Mishra et al. *Boundary Value Problems* 2014, **2014**:182
http://www.boundaryvalueproblems.com/content/2014/1/182

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Green's function for certain domains in the Heisenberg group $\mathbb{H}_n$

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Abstract
We obtain an explicit smooth kernel which works as the Green's function when applied to circular functions for annular and strip shaped domains in the Heisenberg group $\mathbb{H}_n$ by using the Kelvin transform and its generalizations.
MSC: 22E30; 34B27

Keywords: Heisenberg group; sub-Laplacian; Kelvin *R*-transform; Green function

1 Introduction

A Green's function is an integral kernel that can be used to solve inhomogeneous differential equations with boundary conditions. It has interesting physical significance when the involved differential operator is a Laplacian. For example, for the heat conduction equation, the Green's function is proportional to the temperature caused by a concentrated energy source. The theory of the Green's function has been widely studied in the case of domains in Euclidean spaces.

Let Ω be a domain in $\mathbb{R}^n$ with smooth boundary. Let $\frac{\partial}{\partial n}$ and $d\sigma$ denote the outward normal derivative and surface measure, respectively, on the boundary $\partial\Omega$ of Ω. Then a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ is given by

$$u(y) = \int_\Omega G(x,y)\Delta u(x)\,dx - \int_{\partial\Omega}\left(G(x,y)\frac{\partial u(x)}{\partial n} - u(x)\frac{\partial G(x,y)}{\partial n}\right)d\sigma(x),$$

where $G(x,y) = \varphi(\|x-y\|) + \widetilde{g}(x)$, $x \in \bar{\Omega}$, $y \in \Omega$, $x \neq y$, $\widetilde{g}(x)$ is an arbitrary harmonic function in Ω and

$$\varphi(r) = \begin{cases} \frac{r^{2-n}}{(2-n)\sigma_n}, & \text{for } n > 2, \\ \frac{1}{2\pi}\log r, & \text{for } n = 2, \end{cases}$$

σ_n being the volume of the n-sphere in $\mathbb{R}^n$. Suppose that the function $G(x,y)$ satisfies in addition $G(x,y) = 0$ for $x \in \bar{\Omega}$, $y \in \Omega$. Then, in particular the solution of the Dirichlet problem,

$$\Delta u = f \quad \text{on } \Omega, \qquad u = g \quad \text{on } \partial\Omega,$$

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is given by

$$u(y) = \int_{\Omega} G(x,y)f(x)\,dx + \int_{\partial\Omega} g(x)\frac{\partial G(x,y)}{\partial n}\,d\sigma(x).$$

A Green's function $G(x,y)$ for Δ on Ω is a function G as above, *i.e.* $x \to G(x,y)$ is in $C^2(\bar{\Omega} \setminus \{y\})$, $\Delta_x G(x,y) = 0$ for $x \in \Omega$, $G(x,y) = 0$ for $x \in \partial\Omega$, $y \in \Omega$.

The case $n = 2$, as can be seen above, is different from the cases for $n \geq 3$, and is even more interesting. For one can identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and the whole classical complex function theory can used as a tool [1, 2]. In [3, p.386], Courant and Hilbert gave an explicit Green's function for annular domain in $\mathbb{C}$ using infinitely many reflections of the pole with respect to the boundary circles of the domain which we describe in brief here. We take A to be the set $\{z \in \mathbb{C} : r^{\frac{1}{2}} < |z| < r^{-\frac{1}{2}}\}$, where $0 < r < 1$. Since in two dimensions, $\log|z|$ is a fundamental solution for the Laplacian, the Green's function (up to a constant multiple) can be sought in the form

$$G(z,\zeta) = \log|z-\zeta| - h(z,\zeta),$$

where $h(z,\zeta)$ is harmonic in A satisfying

$$\lim_{|\zeta|\to r^{\frac{1}{2}}} \log|z-\zeta| = \lim_{|\zeta|\to r^{\frac{1}{2}}} h(z,\zeta),$$
$$\lim_{|\zeta|\to r^{-\frac{1}{2}}} \log|z-\zeta| = \lim_{|\zeta|\to r^{-\frac{1}{2}}} h(z,\zeta).$$

Since harmonic functions in $\mathbb{C}$ are real parts of analytic functions, we may find G in the form $\log|f(z,\zeta)|$ for suitable analytic function f in ζ. Since $G(z,\zeta)$ must have a pole at $z = \zeta$, $(z-\zeta)$ is an obvious factor of $f(z,\zeta)$. Assuming that $f(z,\bar{\zeta})$ is equal to $\overline{f(z,\zeta)}$, one may arrive at the functional equations

$$\begin{cases} f(z,\zeta)f(z,r/\zeta) = 1, \\ f(z,\zeta)f(z,1/\zeta) = 1. \end{cases} \tag{1}$$

Since f has a simple zero at $\zeta = z$, it must have simple zeros at $r^{\pm 2}z, r^{\pm 4}z, \ldots$ and simple poles at $\frac{r}{z}^{\pm 1}, \frac{r}{z}^{\pm 3}, \ldots$. It may be noted here that the point r/z is obtained by reflecting z with respect to boundary $|\zeta| = r^{\frac{1}{2}}$ and r^3/z is obtained by reflecting r/z with respect to the image of boundary $|\zeta| = r^{-\frac{1}{2}}$ under the first reflection and so on. We may now consider the function

$$F(\zeta) = \left(1 - \frac{\zeta}{z}\right)\frac{\prod_{\nu=1}^{\infty}(1 - r^{2\nu}\frac{z}{\zeta})(1 - r^{2\nu}\frac{\zeta}{z})}{\prod_{\nu=1}^{\infty}(1 - r^{2\nu-1}z\zeta)(1 - r^{2\nu-1}\frac{1}{z\zeta})},$$

which has poles and zeros at exactly the same set of points as f has. Now, $f(z,\zeta)$ may be adopted in the form $a\zeta^b F(\zeta)$, the constants a and b to be determined in a way that f

satisfies the functional equations (1) and

$$\lim_{|\zeta|\to r^{\frac{1}{2}}} f(z,\zeta) = 1,$$

$$\lim_{|\zeta|\to r^{-\frac{1}{2}}} f(z,\zeta) = 1.$$

After an easy calculation, one arrives at

$$a = \sqrt{z}r^{1/4} \quad \text{and} \quad b = -\frac{1}{2} - \frac{\log z}{\log r}.$$

In order to generalise this idea, one may just rewrite the expression of G as a product

$$G(z,\zeta) = Re\left[\log a\zeta^b\left(1-\frac{\zeta}{z}\right)\frac{\prod_{\nu=1}^{\infty}(1-r^{2\nu}\frac{z}{\zeta})(1-r^{2\nu}\frac{\zeta}{z})}{\prod_{\nu=1}^{\infty}(1-r^{2\nu-1}z\zeta)(1-r^{2\nu-1}\frac{1}{z\zeta})}\right]. \tag{2}$$

The expression in (2) indicates that the Green's function for annulus in a general setup must look like an infinite convergent sum of functions harmonic in annulus and having poles at carefully chosen points outside the annulus.

The rich geometric structure of Heisenberg group allows us to construct explicit examples of domains that are relevant in Potential theory. On the Heisenberg group we have an analogue of the Laplacian which was first studied by Folland and Stein [4]. The study of the Green's function on the Heisenberg group became interesting after Folland [5] found a smooth fundamental solution for this operator. Korányi first gave a Green's function for circular data for a certain gauge ball [6] using the Kelvin transform on $\mathbb{H}_n$ [7, 8]. The general Green's function is not known for any domain in $\mathbb{H}_n$, $n > 1$. The Green's function for circular data are studied for various domains, *e.g.*, for half space in [9], for quarter space in [10] and for annulus in [11]. Some estimates for the Green's function and the Poisson kernel on bounded domains of the Heisenberg group have been given in [12] and Green's function on bounded domains for sub-Laplacian on the stratified Lie groups are also studied in [13].

In this article, we have generalised the method of Courant and Hilbert to the case of annular domain in the Heisenberg group. The role of repeated reflections here is being played by repeated Kelvin transforms. A similar idea works for the case of an infinite strip in the Heisenberg group.

2 The Heisenberg group

The Heisenberg group $\mathbb{H}_n$ is the set of points $[z,t] \in \mathbb{C}^n \times \mathbb{R}$ with the multiplication given by

$$[z,t]\cdot[z',t'] = [z+z', t+t'+2\Im(z\cdot\bar{z}')],$$

$z, z' \in \mathbb{C}^n$, $t, t' \in \mathbb{R}$. This multiplication turns $\mathbb{H}_n$ into a Lie group.

A basis of the Lie algebra of $\mathbb{H}_n$ is $\{Z_j, \bar{Z}_j, T : 1 \le j \le n\}$ where

$$Z_j = \partial_{z_j} + i\bar{z}_j\partial_t;$$

$$\bar{Z}_j = \partial_{\bar{z}_j} - iz_j\partial_t;$$

$$T = \partial_t.$$

The sub-Laplacian on $\mathbb{H}_n$ is given by

$$\Delta_0 = 2\sum_{j=1}^{n}(\bar{Z}_j Z_j + Z_j \bar{Z}_j).$$

We shall consider a slightly modified subelliptic operator $L_0 = -\frac{1}{4}\Delta_0$. The natural gauge on $\mathbb{H}_n$ is given by

$$N(z,t) = \left(|z|^4 + t^2\right)^{\frac{1}{4}}.$$

The fundamental solution for L_0 on $\mathbb{H}_n$ with pole at identity is given in [5] as

$$g_e(\xi) = g_e\left([z,t]\right) = a_o\left(|z|^4 + t^2\right)^{-\frac{n}{2}},$$

where

$$a_o = 2^{n-2}\frac{(\Gamma(\frac{n}{2}))^2}{\pi^{n+1}},$$

and $\xi = [z,t]$. The fundamental solution with pole at η is given by

$$g_\eta(\xi) = g_e\left(\xi^{-1}\eta\right).$$

From [9], for $\eta = [\varsigma,\tau]$ and $\xi = [z,t]$,

$$g_\eta(\xi) = a_o\left|C(\eta,\xi) - P(\eta,\xi)\right|^{-n},$$

where

$$C(\eta,\xi) = |z|^2 + |\varsigma|^2 + i(t-\tau) \quad \text{and} \quad P(\eta,\xi) = 2z\cdot\bar{\varsigma}.$$

For an integrable function f on $\mathbb{H}_n$, we denote the average of f by

$$\bar{f}\left([z,t]\right) = \frac{1}{2\pi}\int_0^{2\pi} f\left(\left[e^{i\theta}z,t\right]\right)d\theta.$$

A function f is said to be circular if $f([z,t]) = \bar{f}([z,t])$ for $[z,t] \in \mathbb{H}_n$.

As in [9],

$$\bar{g}_\eta(\xi) = a_o\left|C(\eta,\xi)\right|^{-n}F\left(\frac{n}{2},\frac{n}{2};n;\frac{|P(\eta,\xi)|^2}{|C(\eta,\xi)|^2}\right),$$

where F is the Gaussian hypergeometric function [14].

3 Green's function for annular domain

In this section, D will denote the annulus $\{\xi \in \mathbb{H}_n : 0 < R < N(\xi) < 1\}$.

We will be constructing an explicit Green's function for the domain D on the lines described in Section 1 for the annulus in $\mathbb{C}$. A natural candidate which would generalise the map $f(z) \mapsto f(\frac{1}{z})$ to $\mathbb{R}^n$ is the Kelvin transform, which is defined as follows. Given a function u defined on a set $E \subseteq \mathbb{R}^n \setminus \{0\}$, the Kelvin transform $K[u]$ of u is defined on $E^* = \{x^* : x \in E\}$ by

$$K[u](x) = |x|^{2-n} u(x^*),$$

where x^* is called the inverse of $\mathbb{R}^n \cup \{\infty\}$ relative to the unit sphere. We are fortunate enough to have an analogue of the Kelvin transform for the Heisenberg group given by Korányi [7] and also by Koornwinder [8]. But the property that the classical Kelvin transform of a function agrees with the values of the function at the unit sphere in $\mathbb{R}^n$ is not true in the case of $\mathbb{H}_n$. However, things go well for the class of circular functions. All our construction, therefore, will work for the class of circular functions only. Finding a Green's function which works for all the continuous functions is still an open problem even in the case of the unit gauge ball.

The Kelvin transform on the Heisenberg group has been defined and studied in [7]. For any f on $\mathbb{H}_n$ the Kelvin transform of f is defined by

$$Kf = N^{-2n} f \circ h,$$

where h is the inversion,

$$h([z,t]) = \left[\frac{-z}{|z|^2 - it}, \frac{-t}{|z|^4 + t^2}\right],$$

for $[z,t] \in \mathbb{H}_n \setminus \{e\}$. The Kelvin transform sends a harmonic function on $\mathbb{H}_n \setminus \{e\}$ to a harmonic function. It was shown in [6] that for a circular function f on $\mathbb{H}_n \setminus \{e\}$, we have

$$K(f)(\xi^{-1}) = f(\xi),$$

for all $\xi \in \mathbb{H}_n \setminus \{e\}$ with $N(\xi) = 1$.

From [6, (3.3)] we have, for $\eta \neq e \in \mathbb{H}_n$,

$$K(g_\eta) = N(\eta)^{-2n} g_{\eta^*}, \tag{3}$$

where we wrote η^* for $h(\eta)$.

The Kelvin R-transform on the Heisenberg group was defined and studied in [11]. For f defined on $\mathbb{H}_n \setminus \{e\}$, the Kelvin R-transform is defined as

$$K_R(f) = R^{2n} g_e f \circ h_R,$$

where h_R is the inversion with respect to the Korányi ball of radius R, *i.e.* $\{[z,t] : N(z,t) < R\}$

$$h_R([z,t]) = \left[\frac{-R^2 z}{|z|^2 - it}, \frac{-R^4 t}{|z|^4 + t^2}\right],$$

for $[z,t] \in \mathbb{H}_n \setminus \{e\}$.

The Kelvin R-transform sends a harmonic function on $\mathbb{H}_n \setminus \{e\}$ to a harmonic function. It was shown in [11] that for a circular function f on $\mathbb{H}_n \setminus \{e\}$, and $R > 0$, we have

$$K_R(f)(\xi^{-1}) = f(\xi),$$

for all $\xi \in \mathbb{H}_n$ with $N(\xi) = R$.

From [11, (14)] we have, for $\eta \neq e \in \mathbb{H}_n$,

$$K_R(g_\eta) = R^{2n} N(\eta)^{-2n} g_{\eta^\dagger}, \tag{4}$$

where $\eta^\dagger = h_R(\eta)$.

A Green's function for annular domain was given in [11], however, the function was not continuous. The Green's function constructed below is a smooth function.

For each $\eta \in D$, we define functions $H_k(\eta,\cdot)$, $M_k(\eta,\cdot)$, $U_k(\eta,\cdot)$ and $V_k(\eta,\cdot)$ on D inductively as follows:

$$H_1(\eta,\xi) = K(\bar{g_\eta}) \circ i(\xi),$$
$$M_1(\eta,\xi) = \bar{g_\eta}(\xi),$$
$$U_1(\eta,\xi) = K_R(\bar{g_\eta}) \circ i(\xi),$$
$$V_1(\eta,\xi) = K\bigl(K_R(\bar{g_\eta}) \circ i\bigr) \circ i(\xi).$$

When $H_k(\eta,\xi)$, $M_k(\eta,\xi)$, $U_k(\eta,\xi)$, $V_k(\eta,\xi)$ are defined, define

$$H_{k+1}(\eta,\xi) = K\bigl(K_R\bigl(H_k(\eta,\xi)\bigr) \circ i\bigr) \circ i(\xi),$$
$$M_{k+1}(\eta,\xi) = K_R\bigl(K\bigl(M_k(\eta,\xi)\bigr) \circ i\bigr) \circ i(\xi),$$
$$U_{k+1}(\eta,\xi) = K_R\bigl(K\bigl(U_k(\eta,\xi)\bigr) \circ i\bigr) \circ i(\xi),$$
$$V_{k+1}(\eta,\xi) = K\bigl(K_R\bigl(V_k(\eta,\xi)\bigr) \circ i\bigr) \circ i(\xi),$$

where i denotes inversion in the Heisenberg group *i.e.* $i[z,t] = [-z,-t]$ for $[z,t] \in \mathbb{H}_n$.

For each η, $\xi \in D$, define

$$G(\eta,\xi) = \sum_{k=1}^{\infty} \bigl[M_k(\eta,\xi) - H_k(\eta,\xi)\bigr] + \sum_{k=1}^{\infty} \bigl[V_k(\eta,\xi) - U_k(\eta,\xi)\bigr]. \tag{5}$$

Our main objective of this section is to prove the following theorem.

Theorem 3.1 *The function* $G(\eta,\xi)$ *as defined in* (5) *is a smooth function on* $D = \{\xi \in \mathbb{H}_n : 0 < R < N(\xi) < 1\}$ *and satisfies the following.*

(i) $L_0 G(\eta,\xi) = \delta_\eta$.

(ii) *The limits of the function* $G(\eta,\xi)$ *vanish at the boundaries of the annular domain i.e. at* $N(\xi) = 1$ *and* $N(\xi) = R$.

We begin with some lemmas about the functions $H_k(\eta,\xi)$, $M_k(\eta,\xi)$, $U_k(\eta,\xi)$ and $V_k(\eta,\xi)$.

Lemma 3.2 *For $k \geq 1$, let $H_k(\eta,\xi)$, $M_k(\eta,\xi)$, $U_k(\eta,\xi)$ and $V_k(\eta,\xi)$ be as defined above then*

$$\begin{aligned} H_k(\eta,\xi) &= R^{(2k-2)n}\big(N(\xi)\big)^{-2n}\left|\frac{|z|^2}{|z|^4+t^2}+R^{2(2k-2)}|\varsigma|^2+i\left(\frac{t}{|z|^4+t^2}-R^{2(2k-2)}\tau\right)\right|^{-n} \\ &\quad\times F\left(\frac{n}{2},\frac{n}{2};n;\frac{u_{k,1}}{v_{k,1}}\right), \end{aligned} \tag{6}$$

where

$$\frac{u_{k,1}}{v_{k,1}} = \frac{4R^{2(2k-2)}|z|^2|\varsigma|^2}{1+R^{4(2k-2)}(|\varsigma|^4+\tau^2)(|z|^4+t^2)+2R^{2(2k-2)}(|z|^2|\varsigma|^2-t\tau)},$$

$$M_k(\eta,\xi) = R^{(2k-2)n}\big||z|^2+R^{2(2k-2)}|\varsigma|^2+i\big(t-R^{2(2k-2)}\tau\big)\big|^{-n}F\left(\frac{n}{2},\frac{n}{2};n;\frac{u_{k,2}}{v_{k,2}}\right), \tag{7}$$

where

$$\frac{u_{k,2}}{v_{k,2}} = \frac{4R^{2(2k-2)}|z|^2|\varsigma|^2}{(|z|^4+t^2)+R^{4(2k-2)}(|\varsigma|^4+\tau^2)+2R^{2(2k-2)}(|z|^2|\varsigma|^2-t\tau)},$$

$$U_k(\eta,\xi) = R^{2kn}\big(N(\xi)\big)^{-2n}\left|\frac{R^{4k}|z|^2}{|z|^4+t^2}+|\varsigma|^2+i\left(\frac{R^{4k}t}{|z|^4+t^2}-\tau\right)\right|^{-n}F\left(\frac{n}{2},\frac{n}{2};n;\frac{u_{k,3}}{v_{k,3}}\right), \tag{8}$$

where

$$\frac{u_{k,3}}{v_{k,3}} = \frac{4R^{2(2k)}|z|^2|\varsigma|^2}{R^{2(4k)}+(|\varsigma|^4+\tau^2)(|z|^4+t^2)+2R^{2(4k)}(|z|^2|\varsigma|^2-t\tau)},$$

and

$$V_k(\eta,\xi) = R^{2kn}\big|R^{4k}|z|^2+|\varsigma|^2+i\big(R^{4k}t-\tau\big)\big|^{-n}F\left(\frac{n}{2},\frac{n}{2};n;\frac{u_{k,4}}{v_{k,4}}\right), \tag{9}$$

where

$$\frac{u_{k,4}}{v_{k,4}} = \frac{4R^{2(2k)}|z|^2|\varsigma|^2}{R^{2(4k)}(|z|^4+t^2)+(|\varsigma|^4+\tau^2)+2R^{2(4k)}(|z|^2|\varsigma|^2-t\tau)}.$$

Proof We prove (6) by induction on k.

For $k=1$, $H_1(\eta,\xi) = K(\bar{g_\eta})(-\xi)$.

Assume (6) for $k=l$, we show the validity of (6) for $k=l+1$.

From (3) and (4), we have

$$\begin{aligned} H_{l+1}(\eta,\xi) &= K\big(K_R\big(H_l(\eta,\xi)\big)\circ i\big)(-\xi) \\ &= g_e(-\xi)K_R\big(H_l(\eta,\xi)\big)(h\xi) \\ &= R^{2n}g_e(-\xi)g_e(h\xi)H_l(\eta,\xi)(h_R h\xi) \\ &= R^{2n}H_l(\eta,\xi)\big(R^2z,R^4t\big) \\ &= R^{2n}R^{(2l-2)n}\big(N(\xi)\big)^{-2n}R^{-4n} \\ &\quad\times\left|\frac{R^4|z|^2}{R^8(|z|^4+t^2)}+R^{2(2l-2)}|\varsigma|^2+i\left(\frac{R^4t}{R^8(|z|^4+t^2)}-R^{2(2l-2)}\tau\right)\right|^{-n} \end{aligned}$$

$$\begin{aligned} &\times F\left(\frac{n}{2},\frac{n}{2};n;\frac{a_{l+1}}{b_{l+1}}\right) \\ &= R^{2ln}\big(N(\xi)\big)^{-2n}\left|\frac{|z|^2}{|z|^4+t^2}+R^{2(2l)}|\varsigma|^2+i\left(\frac{t}{|z|^4+t^2}-R^{2(2l)}\tau\right)\right|^{-n} \\ &\times F\left(\frac{n}{2},\frac{n}{2};n;\frac{u_{l+1,1}}{v_{l+1,1}}\right), \end{aligned}$$

where

$$\frac{u_{l+1,1}}{v_{l+1,1}} = \frac{4R^{2(2l)}|z|^2|\varsigma|^2}{1+R^{4(2l)}(|\varsigma|^4+\tau^2)(|z|^4+t^2)+2R^{2(2l)}(|z|^2|\varsigma|^2-t\tau)}.$$

Therefore, by induction (6) follows. Similarly we can prove (7), (8) and (9). □

Next, we state a result about estimation of Gaussian hypergeometric function. The proof can be found in [14].

Lemma 3.3 *The Gaussian hypergeometric function is defined by*

$$F(a,b;c;z) = 1+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n n!},$$

for c neither zero nor a negative integer. If none of a, b, c is zero or a negative integer, the above series has the circle $|z|<1$ *as its circle of convergence. If either or both of a and b is zero or a negative integer, the series terminates, and convergence does not enter the discussion.*

Lemma 3.4 *For each* $\eta,\xi\in D$

$$\sum_{k=1}^{\infty}\big[M_k(\eta,\xi)-H_k(\eta,\xi)\big]+\sum_{k=1}^{\infty}\big[V_k(\eta,\xi)-U_k(\eta,\xi)\big] \tag{10}$$

is absolutely and uniformly convergent on compact neighbourhoods of ξ *(for* $\eta\neq\xi$*). In other words,* $G(\eta,\xi)$ *is a well-defined function for* $\eta\neq\xi$*.*

Proof Firstly, we will prove that the Gaussian hypergeometric functions involved in expression of infinite series (10) are uniformly bounded.

Consider

$$\begin{aligned} v_{k,1}-u_{k,1} &= 1+R^{4(2k-2)}\big(|\varsigma|^4+\tau^2\big)\big(|z|^4+t^2\big)+2R^{2(2k-2)}\big(|z|^2|\varsigma|^2-t\tau\big) \\ &\quad -4R^{2(2k-2)}|z|^2|\varsigma|^2 \\ &= 1+R^{2(2k-2)}\big[R^{2(2k-2)}\big(|\varsigma|^4+\tau^2\big)\big(|z|^4+t^2\big)-2|z|^2|\varsigma|^2-2t\tau\big] \\ &\geq 1-2R^{2(2k-2)}\big[|z|^2|\varsigma|^2+t\tau\big] \\ &\geq 1-4R^{2(2k-2)}. \end{aligned}$$

We can choose k large enough such that $R^{2(2k-2)} < \frac{1}{8}$. Thus for sufficiently large k, $v_{k,1} - u_{k,1} \geq \frac{1}{2}$. Since $u_{k,1}$, $v_{k,1}$ are bounded, the argument $\frac{u_{k,1}}{v_{k,1}}$ of

$$F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{u_{k,1}}{v_{k,1}}\right)$$

is bounded away from 1. Thus, by using Lemma 3.3, $\{F(\frac{n}{2}, \frac{n}{2}; n; \frac{u_{k,1}}{v_{k,1}})\}_{k=1}^{\infty}$ is uniformly bounded for $\eta \neq \xi$, say

$$\left|F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{u_{k,1}}{v_{k,1}}\right)\right| < E_1.$$

Similarly we have constants E_2, E_3, E_4 such that

$$\left|F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{u_{k,i}}{v_{k,i}}\right)\right| < E_i \quad \text{for } i = 2, 3, 4.$$

We assert that both series in (10) are uniformly and absolutely convergent on compact neighbourhoods of ξ.

We have

$$\begin{aligned}\left|M_k(\eta,\xi)\right| &= \left|R^{(2k-2)n}\right|\left||z|^2 + R^{2(2k-2)}|\varsigma|^2 + i\left(t - R^{2(2k-2)}\tau\right)\right|^{-n}\left|F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{u_{k,2}}{v_{k,2}}\right)\right| \\ &\leq \left[2\left(1 + |z|^2|\varsigma|^2 + |t\tau|\right)\right]^{\frac{-n}{2}} \cdot E_2 \cdot \left|R^{(2k-2)n}\right|.\end{aligned}$$

Since $R < 1$, the series $\sum_{k=1}^{\infty} R^{2kn}$ is convergent and so $\sum_{k=1}^{\infty} |M_k(\eta,\xi)|$ is uniformly convergent on compact neighbourhood of $\xi = [z,t]$.

Similar estimates show that $\sum_{k=1}^{\infty} |H_k(\eta,\xi)|$ is uniformly convergent on a compact neighbourhood of $\xi = [z,t]$.

Hence $\sum_{k=1}^{\infty} [M_k(\eta,\xi) - H_k(\eta,\xi)]$ is absolutely and uniformly convergent on compact neighbourhoods of ξ (for $\eta \neq \xi$).

We have

$$\begin{aligned}\left|V_k(\eta,\xi)\right| &= \left|R^{2kn}\right|\left|R^{4k}|z|^2 + |\varsigma|^2 + i\left(R^{4k}t - \tau\right)\right|^{-n}\left|F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{u_{k,4}}{v_{k,4}}\right)\right| \\ &\leq \left[2\left(1 + |z|^2|\varsigma|^2 + |t\tau|\right)\right]^{\frac{-n}{2}} \cdot E_4 \cdot \left|R^{2kn}\right|.\end{aligned}$$

Since $R < 1$, the series $\sum_{k=1}^{\infty} R^{2kn}$ is convergent and so $\sum_{k=1}^{\infty} |V_k(\eta,\xi)|$ is uniformly convergent on compact neighbourhood of $\xi = [z,t]$.

Similar estimates show that $\sum_{k=1}^{\infty} |U_k(\eta,\xi)|$ is uniformly convergent on a compact neighbourhood of $\xi = [z,t]$.

Hence $\sum_{k=1}^{\infty} [V_k(\eta,\xi) - U_k(\eta,\xi)]$ is absolutely and uniformly convergent on compact neighbourhoods of ξ (for $\eta \neq \xi$). □

Lemma 3.5 *For each $\eta \in D$ and $k \geq 1$,*

$$\lim_{N(\xi)\to 1}\left(M_k(\eta,\xi) - H_k(\eta,\xi)\right) = 0 \quad \textit{and} \quad \lim_{N(\xi)\to 1}\left(V_k(\eta,\xi) - U_k(\eta,\xi)\right) = 0.$$

Also,

$$\lim_{N(\xi)\to R}\big(M_{k+1}(\eta,\xi)-H_k(\eta,\xi)\big)=0 \quad \textit{and} \quad \lim_{N(\xi)\to R}\big(V_k(\eta,\xi)-U_{k+1}(\eta,\xi)\big)=0.$$

Proof By using (6) and (7), we get

$$\begin{aligned}\lim_{N(\xi)\to 1} M_k(\eta,\xi) &= R^{(2k-2)n}\big(1+R^{4(2k-2)}\big(|\varsigma|^4+\tau^2\big)+2R^{2(2k-2)}\big(|z|^2|\varsigma|^2-t\tau\big)\big)^{-\frac{n}{2}} \\ &\quad\times F\left(\frac{n}{2},\frac{n}{2};n;\frac{4R^{2(2k-2)}|z|^2|\varsigma|^2}{1+R^{4(2k-2)}(|\varsigma|^4+\tau^2)+2R^{2(2k-2)}(|z|^2|\varsigma|^2-t\tau)}\right) \\ &= \lim_{N(\xi)\to 1} H_k(\eta,\xi).\end{aligned}$$

So, $\lim_{N(\xi)\to 1}(M_k(\eta,\xi)-H_k(\eta,\xi))=0$. Similarly, by using (8) and (9), we can prove that $\lim_{N(\xi)\to 1}(V_k(\eta,\xi)-U_k(\eta,\xi))=0$.

Now we have

$$M_{k+1}(\eta,\xi)=R^{2kn}\big||z|^2+R^{4k}|\varsigma|^2+i\big(t-R^{4k}\tau\big)\big|^{-n}F\left(\frac{n}{2},\frac{n}{2};n;\frac{u_{k+1,2}}{v_{k+1,2}}\right),$$

where

$$\frac{u_{k+1,2}}{v_{k+1,2}}=\frac{4R^{4k}|z|^2|\varsigma|^2}{(|z|^4+t^2)+R^{2.4k}(|\varsigma|^4+\tau^2)+2R^{4k}(|z|^2|\varsigma|^2-t\tau)},$$

$$\begin{aligned}\lim_{N(\xi)\to R} M_{k+1}(\eta,\xi) &= R^{2kn}\big(R^4+R^{2.4k}\big(|\varsigma|^4+\tau^2\big)+2R^{4k}\big(|z|^2|\varsigma|^2-t\tau\big)\big)^{-\frac{n}{2}} \\ &\quad\times F\left(\frac{n}{2},\frac{n}{2};n;\frac{4R^{4k}|z|^2|\varsigma|^2}{R^4+R^{2.4k}(|\varsigma|^4+\tau^2)+2R^{4k}(|z|^2|\varsigma|^2-t\tau)}\right) \\ &= \lim_{N(\xi)\to R} H_k(\eta,\xi).\end{aligned}$$

So, $\lim_{N(\xi)\to R}(M_{k+1}(\eta,\xi)-H_k(\eta,\xi))=0$. Similarly, by using (8) and (9), we can prove that $\lim_{N(\xi)\to R}(V_k(\eta,\xi)-U_{k+1}(\eta,\xi))=0$. □

Proof of Theorem 3.1 First note that $H_k(\eta,\xi)$, $M_k(\eta,\xi)$, $k>1$, $U_k(\eta,\xi)$ and $V_k(\eta,\xi)$ are all harmonic functions on D (this follows from the definition of these functions and properties of the Kelvin transforms K and K_R). We first apply term by term the Laplacian to the series (5) and we have

$$\begin{aligned}\sum_{k=1}^{\infty} L_0\big[M_k(\eta,\xi)-H_k(\eta,\xi)\big]+\sum_{k=1}^{\infty} L_0\big[V_k(\eta,\xi)-U_k(\eta,\xi)\big] &= L_0 M_1(\eta,\xi) \\ &= L_0\bar{g}_\eta(\xi).\end{aligned}$$

Since the series obtained, after applying the Laplacian term by term, is a single term it is uniformly convergent on compact sets away from η. We conclude that the limit of the series (5) is a C^2 function and we can actually apply the Laplacian to the series (5) term by term and we get

$$L_0 G(\eta,\xi)=L_0\bar{g}_\eta(\xi)=\delta_\eta.$$

From Lemma 3.5 we get

$$\lim_{N(\xi)\to 1}\left(\sum_{k=1}^{\infty}\left[M_k(\eta,\xi)-H_k(\eta,\xi)\right]+\sum_{k=1}^{\infty}\left[V_k(\eta,\xi)-U_k(\eta,\xi)\right]\right)=0.$$

Also,

$$\begin{aligned}
&\lim_{N(\xi)\to R}\left(\sum_{k=1}^{\infty}\left[M_k(\eta,\xi)-H_k(\eta,\xi)\right]+\sum_{k=1}^{\infty}\left[V_k(\eta,\xi)-U_k(\eta,\xi)\right]\right)\\
&\quad=\lim_{N(\xi)\to R}\left(\left(M_1(\eta,\xi)-U_1(\eta,\xi)\right)+\sum_{k=1}^{\infty}\left(M_{k+1}(\eta,\xi)-H_k(\eta,\xi)\right)\right.\\
&\qquad\left.+\sum_{k=1}^{\infty}\left(V_k(\eta,\xi)-U_{k+1}(\eta,\xi)\right)\right)\\
&\quad=\lim_{N(\xi)\to R}\left(M_1(\eta,\xi)-U_1(\eta,\xi)\right)\\
&\qquad+\sum_{k=1}^{\infty}\left(\lim_{N(\xi)\to R}\left(M_{k+1}(\eta,\xi)-H_k(\eta,\xi)\right)\right)\\
&\qquad+\sum_{k=1}^{\infty}\left(\lim_{N(\xi)\to R}\left(V_k(\eta,\xi)-U_{k+1}(\eta,\xi)\right)\right).
\end{aligned}$$

We have $M_1(\eta,\xi)=\bar{g_\eta}(\xi)$ and $U_1(\eta,\xi)=K_R(\bar{g_\eta})(-\xi)$.

Since $g_\eta(\xi)$ is circular, therefore, by (4), $M_1(\eta,\xi)-U_1(\eta,\xi)=0$ and from Lemma 3.5 we get

$$\lim_{N(\xi)\to R}\left(\sum_{k=1}^{\infty}\left[M_k(\eta,\xi)-H_k(\eta,\xi)\right]+\sum_{k=1}^{\infty}\left[V_k(\eta,\xi)-U_k(\eta,\xi)\right]\right)=0.$$

Hence the theorem. □

The Poisson kernel is the normal derivative of the Green's function and, from [6], is given by

$$P(\eta,\xi)=-\frac{1}{4}\frac{\partial}{\partial n_0}G(\eta,\xi),\quad \xi\in\partial D,$$

where

$$\frac{\partial}{\partial n_0}=\begin{cases}\frac{1}{|z|}(\bar{A}E+A\bar{E}) & \text{at } (\partial D)_1 \textit{ i.e.} \text{ at the boundary } \{\xi\in\mathbb{H}_n : N(\xi)=1\},\\ \frac{-1}{R^2|z|}(\bar{A}E+A\bar{E}) & \text{at } (\partial D)_2 \textit{ i.e.} \text{ at the boundary } \{\xi\in\mathbb{H}_n : N(\xi)=R\},\end{cases}$$

$$A=|z|^2-it\quad\text{and}\quad E=\sum z_jZ_j.$$

Making use of the identity

$$F\left(\frac{n}{2},\frac{n}{2};n;x\right)+\frac{2}{n}xF'\left(\frac{n}{2},\frac{n}{2};n;x\right)=F\left(\frac{n}{2}+1,\frac{n}{2};n;x\right),$$

which is obvious from the Euler integral representation, a computation gives $P(\eta,\xi)$ at $(\partial D)_1$ by

$$\begin{aligned}
P(\eta,\xi) &= \sum_{i=1}^{4}\sum_{k=1}^{\infty} R^{2kn}|z|\frac{n}{2}(\nu_{k;i})^{-\frac{n}{2}-1}F\left(\frac{n}{2}+1,\frac{n}{2};n;\frac{u_{k,i}}{\nu_{k,i}}\right)\\
&\quad\times\Bigg[R^{-2n}\big(N(\xi)\big)^{-2n}\left(\frac{2|z|^2+t^2}{(|z|^4+t^2)^2}\big(1+2R^{2(2k-2)}\big(|z|^2|\varsigma|^2-t\tau\big)\big)\right.\\
&\quad\left.-\frac{R^{2(2k-2)}}{|z|^4+t^2}\big(2|\varsigma|^2-t\tau\big)\right)\delta_{1i}+R^{-2n}\big(\big(|z|^2+t^2\big)\\
&\quad+R^{2(2k-2)}\big(2|\varsigma|^2-t\tau\big)\big)\delta_{2i}+R^{4k}\big(N(\xi)\big)^{-2n}\\
&\quad\times\left(\frac{2|z|^2+t^2}{(|z|^4+t^2)^2}\big(R^{4k}+2\big(|z|^2|\varsigma|^2-t\tau\big)\big)-\frac{1}{|z|^4+t^2}\big(2|\varsigma|^2-t\tau\big)\right)\delta_{3i}\\
&\quad+R^{4k}\big(R^{4k}\big(|z|^2+t^2\big)+\big(2|\varsigma|^2-t\tau\big)\big)\delta_{4i}\Bigg],
\end{aligned}$$

where δ_{ai} denotes the Kronecker delta function of the indices a and i. The Poisson kernel $P(\eta,\xi)$ at $(\partial D)_2$ is given by

$$\begin{aligned}
P(\eta,\xi) &= \sum_{i=1}^{4}\sum_{k=1}^{\infty} -R^{(2kn-2)}|z|\frac{n}{2}(\nu_{k;i})^{-\frac{n}{2}-1}F\left(\frac{n}{2}+1,\frac{n}{2};n;\frac{u_{k,i}}{\nu_{k,i}}\right)\\
&\quad\times\Bigg[R^{-2n}\big(N(\xi)\big)^{-2n}\left(\frac{2|z|^2+t^2}{(|z|^4+t^2)^2}\big(1+2R^{2(2k-2)}\big(|z|^2|\varsigma|^2-t\tau\big)\big)\right.\\
&\quad\left.-\frac{R^{2(2k-2)}}{|z|^4+t^2}\big(2|\varsigma|^2-t\tau\big)\right)\delta_{1i}+R^{-2n}\big(\big(|z|^2+t^2\big)\\
&\quad+R^{2(2k-2)}\big(2|\varsigma|^2-t\tau\big)\big)\delta_{2i}+R^{4k}\big(N(\xi)\big)^{-2n}\\
&\quad\times\left(\frac{2|z|^2+t^2}{(|z|^4+t^2)^2}\big(R^{4k}+2\big(|z|^2|\varsigma|^2-t\tau\big)\big)-\frac{1}{|z|^4+t^2}\big(2|\varsigma|^2-t\tau\big)\right)\delta_{3i}\\
&\quad+R^{4k}\big(R^{4k}\big(|z|^2+t^2\big)+\big(2|\varsigma|^2-t\tau\big)\big)\delta_{4i}\Bigg].
\end{aligned}$$

Theorem 3.6 *The Green's function and Poisson kernel which we have obtained above solves the Dirichlet boundary value problem for D and the solution for BVP*

$$\begin{aligned}
&L_0u=f \quad \text{in } D,\\
&u=h \quad \text{on } \partial D,
\end{aligned}$$

is given by

$$u(\eta)=\int_D G(\eta,\xi)f(\xi)\,d\nu(\xi)+\int_{(\partial D)_1}P(\eta,\xi)h(\xi)\,d\sigma(\xi)+\int_{(\partial D)_2}P(\eta,\xi)h(\xi)\,d\sigma(\xi),$$

where f and h are circular functions.

4 Green's function for infinite strip

In this section, I will denote the infinite strip $\{\xi = [z', t'] \in \mathbb{H}_n : 0 < t' < 1\}$. Denote, by $H(t)$ the function of (η, ξ),

$$H(t) = a_0|C_t|^{-n} F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{|P|^2}{|C_t|^2}\right),$$

where $C_{\pm t}$ and P are defined as follows:

$$C_{\pm t} = |z|^2 + |z'|^2 + i(t' \pm t),$$
$$P = 2z \cdot \bar{z}',$$

for $\xi = [z', t'] \in \mathbb{H}_n$, $\eta = [z, t] \in \mathbb{H}_n$. A differential operator, whenever applied to function $H(t)$ will be with respect to the variable ξ.

Consider the series

$$\sum_{m=0}^{\infty} H(2m - t), \qquad \sum_{m=0}^{\infty} H(2m + t), \qquad \sum_{m=1}^{\infty} H(-2m - t), \qquad \sum_{m=1}^{\infty} H(-2m + t).$$

We first show that the four series are uniformly convergent on compact neighbourhoods of ξ. For this firstly we show that the sequences of functions

$$F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{|P|^2}{|C_{2m-t}|^2}\right), \qquad F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{|P|^2}{|C_{2m+t}|^2}\right),$$
$$F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{|P|^2}{|C_{-2m-t}|^2}\right), \qquad F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{|P|^2}{|C_{-2m+t}|^2}\right)$$

are uniformly bounded.

Consider the argument $u_{m,1}$ of the hypergeometric function $F(\frac{n}{2}, \frac{n}{2}; n; \frac{|P|^2}{|C_{2m-t}|^2})$ *i.e.*

$$\begin{aligned}
|u_{m,1}| = \left|\frac{|P|^2}{|C_{2m-t}|^2}\right| &= \frac{4|z|^2|z'|^2}{||z|^2 + |z'|^2 + i(t' + 2m - t)|^2} \\
&= \frac{4|z|^2|z'|^2}{|z|^4 + |z'|^4 + 2|z|^2|z'|^2 + (t' + 2m - t)^2} \\
&\leq \frac{4|z|^2|z'|^2}{|z|^4 + |z'|^4 + 2|z|^2|z'|^2}.
\end{aligned}$$

So,

$$\begin{aligned}
1 - \left|\frac{|P|^2}{|C_{2m-t}|^2}\right| &\geq 1 - \frac{4|z|^2|z'|^2}{|z|^4 + |z'|^4 + 2|z|^2|z'|^2} \\
&= \frac{|z|^4 + |z'|^4 - 2|z|^2|z'|^2}{|z|^4 + |z'|^4 + 2|z|^2|z'|^2} \\
&= \frac{(|z|^2 - |z'|^2)^2}{|z|^4 + |z'|^4 + 2|z|^2|z'|^2}.
\end{aligned}$$

We can choose a suitable compact neighbourhood of ξ such that $|z|^2 - |z'|^2 > \epsilon$ for some $\epsilon > 0$. So, we have

$$1 - \frac{|P|^2}{|C_{2m-t}|^2} \geq \frac{\epsilon^2}{|z|^4 + |z'|^4 + 2|z|^2|z'|^2} > \epsilon_1,$$

for some $\epsilon_1 > 0$.

This implies that $\frac{|P|^2}{|C_{2m-t}|^2} < 1 - \epsilon_1$ *i.e.* the argument $u_{m,1}$ of the hypergeometric function $F(\frac{n}{2}, \frac{n}{2}; n; u_{m,1})$ is bounded away from 1. Therefore, $\{F(\frac{n}{2}, \frac{n}{2}; n; u_{m,1})\}_{m=1}^{\infty}$ is uniformly bounded, say

$$\left| F\left(\frac{n}{2}, \frac{n}{2}; n; u_{m,1}\right)\right| < S.$$

Similarly, it can be shown that

$$\left\{ F\left(\frac{n}{2}, \frac{n}{2}; n; |u_{m,i}|\right)\right\}_{m=0}^{\infty}, \quad i = 2, 3, 4,$$

are uniformly bounded, where $u_{m,2} = \frac{|P|^2}{|C_{2m+t}|^2}$, $u_{m,3} = \frac{|P|^2}{|C_{-2m-t}|^2}$ and $u_{m,4} = \frac{|P|^2}{|C_{-2m+t}|^2}$.

Next, consider the term $|C_{2m-t}|^{-n}$, $n > 1$.

$$\begin{aligned} |C_{2m-t}|^{-n} &= \left| |z|^2 + \left|z'\right|^2 + i\left(t' + 2m - t\right)\right|^{-n} \\ &= (m)^{-n} \left| \frac{|z|^2 + |z'|^2}{m} + i\left(\frac{t' - t}{m} + 2\right)\right|^{-n} \\ &\leq (2m)^{-n} \quad \text{for sufficiently large } m \end{aligned}$$

on any compact neighbourhood of $\xi = [z, t]$, for fixed $\eta = [z', t']$.

Consider

$$\begin{aligned} \left|H(2m - t)\right| &= \left| F\left(\frac{n}{2}, \frac{n}{2}; n; \frac{|Q|^2}{|C_{2m-t}|^2}\right) \cdot a_0 |C_{2m-t}|^{-n}\right| \\ &\leq S \cdot |a_0| (2m)^{-n}, \end{aligned}$$

on compact neighbourhood of ξ for fixed η. Since $m > 0$, the series $\sum_{m=1}^{\infty} (2m)^{-n}$ is convergent and therefore, $\sum_{m=0}^{\infty} H(2m - t)$ is uniformly convergent on compact neighbourhoods of ξ.

Similarly, $\sum_{m=0}^{\infty} H(2m + t)$, $\sum_{m=1}^{\infty} H(-2m - t)$, $\sum_{m=1}^{\infty} H(-2m + t)$ are uniformly convergent on compact neighbourhoods of ξ.

Define

$$G_I(\eta, \xi) = \sum_{m=0}^{\infty} \left[H(2m - t) - H(2m + t)\right] + \sum_{m=1}^{\infty} \left[H(-2m - t) - H(-2m + t)\right]. \tag{11}$$

For each η, $G_I(\eta, \xi)$ is a well-defined function. Now, we claim that $G_I(\eta, \xi)$ works as a Green's function for the domain I when applied to circular functions.

Theorem 4.1 *The function $G_I(\eta, \xi)$ is a smooth function on $I = \{\xi = [z', t'] \in \mathbb{H}_n : 0 < t' < 1\}$ and satisfies the following.*

(i) $L_0 G_I(\eta,\xi) = \delta_\eta$.
(ii) *The limits of the function* $G_I(\eta,\xi)$ *vanish at the boundaries of infinite strip i.e. at* $t' = 0$ *and* $t' = 1$.

Proof First note that for $m > 0$, $H(2m+t)$, $H(2m-t)$, $H(-2m-t)$ and $H(-2m+t)$ are all harmonic functions on I and $H(t)$ is also harmonic. We first apply the Laplacian term by term to the series (11)

$$L_0 G_I(\eta,\xi) = L_0 H(-t) = \delta_\eta.$$

It can easily be seen that for $t' = 0$, $|C_{-2m+t}|^2 = |C_{2m-t}|^2$.

So $H(-2m+t) = H(2m-t)$ for $m = 1,2,\ldots$. Similarly $H(2m+t) = H(-2m-t)$ for $m = 1,2,\ldots$ and $H(t) = H(-t)$. Therefore,

$$\lim_{t'\to 0}\left(\sum_{m=0}^{\infty}\bigl[H(2m-t) - H(2m+t)\bigr] + \sum_{m=1}^{\infty}\bigl[H(-2m-t) - H(-2m+t)\bigr]\right) = 0.$$

Moreover,

$$\begin{aligned}
&\lim_{t'\to 1}\left(\sum_{m=0}^{\infty}\bigl[H(2m-t) - H(2m+t)\bigr] + \sum_{m=1}^{\infty}\bigl[H(-2m-t) - H(-2m+t)\bigr]\right)\\
&\quad = \lim_{t'\to 1}\Biggl(\bigl(H(-2-t) - H(t)\bigr) + \sum_{m=0}^{\infty}\bigl[H(2m-t) - H\bigl(-2(m+1)+t\bigr)\bigr]\\
&\qquad + \sum_{m=1}^{\infty}\bigl[H\bigl(-2(m+1)-t\bigr) - H(2m+t)\bigr]\Biggr)\\
&\quad = \Biggl(\lim_{t'\to 1}\bigl(H(-2-t) - H(t)\bigr) + \sum_{m=0}^{\infty}\lim_{t'\to 1}\bigl[H(2m-t) - H\bigl(-2(m+1)+t\bigr)\bigr]\\
&\qquad + \sum_{m=1}^{\infty}\lim_{t'\to 1}\bigl[H\bigl(-2(m+1)-t\bigr) - H(2m+t)\bigr]\Biggr).
\end{aligned}$$

It can easily be seen that for $t' = 1$, $H(t) = H(-2-t)$, $H(2m-t) = H(-2(m+1)+t)$ for $m = 0,1,\ldots$ and $H(2m+t) = H(-2(m+1)-t)$ for $m = 1,2,\ldots$.

Therefore,

$$\lim_{t'\to 0}\left(\sum_{m=0}^{\infty}\bigl[H(2m-t) - H(2m+t)\bigr] + \sum_{m=1}^{\infty}\bigl[H(-2m-t) - H(-2m+t)\bigr]\right) = 0.$$

Hence the theorem. □

The Dirichlet BVP for circular data similar to that in Theorem 3.6 on I can be solved by obtaining a Poisson kernel on I.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors, MMM, AK and SD, contributed to each part of this work equally and read and approved the final version of the manuscript.

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Acknowledgements

The authors are thankful to PK Ratnakumar, Harish Chandra Research Institute, Allahabad, India, for his valuable discussion. The third author was supported by the Junior Research Fellowship of Council of Scientific and Industrial Research, India (Grant no. 09/045(1152)/2012-EMR-I). The authors would like to thank the referees for many suggestions, which helped in improving exposition.

Received: 2 December 2013 Accepted: 9 July 2014 Published online: 25 September 2014

References

1. Begehr, H, Venegas, CJ: Iterated Neumann problem for the higher order Poisson equation. Math. Nachr. **279**(1-2), 38-57 (2006)
2. Begehr, H, Vaitekhovich, T: Harmonic boundary value problems in half disc and half ring. Funct. Approx. Comment. Math. **40**(2), 251-282 (2009)
3. Courant, R, Hilbert, D: Methods of Mathematical Physics, vol. 1. Wiley, Weinheim (2008)
4. Folland, GB, Stein, EM: Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group. Commun. Pure Appl. Math. **27**, 429-522 (1974)
5. Folland, GB: A fundamental solution for a subelliptic operator. Bull. Am. Math. Soc. **79**, 373-376 (1973)
6. Korányi, A, Riemann, HM: Horizontal normal vectors and conformal capacity of spherical rings in the Heisenberg group. Bull. Sci. Math. Ser. (2) **111**, 3-21 (1987)
7. Korányi, A: Kelvin transforms and harmonic polynomials on the Heisenberg group. J. Funct. Anal. **49**, 177-185 (1982)
8. Greiner, PC, Koornwinder, TH: Variations on the Heisenberg harmonics. Report ZW 186/83, Mathematisch Centrum, Amsterdam (1983)
9. Korányi, A: Poisson formulas for circular functions and some groups of type H. Sci. China Ser. A **49**, 1683-1695 (2006)
10. Kumar, A, Mishra, MM: Green's functions on the Heisenberg group. Analysis **30**, 147-155 (2010)
11. Kumar, A, Mishra, MM: Green functions and related boundary value problems on the Heisenberg group. Complex Var. Elliptic Equ. **58**, 547-556 (2013)
12. Uguzzoni, F, Lanconelli, E: On the Poisson kernel for the Kohn Laplacian. Rend. Mat. Appl. **17**(4), 659-677 (1997)
13. Bonfiglioli, A, Lanconelli, E, Uguzzoni, F: Stratified Lie Groups and Potential Theory for Their Sub-Laplacians. Springer Monograph in Mathematics. Springer, Berlin (2007)
14. Rainville, ED: Special Functions. MacMillan, New York (1960)

doi:10.1186/s13661-014-0182-7
Cite this article as: Mishra et al.: Green's function for certain domains in the Heisenberg group $\mathbb{H}_n$. *Boundary Value Problems* 2014 **2014**:182.