

RESEARCH

Open Access

Existence and multiplicity results for a degenerate quasilinear elliptic system near resonance

Yu-Cheng An^{1,2}, Xiong Lu¹ and Hong-Min Suo^{1*}

*Correspondence: gzmymx88@sina.com
¹School of Sciences, Guizhou Minzu University, Guiyang, 550025, P.R. China
Full list of author information is available at the end of the article

Abstract

We establish existence and multiplicity results for weak solutions of a degenerate quasilinear elliptic system. By using Ekeland's variational principle, the mountain pass theorem and the saddle point theorem in critical point theory, we obtain the existence of one or three solutions for an elliptic system with Dirichlet boundary conditions under some restriction on λ . This kind of results was firstly obtained by Mawhin and Schmitt (Ann. Pol. Math. 51:241-248, 1990) for a semilinear two-point boundary value problem.

Keywords: degenerate elliptic system; Ekeland's variational principle; mountain pass theorem; saddle point theorem

1 Introduction and main results

In this paper, we consider the following degenerate quasilinear elliptic systems:

$$\begin{cases} -\nabla(h_1(x)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^\alpha|v|^\beta v + F_u(x, u, v), & x \in \Omega, \\ -\nabla(h_2(x)|\nabla v|^{q-2}\nabla v) = \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^\alpha|v|^\beta v + F_v(x, u, v), & x \in \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \quad (1)$$

where Ω is a bounded and connected subset of R^N ($N \geq 2$), λ is a nonnegative parameter, $F \in C^1(\overline{\Omega} \times R^2, R)$, $\nabla F = (F_u, F_v)$ denotes the gradient of F with respect to $(u, v) \in R^2$.

(f1) There exists a positive constant C_1 such that, for some $\theta \in (1, p)$, $\tau \in (1, q)$,

$$\begin{aligned} |F_u(x, u, v)| &\leq C_1(1 + |u|^{\theta-1} + |v|^{\tau(\theta-1)/\theta}), \\ |F_v(x, u, v)| &\leq C_1(1 + |v|^{\tau-1} + |u|^{\theta(\tau-1)/\tau}), \end{aligned}$$

for all $(u, v) \in R^2$ and $x \in \Omega$.

(H) $p > 1, q > 1, \alpha \geq 0, \beta \geq 0$, such that

$$\frac{\alpha + 1}{p} + \frac{\beta + 1}{q} = 1.$$

The degeneracy of this system is considered in the sense that the measurable, nonnegative diffusion coefficients h_1, h_2 are allowed to vanish in Ω (as well as at the boundary $\partial\Omega$)

and/or to blow up in $\overline{\Omega}$. The consideration of suitable assumptions on the diffusion coefficients will be based on [1], where the degenerate scalar equation was studied. We introduce the function space $(N)_p$ which consists of functions $h : \Omega \subset R^N \rightarrow R$, such that $h \in L^1(\Omega)$, $h^{-\frac{1}{p-1}} \in L^1(\Omega)$ and $h^{-s} \in L^1(\Omega)$, for some $p > 1$, $s > \max\{\frac{N}{p}, \frac{1}{p-1}\}$ satisfying $ps \leq N(s + 1)$.

Then for the weight functions h_1, h_2 we assume the following hypothesis:

- (N) There exist functions m_1 satisfying condition $(N)_p$, for some s_p , and m_2 satisfying condition $(N)_q$, for some s_q , such that

$$\frac{m_1(x)}{k_1} \leq h_1(x) \leq k_1 m_1(x), \quad \frac{m_2(x)}{k_2} \leq h_2(x) \leq k_2 m_2(x), \tag{2}$$

a.e. in Ω , for some constants $k_1 > 1$ and $k_2 > 1$.

The mathematical modeling of various physical processes, ranging from physics to biology, where spatial heterogeneity has a primary role, is reduced to nonlinear evolution equations with variable diffusion or dispersion. Note that problem (1) is closely related (see [1]) to the following system:

$$\begin{cases} -\nabla(h_1(x, u, v)|\nabla u|^{p-2}\nabla u) = f(\lambda, x, u, v, \nabla u, \nabla v), & x \in \Omega, \\ -\nabla(h_2(x, u, v)|\nabla v|^{q-2}\nabla v) = g(\lambda, x, u, v, \nabla u, \nabla v), & x \in \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases}$$

This system has successfully dealt with a variety of physical phenomena, such as the heat propagation in heterogeneous materials, the transport of electron temperature in a confined plasma, the propagation of varying amplitude waves in a nonlinear medium, the electromagnetic phenomena in nonhomogeneous superconductors, the dynamics of Josephson functions, and the spread of microorganisms. For more details as regards this type of system, see [2–4] and the references therein.

An example of the physical motivation of the assumptions (N), $(N)_p$ may be found in [3]. These assumptions are related to the modeling of reaction diffusion processes in composite materials occupying a bounded domain Ω , which at some points behave as perfect insulators. When at some points the medium is perfectly insulating, it is natural to assume that $h_1(x)$ and/or $h_2(x)$ vanish in Ω . For more information, we refer the interesting reader to [4] and the references therein.

For the perturbed problem, Mawhin and Schmitt [5] first considered the two-point boundary value problem

$$-u'' - \lambda u = f(x, u) + h(x), \quad u(0) = u(\pi) = 0. \tag{3}$$

Under the assumption that f is bounded and satisfies a sign condition, if the parameter λ is sufficiently close to λ_1 from left, problem (3) has at least three solutions, if $\lambda_1 \leq \lambda < \lambda_2$, problem (3) has at least one solution, where λ_1, λ_2 are the first, second eigenvalues of the corresponding linear problem. Ma *et al.* [6] considered the boundary value problem $\Delta u + \lambda u + f(x, u) = h(x)$ defined on a bounded open set $\Omega \subset R^N$, no matter whether the boundary conditions are Dirichlet or Neumann conditions, as the parameter λ approaches λ_1 from left, there exist three solutions. Moreover, the existence of three solutions was obtained for the quasilinear problem in bounded domains as the parameter λ approaches λ_1 from left in [7, 8], these results were extended to the perturbed p -Laplacian equation

in R^N . Especially, the existence of three solutions has been extended to cooperative elliptic systems in [9]. Motivated by the above idea, we have the goal in this paper of extending these results to some degenerate quasilinear elliptic systems with the Dirichlet boundary conditions.

Next, we recall the basic results of the following eigenvalue problem (for details see [2]):

$$\begin{cases} -\nabla(h_1(x)|\nabla u|^{p-2}\nabla u) = \lambda a(x)|u|^{p-2}u + \lambda b(x)|u|^\alpha|v|^\beta v, & x \in \Omega, \\ -\nabla(h_2(x)|\nabla v|^{q-2}\nabla v) = \lambda d(x)|v|^{q-2}v + \lambda b(x)|u|^\alpha|v|^\beta u, & x \in \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \quad (4)$$

where h_1, h_2 satisfy (N), and the coefficient functions a, d , and b satisfy conditions (A), (D), and (B), respectively (the conditions (A), (D), (B) see Section 2). Zographopoulos proved the theorem below.

Theorem 1.1 *Let Ω be a bounded domain of R^N ($N \geq 2$). Assume that hypotheses (H) and (N) are satisfied and the coefficient functions a, d and b satisfy conditions (A), (D) and (B), respectively. Denote by L the set*

$$L = \left\{ (u, v) \in Z : \frac{\alpha + 1}{p} \int_{\Omega} a(x)|u|^p dx + \frac{\beta + 1}{q} \int_{\Omega} d(x)|v|^q dx + \int_{\Omega} b(x)|u|^\alpha|v|^\beta uv dx = 1 \right\},$$

where $Z = D_0^{1,p}(\Omega, h_1) \times D_0^{1,q}(\Omega, h_2)$ (we will determine in Section 2 what Z means). Then the system (4) admits a positive principal eigenvalue λ_1 , satisfying

$$\lambda_1 = \inf_{(u,v) \in L} \left\{ \frac{\alpha + 1}{p} \int_{\Omega} h_1(x)|\nabla u|^p dx + \frac{\beta + 1}{q} \int_{\Omega} h_2(x)|\nabla v|^q dx \right\}. \quad (5)$$

The associated normalized eigenfunction $\phi_1 = (u_1, v_1)$ belongs to Z and each component is nonnegative. In addition,

- (i) the set of all eigenfunctions corresponding to the principal eigenvalue λ_1 forms a one-dimensional manifold $E_1 \subset Z$, which is defined by

$$E_1 = \{ (c_1 u_1, c_1^{p/q} v_1) : c_1 \in R \}.$$

- (ii) λ_1 is the only eigenvalue of (4) to which corresponds a componentwise nonnegative eigenfunction.
- (iii) λ_1 is isolated in the following sense: there exists $\eta > 0$, such that the interval $(0, \lambda_1 + \eta)$ does not contain any other eigenvalue than λ_1 .

Now we are ready to state our main results. For the related definitions we refer to Section 2.

Theorem 1.2 *Let the hypotheses of Theorem 1.1 and (f1) be satisfied. In addition,*

$$\lim_{|t| \rightarrow \infty} F(x, t\phi_1) = +\infty \quad (6)$$

uniformly in $x \in \Omega$. Then, for $\lambda < \lambda_1$ sufficiently close to λ_1 , the system (1) has at least three solutions, where $\lambda_1 > 0$ is the first eigenvalue of the system (4).

Theorem 1.3 *Let the hypotheses of Theorem 1.1 and (f1) be satisfied. In addition,*

$$\lim_{|(u,v)| \rightarrow \infty} \left(\frac{1}{p} F_u(x, u, v) + \frac{1}{q} F_v(x, u, v) - F(x, u, v) \right) = \infty \tag{7}$$

uniformly in $x \in \Omega$. Then for $\lambda \in (\lambda_1, \lambda_2)$, the system (1) has at least one solution (we will determine in Section 3 what λ_2 mean).

2 Space and operator setting

Let $h(x)$ be a nonnegative weight function in Ω which satisfies condition $(N)_p$. We consider the weighted Sobolev space $D_0^{1,p}(\Omega, h)$ to be defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{D_0^{1,p}(\Omega, h)} = \left(\int_{\Omega} h(x) |\nabla u|^p dx \right)^{1/p}.$$

The space $D_0^{1,p}(\Omega, h)$ is a reflexive Banach space. For a discussion of the space setting we refer to [1] and the references therein. Let

$$p_s^* = \frac{Nps}{N(s+1) - ps}.$$

Lemma 2.1 [1] *Assume that Ω is a bounded domain in R^N and the weight h satisfies $(N)_p$. Then the following embeddings hold:*

- (I) $D_0^{1,p}(\Omega, h) \hookrightarrow L^{p_s^*}(\Omega)$ continuously for $1 < p_s^* < N$,
- (II) $D_0^{1,p}(\Omega, h) \hookrightarrow L^r(\Omega)$ compactly for any $r \in [1, p_s^*]$.

In the sequel we denote by p^* and q^* the quantities $p_{s_p}^*$ and $q_{s_q}^*$, respectively, where s_p and s_q are induced by condition (N), recall that h_1, h_2 satisfy (N). The assumptions concerning the coefficient functions of systems (1) and systems (4) are the following.

- (A) $a \in L^{\frac{p^*}{p^*-p}}(\Omega)$ and either there exists $\Omega_a^+ \subset \Omega$ of positive Lebesgue measure, i.e., $|\Omega_a^+| > 0$, such that $a(x) > 0$, for all $x \in \Omega_a^+$, neither $a(x) \equiv 0$, in Ω .
- (D) $d \in L^{\frac{q^*}{q^*-p}}(\Omega)$ and either there exists $\Omega_d^+ \subset \Omega$ of positive Lebesgue measure, i.e., $|\Omega_d^+| > 0$, such that $d(x) > 0$, for all $x \in \Omega_d^+$, neither $d(x) \equiv 0$, in Ω .
- (B) $b(x) \geq 0$, a.e. in Ω , $b \not\equiv 0$ and $b \in L^\omega(\Omega)$, where $\omega = (1 - \frac{\alpha+1}{p^*} - \frac{\beta+1}{q^*})^{-1}$.

The space setting for our problem is the product space $Z = D_0^{1,p}(\Omega, h_1) \times D_0^{1,q}(\Omega, h_2)$ equipped with the norm

$$\|z\| = \|u\|_{D_0^{1,p}(\Omega, h_1)} + \|v\|_{D_0^{1,q}(\Omega, h_2)}, \quad z = (u, v) \in Z.$$

Observe that inequalities (2) in condition (N) imply that the functional spaces $D_0^{1,p}(\Omega, h_1) \times D_0^{1,q}(\Omega, h_2)$ and $D_0^{1,p}(\Omega, m_1) \times D_0^{1,q}(\Omega, m_2)$ are equivalent. Next, let us introduce the functionals $I, J : Z \rightarrow R$ in the following way:

$$I(u, v) = \frac{\alpha+1}{p} \int_{\Omega} h_1(x) |\nabla u|^p dx + \frac{\beta+1}{q} \int_{\Omega} h_2(x) |\nabla v|^q dx,$$

$$J(u, v) = \frac{\alpha+1}{p} \int_{\Omega} a(x) |u|^p dx + \frac{\beta+1}{q} \int_{\Omega} d(x) |v|^q dx + \int_{\Omega} b(x) |u|^\alpha |v|^\beta uv dx.$$

It is a standard procedure (see [10], Lemma 2.1) to prove the following properties of the functionals.

Lemma 2.2 *The functionals I, J are well defined. Moreover, I is continuous and J is compact.*

Notation For simplicity we use the symbol $\|\cdot\|_{h_1}$ for the norm $\|\cdot\|_{D_0^{1,p}(\Omega, h_1)}$ and $\|\cdot\|_{h_2}$ for the norm $\|\cdot\|_{D_0^{1,q}(\Omega, h_2)}$.

3 Proof of theorems

Let $\Phi_\lambda : Z \rightarrow \mathbb{R}$ be the functional defined by

$$\Phi_\lambda(u, v) = I(u, v) - \lambda J(u, v) - N(u, v), \quad \forall (u, v) \in Z,$$

where

$$N(u, v) = \int_\Omega F(x, u, v) \, dx.$$

Since the potential F has sublinear growth, by a standard argument, it follows that $\Phi_\lambda \in C^1(Z, \mathbb{R})$. In addition, $(u, v) \in Z$ is a weak solution of systems (1) if and only if (u, v) is a critical point of Φ_λ .

By (5) we can deduce that

$$I(u, v) \geq \lambda_1 J(u, v), \quad \forall (u, v) \in Z. \tag{8}$$

Let $V = \text{span}\{\phi_1\}$ and $W = \{z \in Z : \langle J'(\phi_1), z \rangle = 0\}$, we have from the simplicity of λ_1 , $Z = V \oplus W$. Then, since λ_1 is also isolated, we have

$$\lambda_2 = \inf_{(u,v) \in W} \{I(u, v) : J(u, v) = 1\},$$

which satisfies $\lambda_1 < \lambda_2$. In addition,

$$I(u, v) \geq \lambda_2 J(u, v), \quad \forall (u, v) \in W. \tag{9}$$

On the other hand, from the condition (f1) and by Young's inequality, for every fixed $(u_0, v_0) \in \mathbb{R}^2$, it follows that

$$\begin{aligned} & |F(x, u_0, v_0)| \\ &= \left| \int_0^1 (F_u(x, ru_0, rv_0)u_0 + F_v(x, ru_0, rv_0)v_0) \, dr + F(x, 0, 0) \right| \\ &\leq \int_0^1 (|F_u(x, ru_0, rv_0)||u_0| + |F_v(x, ru_0, rv_0)||v_0|) \, dr + |F(x, 0, 0)| \\ &\leq C_1 \int_0^1 (|u_0| + |u_0|^\theta r^{\theta-1} + |rv_0|^{\tau(\theta-1)/\theta} |u_0| + |v_0| + |v_0|^\tau r^{\tau-1} \\ &\quad + |ru_0|^{\theta(\tau-1)/\tau} |v_0|) \, dr + |F(x, 0, 0)| \end{aligned}$$

$$\begin{aligned} &\leq |F(x, 0, 0)| + C_1 \left(|u_0| + |v_0| + \frac{1}{\theta} |u_0|^\theta + \frac{1}{\tau} |u_0|^\tau \right. \\ &\quad \left. + \frac{\theta}{\theta + \tau(\theta - 1)} |v_0|^{\tau(\theta-1)/\theta} |u_0| + \frac{\tau}{\tau + \theta(\tau - 1)} |u_0|^{\theta(\tau-1)/\tau} |v_0| \right) \\ &\leq C_2 (|u_0| + |v_0| + |u_0|^\theta + |v_0|^\tau + |F(x, 0, 0)|), \end{aligned} \tag{10}$$

where C_2 is a positive constant independent to (u_0, v_0) , therefore, from Lemma 2.1, we have

$$\begin{aligned} \left| \int_{\Omega} F(x, u, v) dx \right| &\leq \int_{\Omega} |F(x, u, v)| dx \\ &\leq C_2 \int_{\Omega} (|u| + |v| + |u|^\theta + |v|^\tau) dx + C_2 \int_{\Omega} |F(x, 0, 0)| dx \\ &\leq C_2 (\|u\|_{L^1} + \|v\|_{L^1} + \|u\|_{L^\theta}^\theta + \|v\|_{L^\tau}^\tau) + C_4 \\ &\leq C_3 (\|u\|_{h_1} + \|v\|_{h_2} + \|u\|_{h_1}^\theta + \|v\|_{h_2}^\tau) + C_4 \end{aligned} \tag{11}$$

for all $(u, v) \in Z$, where $C_4 = C_2 \int_{\Omega} |F(x, 0, 0)| dx$ and C_3 is a positive constant.

Next, we will prove Theorem 1.2 by using Ekeland's variational principle and the mountain pass theorem and Theorem 1.3 by using the saddle point theorem.

Proof of Theorem 1.2 The proof will be divided into four steps.

Step 1. The functional Φ_λ is coercive in Z , Φ_λ is bounded from below on W and there is a constant M , independent of λ , such that $\inf_{z \in W} \Phi_\lambda(z) \geq M$.

For $\lambda < \lambda_1$, from (8) and (11), we get

$$\begin{aligned} \Phi_\lambda(u, v) &\geq a_1 \left(1 - \frac{\lambda}{\lambda_1} \right) (\|u\|_{h_1}^p + \|v\|_{h_2}^q) \\ &\quad - C_3 (\|u\|_{h_1} + \|v\|_{h_2} + \|u\|_{h_1}^\theta + \|v\|_{h_2}^\tau) - C_4, \end{aligned} \tag{12}$$

for all $(u, v) \in Z$, where $a_1 = \min\{(1 + \alpha)/p, (1 + \beta)/q\} > 0$, which shows that Φ_λ is coercive in Z .

Similarly, from (9) and (11), we obtain

$$\begin{aligned} \Phi_\lambda(u, v) &\geq a_1 \left(1 - \frac{\lambda_1}{\lambda_2} \right) (\|u\|_{h_1}^p + \|v\|_{h_2}^q) \\ &\quad - C_3 (\|u\|_{h_1} + \|v\|_{h_2} + \|u\|_{h_1}^\theta + \|v\|_{h_2}^\tau) - C_4, \end{aligned} \tag{13}$$

for all $(u, v) \in W$. Hence Φ_λ is coercive in W and Φ_λ is bounded from below on W . Moreover, there is a constant M , independent of λ , such that $\inf_{z \in W} \Phi_\lambda(z) \geq M$.

Step 2. If the parameter λ is sufficiently close to λ_1 from the left, we have $t^- < 0 < t^+$ such that $\Phi_\lambda(t^\pm \phi_1) < M$.

We choose $t^+ > 0$ sufficiently large, we get from (6) that $N(t^+ \phi_1) > -M + 1$, so that

$$\begin{aligned} \Phi_\lambda(t^+ \phi_1) &< \left(1 - \frac{\lambda}{\lambda_1} \right) I(t^+ \phi_1) + M - 1 \\ &\leq a_2 \left(1 - \frac{\lambda}{\lambda_1} \right) (\|t^+ u_1\|_{h_1}^p + \|t^+ v_1\|_{h_2}^q) + M - 1, \end{aligned} \tag{14}$$

where $a_2 = \max\{(1 + \alpha)/p, (1 + \beta)/q\}$. Then for λ sufficiently close to λ_1 from the left, $\Phi_\lambda(t^+ \phi_1) < M$. The same conclusion holds for a $t^- < 0$.

Step 3. If $\lambda < \lambda_1$, the functional Φ_λ satisfies the (P.S.) condition. In addition, let

$$\Theta^\pm = \{z \in Z : z = \pm t\phi_1 + \xi, \text{ with } t > 0 \text{ and } \xi \in W\},$$

Φ_λ satisfies $(P.S.)_{c, \Theta^+}$ and $(P.S.)_{c, \Theta^-}$ for all $c < M$.

On one hand, if (u_n, v_n) be a (P.S.) sequence of Φ_λ , that is,

$$\Phi_\lambda(u_n, v_n) \rightarrow c \in R, \quad \|\Phi'_\lambda(u_n, v_n)\|_{H^*} \rightarrow 0 \tag{15}$$

as $n \rightarrow \infty$. From Step 1 and (15), (u_n, v_n) must be bounded in Z , that is, there is $K > 0$ such that

$$\|u_n\|_{h_1} + \|v_n\|_{h_2} \leq K \tag{16}$$

for all $n \in N$. Thus, there is a subsequence of (u_n, v_n) , without any loss of generality still denoted by (u_n, v_n) , and $(u, v) \in Z$ such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in Z , $(u_n, v_n) \rightarrow (u, v)$ strongly in $L^p(\Omega) \times L^q(\Omega)$. Consequently, from (15) and (16), one has

$$\lim_{n \rightarrow \infty} \langle \Phi'_\lambda(u_n, v_n), (u_n - u, 0) \rangle = 0. \tag{17}$$

From the condition (f1), Hölder's inequality, Lemma 2.1, and (16), it follows that

$$\begin{aligned} \left| \int_\Omega F_u(x, u_n, v_n)(u_n - u) dx \right| &\leq C_1 \int_\Omega (1 + |u_n|^{\theta-1} + |v_n|^{\tau(\theta-1)/\theta}) |u_n - u| dx \\ &\leq C_1 (\|u_n\|_{L^\theta}^{\theta-1} + \|v_n\|_{L^\tau}^{\tau(\theta-1)/\theta}) \|u_n - u\|_{L^\theta} \\ &\quad + C_1 |\Omega|^{(p-1)/p} \|u_n - u\|_{L^p} \\ &\leq C_1 S (\|u_n\|_{h_1}^{\theta-1} + \|v_n\|_{h_2}^{\tau(\theta-1)/\theta}) \|u_n - u\|_{L^\theta} \\ &\quad + C_1 |\Omega|^{(p-1)/p} \|u_n - u\|_{L^p} \\ &\rightarrow 0 \end{aligned} \tag{18}$$

as $n \rightarrow \infty$, where S is the embedding constant. Combining (17), (18), and Lemma 2.2, we get

$$\lim_{n \rightarrow \infty} \int_\Omega h_1(x) |\nabla u_n|^{p-2} \nabla u_n (\nabla u_n - \nabla u) dx = 0.$$

Similarly, we also obtain

$$\lim_{n \rightarrow \infty} \int_\Omega h_1(x) |\nabla u|^{p-2} \nabla u (\nabla u_n - \nabla u) dx = 0,$$

hence, we conclude that

$$\lim_{n \rightarrow \infty} \int_\Omega h_1(x) (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u) (\nabla u_n - \nabla u) dx = 0.$$

Therefore, using a well-known lemma, see for instance [11, Lemma 3.1] or [12, Lemma 4.1], we get $u_n \rightarrow u$ in $D_0^{1,p}(\Omega, h_1)$ as $n \rightarrow \infty$. Similarly, we get $v_n \rightarrow v$ in $D_0^{1,q}(\Omega, h_2)$ as $n \rightarrow \infty$, that is, (u_n, v_n) has a convergent subsequence.

On the other hand, let $\{z_n\} \subset \Theta^+$ satisfy $\Phi_\lambda(z_n) \rightarrow c < M$ and $\Phi'_\lambda(z_n) \rightarrow 0$ as $n \rightarrow \infty$. Since Φ_λ is coercive and the potential F satisfies the condition (f1), there is $z \in Z$ such that $z_n \rightarrow z$ strongly in Z . If $z \in \partial\Theta^+ = W$, from the second conclusion of Step 1, we get $\Phi_\lambda(z_n) \rightarrow c \geq M$, which is impossible. Hence $z \in \Theta^+$ and Φ_λ satisfies the $(P.S.)_{c, \Theta^+}$ condition. Similarly we see that $(P.S.)_{c, \Theta^-}$ holds for all $c < M$.

Step 4. Three solutions are obtained.

If $\lambda < \lambda_1$ is sufficiently close λ_1 , from Step 1 and Step 2, we get $-\infty < \inf_{\Theta^\pm} \Phi_\lambda < M$, which implies that Φ_λ is bounded below in Θ^+ . Consequently, from Ekeland's variational principle, there exists $\{z_n\} \subset \Theta^+$ such that $\Phi_\lambda(z_n) \rightarrow \inf_{\Theta^+} \Phi_\lambda$ and $\Phi'_\lambda(z_n) \rightarrow 0$ as $n \rightarrow \infty$. Since Φ_λ satisfies $(P.S.)_{c, \Theta^+}$ for all $c < M$, there is $z^+ \in \Theta^+$ such that $\Phi_\lambda(z) = \inf_{\Theta^+} \Phi_\lambda$, that is, the infimum is attained in Θ^+ . A similar conclusion holds in Θ^- . So Φ_λ has two distinct critical points, denoted by z^+ and z^- .

To fix ideas, suppose that $\Phi_\lambda(z^-) \leq \Phi_\lambda(z^+)$, if z^+ is not an isolated critical point, then Φ_λ has at least three solutions. Otherwise, putting

$$\Psi(z) = \Phi_\lambda(z + z^+) - \Phi_\lambda(z^+), \quad e = z^- - z^+,$$

we have $\Psi(0) = 0$, $\Psi(e) \leq 0$, and there exist $r > 0$, $\rho > 0$ such that $\Psi(z) \geq \rho$ if $\|z\| = r$. Then, since $\Phi'_\lambda = \Psi'$ and Ψ also satisfies the $(P.S.)$ condition, from the mountain pass theorem, the number

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi_\lambda(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1], W) : \gamma(0) = z^+, \gamma(1) = z^-\},$$

is a critical value of Φ_λ . Noting that all paths joining z^+ to z^- pass through W , we have $c \geq M$. Therefore we have obtained a third critical point of Φ_λ . The proof is now complete. \square

Proof of Theorem 1.3 The proof will be divided into two steps.

Step 1 (the $(P.S.)_c$ condition). Let $\{z_n = (u_n, v_n)\} \subset Z$ be such that there exists $c > 0$ such that

$$|\Phi_\lambda(z_n)| \leq c, \quad \forall n \in N. \tag{19}$$

and there exists a strictly decreasing sequence $\{\varepsilon_n\}_{n=1}^\infty$ with $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ such that

$$|\langle \Phi'_\lambda(z_n), z \rangle| \leq \varepsilon_n \|z\|, \quad \forall z \in Z. \tag{20}$$

We first prove that $\{z_n\}$ is bounded in Z , then by a standard argument, $\{z_n\}$ has a convergent subsequence. Suppose by contradiction that $\|z_n\| \rightarrow \infty$, we have

$$\begin{aligned} c + \varepsilon_n \| (u_n, v_n) \| &\geq \left| \Phi_\lambda(u_n, v_n) - \left\langle \Phi'_\lambda \left((u_n, v_n), \left(\frac{u_n}{p}, \frac{v_n}{q} \right) \right) \right\rangle \right| \\ &= \left| \int_\Omega \left(\frac{1}{p} F_u(x, u_n, v_n) + \frac{1}{q} F_v(x, u_n, v_n) - F(x, u_n, v_n) \right) dx \right|. \end{aligned}$$

Letting n tend to infinity, we get

$$\lim_{n \rightarrow \infty} \left| \int_\Omega \left(\frac{1}{p} F_u(x, u_n, v_n) + \frac{1}{q} F_v(x, u_n, v_n) - F(x, u_n, v_n) \right) dx \right| < C,$$

where C is a constant, which contradicts (7). Thus, $\{z_n\}$ is bounded. Since the nonlinearity F satisfies the conditions (f1), by a standard argument, $\{z_n\}$ has a convergent subsequence.

Step 2 (the saddle point theorem). It is well known that the (P.S.) condition can be replaced by the (P.S.)_c condition in the saddle point theorem of Rabinowitz (see [13, 14]). Then to conclude that Φ_λ has a critical point it suffices to show that

$$\lim_{\|z\| \rightarrow \infty} \Phi_\lambda(z) = -\infty, \quad \lim_{\|w\| \rightarrow \infty} \Phi_\lambda(w) = +\infty, \tag{21}$$

where $z \in V = \text{span}\{\phi_1\}$ and $w \in W$.

For $\lambda_1 < \lambda < \lambda_2$, from (11), we get

$$\begin{aligned} \Phi_\lambda(t\phi_1) &= I(t\phi_1) - \lambda J(t\phi_1) - N(t\phi_1) \\ &= \left(1 - \frac{\lambda}{\lambda_1} \right) I(t\phi_1) - N(t\phi_1) \\ &\leq a_1 \left(1 - \frac{\lambda}{\lambda_1} \right) (\|tu_1\|_{h_1}^p + \|tv_1\|_{h_2}^q) \\ &\quad + C_3 (\|tu_1\|_{h_1} + \|tv_1\|_{h_2} + \|tu_1\|_{h_1}^\theta + \|tv_1\|_{h_2}^\tau) + C_4. \end{aligned}$$

Since $\theta \in (1, p)$, $\tau \in (1, q)$, the first part of (21) holds.

For $\lambda_1 < \lambda < \lambda_2$ and for any $w \in W$, from (13), we obtain the second statement of (21). The proof is now complete. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

¹School of Sciences, Guizhou Minzu University, Guiyang, 550025, P.R. China. ²School of Sciences, Bijie University, Bijie, 551700, P.R. China.

Acknowledgements

This work was supported by the Science and Technology Foundation of Guizhou Province (No. LKB[2012]19; No. [2013]2141).

References

1. Drábek, P, Kufner, A, Nicolosi, F: Quasilinear Elliptic Equations with Degenerations and Singularities. de Gruyter, Berlin (1997)
2. Zographopoulos, NB: On the principal eigenvalue of degenerate quasilinear elliptic systems. *Math. Nachr.* **281**, 1351-1365 (2008)
3. Dautray, R, Lions, JL: *Mathematical Analysis and Numerical Methods for Science and Technology*. Springer, Berlin (1990)
4. Karachalios, NI, Zographopoulos, NB: On the dynamics of a degenerate parabolic equation: global bifurcation of stationary states and convergence. *Calc. Var. Partial Differ. Equ.* **25**, 361-393 (2006)
5. Mawhin, J, Schmitt, K: Nonlinear eigenvalue problems with the parameter near resonance. *Ann. Pol. Math.* **51**, 241-248 (1990)
6. Ma, TF, Ramost, M, Sanchez, L: Multiple solutions for a class of nonlinear boundary value problem near resonance: a variational approach. *Nonlinear Anal.* **30**, 3301-3311 (1997)
7. Ma, TF, Pelicer, ML: Perturbations near resonance for the p -Laplacian in R^N . *Abstr. Appl. Anal.* **7**, 323-334 (2002)
8. Ma, TF, Ramost, M: Three solutions of a quasilinear elliptic problem near resonance. *Math. Slovaca* **47**, 451-457 (1997)
9. Ou, ZQ, Tang, CL: Existence and multiplicity results for some elliptic systems at resonance. *Nonlinear Anal.* **71**, 2660-2666 (2009)
10. Drábek, P, Stavrakakis, NM, Zographopoulos, NB: Multiple nonsemitrivial solutions for quasilinear elliptic systems. *Differ. Integral Equ.* **16**, 1519-1532 (2003)
11. Brock, F, Iturriaga, J, Sánchez, J, Ubilla, P: Existence of positive solution for p -Laplacian problems with weight. *Commun. Pure Appl. Anal.* **5**, 941-952 (2006)
12. Ghoussoub, N, Yuan, C: Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents. *Trans. Am. Math. Soc.* **352**, 5703-5743 (2000)
13. Bartolo, P, Benci, V, Fortunato, D: Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity. *Nonlinear Anal.* **7**, 981-1012 (1983)
14. Rabinowitz, PH: *Minimax Methods in Critical Point Theory with Applications to Differential Equations*. CBMS (1986)

doi:10.1186/s13661-014-0184-5

Cite this article as: An et al.: Existence and multiplicity results for a degenerate quasilinear elliptic system near resonance. *Boundary Value Problems* 2014 **2014**:184.

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Immediate publication on acceptance
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com
