } r \in \bigl[0, h^k(t)\bigr].$$

To simplify the discussion, we omit k from u^k, h^k, h^k_∞, and μ_k in the following argument.

One calculates directly

$$\begin{aligned} \frac{d}{dt}\int_0^{h(t)} r^{n-1}u(t,r)\,dr &= \int_0^{h(t)} r^{n-1}u_t(t,r)\,dr + h^{n-1}(t)h'(t)u\bigl(t,h(t)\bigr) \\ &= d\int_0^{h(t)} r^{n-1}\Delta u\,dr + \int_0^{h(t)} \alpha(r)u - \beta(r)f(u)r^{n-1}\,dr \\ &= d\int_0^{h(t)} \bigl(r^{n-1}u_r(r)\bigr)_r\,dr + \int_0^{h(t)} \alpha(r)u - \beta(r)f(u)r^{n-1}\,dr \\ &= -\frac{d}{\mu}h^{n-1}(t)h'(t) + \int_0^{h(t)} \alpha(r)u - \beta(r)f(u)r^{n-1}\,dr. \end{aligned}$$

Integrating from T to $t > T$ implies

$$\int_0^{h(t)} r^{n-1}u(t,r)\,dx = \int_0^{h(T)} r^{n-1}u(T,r)\,dr + \frac{d}{n\mu}\bigl(h(T)^n - h(t)^n\bigr) + \int_T^t \int_0^{h(s)} \alpha(r)u - \beta(r)f(u)r^{n-1}\,dr\,ds.$$

For $t \geq T$ and $r \in [0, h(t)]$, due to $0 < \frac{f(u)}{u(t,r)} \leq \frac{\kappa_1}{\kappa_2}$, we have

$$\alpha(r) - \beta(r)\frac{f(u)}{u(t,r)} \geq \kappa_1 - \kappa_2\frac{f(u)}{u(t,r)} \geq 0.$$

Then

$$\int_0^{h(t)} r^{n-1}u(t,r)\,dx \geq \frac{d}{n\mu}\bigl(h(T)^n - h(t)^n\bigr) + \int_0^{h(T)} r^{n-1}u(T,r)\,dr.$$

Let $t \to +\infty$, since (38) and (39), we obtain

$$\frac{d}{n\mu}\bigl(h(T)^n - h_\infty^n\bigr) + \int_0^{h(T)} r^{n-1}u(t,r)\,dr \leq 0$$

and hence

$$\mu \leq \frac{d[(R^*)^n - h(T)^n]}{n\int_0^{h(T)} r^{n-1}u(T,r)\,dr}. \tag{40}$$

Using Lemma 3.5, $u^k(t,x)$ and $h^k(t)$ are increasing in k. Thus

$$u^k(t,x) \geq u^1(t,x) \quad \text{and} \quad h^k(t) \geq h^1(t).$$

Thus, from (40) we deduce

$$\mu \leq \frac{d[(R^*)^n - h^1(T)^n]}{n\int_0^{h^1(T)} r^{n-1}u^1(T,r)\,dr} < +\infty.$$

This contradicts $\mu_k \to +\infty$ as $k \to \infty$. □

Lemma 3.7 *Suppose $h_0 < R^*$, then there exists $\mu_0 > 0$ depending on u_0 such that vanishing occurs when $\mu \leq \mu_0$.*

Proof At first, we will construct a suitable upper solution of the problem (2), then we use Lemma 3.5 to obtain the conclusion. For $t > 0$ and $r \in [0, \sigma(t)]$, define

$$\sigma(t) = h_0\left(1 + \delta - \frac{\delta}{2}e^{-\gamma t}\right), \qquad w(t,r) = Me^{-\gamma t}V\left(\frac{h_0}{\sigma(t)}r\right),$$

where M, δ, γ are positive constants to be chosen later and $V(|x|)$ is the first eigenfunction of the following problem:

$$\begin{cases} -d\Delta V = \lambda_1(d,\alpha,h_0)\alpha(|x|)V & \text{in } B_{h_0}, \\ V = 0 & \text{on } \partial B_{h_0} \end{cases}$$

with $V \geq 0$ and $\|V\|_\infty = 1$. Due to $h_0 < R^*$, we have

$$1 < \lambda_1(d, \alpha, h_0).$$

Since we have the fact that $V'(0) = 0$ and

$$-d\big(r^{n-1}V'\big)' = r^{n-1}\lambda_1(d, \alpha, h_0)\alpha(r)V > 0 \quad \text{for } 0 < r < h_0,$$

we have

$$V'(r) < 0 \quad \text{for } 0 < r \leq h_0.$$

Let $\tau(t) = 1 + \delta - \frac{\delta}{2}e^{-\gamma t}$, then $\sigma(t) = h_0\tau(t)$. Direct calculation gives

$$\begin{aligned}
&w_t - d\Delta w - \alpha(r)w - \beta(r)f(w) \\
&\quad = Me^{-\gamma t}\bigg(-\gamma V - r\tau^{-2}\tau'(t)V' - dt^{-2}V'' \\
&\qquad - d(N-1)r^{-1}\tau^{-1}V' - V\bigg(\alpha(r) - \beta(r)\frac{f(Me^{-\gamma t}V)}{Me^{-\gamma t}V}\bigg)\bigg) \\
&\quad = Me^{-\gamma t}\bigg(-\gamma V - r\tau^{-2}\tau'(t)V' + \tau^{-2}\lambda_1(d, \alpha, h_0)\alpha\bigg(\frac{r}{\tau}\bigg)V \\
&\qquad - V\bigg(\alpha(r) - \beta(r)\frac{f(Me^{-\gamma t}V)}{Me^{-\gamma t}V}\bigg)\bigg) \\
&\quad \geq Me^{-\gamma t}V\bigg(-\gamma + \tau^{-2}\lambda_1(d, \alpha, h_0)\alpha\bigg(\frac{r}{\tau}\bigg) - \alpha(r) + \beta(r)\frac{f(Me^{-\gamma t}V)}{Me^{-\gamma t}V}\bigg) \\
&\quad \geq Me^{-\gamma t}V\bigg(-\gamma + \frac{\lambda_1(d, \alpha, h_0)}{(1+\delta)^2}\alpha\bigg(\frac{r}{\tau}\bigg) - \alpha(r) + \beta(r)\frac{f(Me^{-\gamma t}V)}{Me^{-\gamma t}V}\bigg) \\
&\quad = Me^{-\gamma t}V\bigg(-\gamma + \bigg(\frac{\lambda_1(d, \alpha, h_0)}{(1+\delta)^2}\frac{\alpha(\frac{r}{\tau})}{\alpha(r)} - 1\bigg)\alpha(r) + \beta(r)\frac{f(Me^{-\gamma t}V)}{Me^{-\gamma t}V}\bigg).
\end{aligned}$$

Due to condition (A1), we easily obtain $\frac{f(Me^{-\gamma t}V)}{Me^{-\gamma t}V} > 0$. Since $1 < \lambda_1(d, \alpha, h_0)$, we can choose $\delta > 0$ sufficiently small such that

$$\varrho := \min_{t>0, r\in[0,\sigma(t)]} \frac{\lambda_1(d, \alpha, h_0)\alpha(\frac{r}{\tau})}{(1+\delta)^2\alpha(r)} - 1 > 0. \tag{41}$$

Let $\gamma = \varrho\kappa_1$, we have

$$w_t - d\Delta w - \alpha(r)w - \beta(r)f(w) \geq 0 \quad \text{for } t > 0 \text{ and } r \in \big[0, \sigma(t)\big].$$

Now, we choose $M > 0$ sufficiently large such that

$$u_0(r) \leq MV\bigg(\frac{r}{(1+\delta/2)}\bigg) = w(0, r) \quad \text{for } r \in [0, h_0].$$

Direct calculation implies

$$\sigma'(t) = \frac{1}{2}h_0\gamma\delta e^{-\gamma t},$$
$$-\mu w_r\bigl(t,\sigma(t)\bigr) = \mu M e^{-\gamma t}\frac{h_0}{\sigma(t)}\bigl|V_r(h_0)\bigr| \le \mu M e^{-\gamma t}\frac{|V_r(h_0)|}{1+\delta/2}.$$

Thus if we let

$$\mu_0 = \frac{\delta(1+\delta/2)\gamma h_0}{2M|V_r(h_0)|},$$

then for any $0 < \mu \le \mu_0$, we have

$$\sigma'(t) \ge -\mu w_r\bigl(t,\sigma(t)\bigr),$$

hence (w,σ) satisfies

$$\begin{cases} w_t - d\Delta w \ge \alpha(r)w - \beta(r)f(w), & t > 0, 0 < r < \sigma(t), \\ w = 0, \quad \sigma'(t) \ge -\mu w_r, & t > 0, r = \sigma(t), \\ w_r(t,0) = 0, & t > 0, \\ \sigma(0) = (1 + \frac{\sigma}{2})h_0 > h_0. \end{cases}$$

Hence, from Lemma 3.5, we have $h(t) \le \sigma(t)$ and $u(t,r) \le w(t,r)$ for $0 \le r \le h(t)$, $t > 0$. This implies $h_\infty \le \lim_{t\to\infty}\sigma(t) = h_0(1+\delta) < \infty$. □

In the same way as the proof of Theorem 2.1 in [8], we can prove the following theorem.

Theorem 3.8 *If $h_0 < R^*$, then there exists a $\mu^* > 0$ depending on u_0 such that spreading occurs when $\mu > \mu^*$, and vanishing happens when $\mu \le \mu^*$.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XD provided the main ideas and lead to write the paper. YL and XJ proved the local existence part. LM and KS proved the rest. HQ checked all the arguments. All authors read and approved the final manuscript.

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