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Nonlinear fourth order boundary value problem

Şeyhmus Yardımcı* and Ekin Uğurlu

*Correspondence:
yardimci@science.ankara.edu.tr
Department of Mathematics,
Ankara University, Tandoğan,
Ankara, 06100, Turkey

Abstract
In this paper we consider a nonlinear boundary value problem generated by a fourth order differential equation on the semi-infinite interval in which the lim-4 case holds for fourth order differential expression at infinity. Using the well-known Banach and Schauder fixed point theorems we prove the existence and uniqueness theorems for the nonlinear boundary value problem.
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1 Introduction

In the literature a kind of first order nonlinear boundary value problems is of the form

$$y' = A(x)y + F(x,y), \tag{1}$$

$$Ty = r, \tag{2}$$

where A is a $n \times n$ matrix defined on some interval $I \subset \mathbb{R}$, F is a $n \times 1$ vector which is continuous on $I \times S$, $S \subset \mathbb{R}^n$, T is a bounded linear operator defined on the space of bounded and continuous $\mathbb{R}^n$-valued functions on I and r is a $n \times 1$ vector in $\mathbb{R}^n$. Existence and uniqueness theorems of the solutions of the problem (1), (2) have been obtained in many papers. These results can be found in [1] and references therein. Similar nonlinear boundary value problem has been studied by Agarwal *et al.* [2] on a time scale $[0,\infty)_{\mathbb{T}} = [0,\infty) \cap \mathbb{T}$ as

$$y^{\Delta} = A(x)y + F(x,y), \tag{3}$$

$$Ly = l, \tag{4}$$

where A is a $n \times n$ bounded matrix, L is a bounded linear operator on $[0,\infty)_{\mathbb{T}}$ and l is a $n \times 1$ vector in $\mathbb{R}^n$. They have introduced existence results for the problem (3), (4).

Second order nonlinear boundary value problems have been investigated by many authors. For example, Baxley has considered second order nonlinear boundary value problem on the semi-infinite interval $[0,\infty)$ as

$$y'' = f(x,y,y'),$$

$$y(0) = y_0, \qquad y'(0) = y_1,$$

as well as on the interval $[0,b]$, where $b<\infty$ [3] (further see [4]). We should note that there are several works in the field of the existence and uniqueness theorems of the second order nonlinear boundary value problems. Some of them can be found in, for example, [5–10]. In particular, in [11] Guseinov and Yaslan have investigated the existence and uniqueness of the solutions of the second order nonlinear boundary value problem on the semi-infinite interval $[0,\infty)$

$$-\bigl(p(x)y'\bigr)' + q(x)y = f(x,y), \quad x \in [0,\infty), \tag{5}$$

$$\alpha y(0) + \beta\bigl(py'\bigr)(0) = d_1, \qquad \gamma W_\infty(y,u) + \delta W_\infty(y,v) = d_2, \tag{6}$$

where p, q are real-valued, measurable functions on $[0,\infty)$ such that Weyl's limit-circle case holds [12–15] for the equation

$$-\bigl(p(x)y'\bigr)' + q(x)y = 0, \quad x \in [0,\infty), \tag{7}$$

α, β, γ, δ are real numbers satisfying $\alpha\delta - \beta\gamma \neq 0$ and d_1, d_2 are arbitrary real numbers, $W_x(y,z)$ denotes the Wronskian of the solutions of (7) and u and v are the solutions of (7). Also they have studied the following nonlinear boundary value problem on the infinite interval $(-\infty,\infty)$:

$$-\bigl(p(x)y'\bigr)' + q(x)y = f(x,y), \quad x \in (-\infty,\infty),$$

$$\alpha W_{-\infty}(y,u) + \beta W_{-\infty}(y,v) = d_1, \qquad \gamma W_\infty(y,u) + \delta W_\infty(y,v) = d_2,$$

where p, q are real-valued, measurable functions on $(-\infty,\infty)$ such that Weyl's limit-circle case holds for the equation

$$-\bigl(p(x)y'\bigr)' + q(x)y = 0, \quad x \in (-\infty,\infty), \tag{8}$$

α, β, γ, δ are real numbers satisfying $\alpha\delta - \beta\gamma \neq 0$ and d_1, d_2 are arbitrary real numbers, $W_x(y,z)$ denotes the Wronskian of the solutions of (8) and u and v are the solutions of (8).

In fact, Weyl showed that [15] at least one of the linearly independent solutions of the equation

$$-\bigl(p(x)y'\bigr)' + q(x)y = \lambda y \tag{9}$$

is in a squarely integrable space on $[0,\infty)$, where λ is a complex parameter. This result follows from the convergence of the corresponding nested circles. These circles either converge to a circle or a point. In the primary case, two linearly independent solutions of (9) and any combinations of them belong to the squarely integrable space and (9) is said to be the limit-circle case. Otherwise (9) is said to be of limit-point case. These Weyl's results have been generalized to the fourth order case as well as $2n$th order case by Everitt [16–18] (further see [13]). Moreover, we should note that limit-point/circle classifications does not depend on the spectral parameter λ. Using these ideas we generalize the results of (5), (6) to the fourth order case as given in (10), (13)-(16). Using Banach and Schauder fixed point theorems we establish the existence and uniqueness theorems for the singular fourth order nonlinear boundary value problem (10), (13)-(16) in the lim-4 case.

2 Nonlinear problem

We consider the fourth order nonlinear differential equation as

$$\bigl(q_2(x)y^{(2)}\bigr)^{(2)} - \bigl(q_1(x)y^{(1)}\bigr)^{(1)} + q_0(x)y = f(x,y), \tag{10}$$

where $x \in I := [0,\infty)$, y is the desired solution, q_0, q_1, $q_1^{(1)}$, q_2, $q_2^{(1)}$, $q_2^{(2)}$ are the real-valued, continuous functions and $q_2 > 0$ on I. Further we assume that $f(x,t)$ is real-valued and continuous function on $(x,t) \in I \times \mathbb{R}$, and for $(x,t) \in I \times \mathbb{R}$, $f(x,t)$ satisfies the following condition:

$$\bigl|f(x,t)\bigr| \le g(x) + a|t|, \tag{11}$$

where $g(x) \ge 0$ and $a > 0$.

It is well known that the rth quasi-derivative of a function y can be defined as follows [13]:

$$\begin{aligned}
&y^{[0]} = y,\\
&y^{[1]} = y^{(1)},\\
&y^{[2]} = q_2 y^{(2)},\\
&y^{[3]} = q_1 y^{(1)} - \bigl(q_2 y^{(2)}\bigr)^{(1)},\\
&y^{[4]} = q_0 y - \bigl(q_1 y^{(1)}\bigr)^{(1)} + \bigl(q_2 y^{(2)}\bigr)^{(2)}.
\end{aligned}$$

Therefore (10) can be rewritten as

$$y^{[4]} = f(x,y).$$

Let $L^2(I)$ denote the Hilbert space consisting of all real-valued functions y such that $\int_0^\infty |y(x)|^2\,dx < \infty$ with the inner product $(y,\chi) = \int_0^\infty y(x)\chi(x)\,dx$ and the norm $\|y\| = (y,y)^{\frac{1}{2}}$.

We assume that $g \in L^2(I)$ and the lim-4 case conditions are satisfied for the equation $y^{[4]} = 0$, $x \in I$ [15–18]. In other words, we assume that four linearly independent solutions of $y^{[4]} = 0$, $x \in I$, belong to $L^2(I)$. In the literature there are sufficient conditions in which the lim-4 case holds for $y^{[4]} = 0$, $x \in I$ [19–22]. For example, in 1977, Eastham [21] proved that the equation

$$\bigl(cx^{\gamma} y^{(2)}\bigr)^{(2)} + \bigl(ax^{\alpha} y^{(1)}\bigr)^{(1)} + bx^{\beta} y = 0, \quad x \in [0,\infty),$$

has four linearly independent solutions belonging to $L^2[0,\infty)$ if

$$\alpha = \gamma = \beta + 2 \quad \text{and} \quad \frac{b}{a} > \frac{\beta}{2} + \frac{1}{4}$$

or

$$\alpha - \beta = 2, \qquad \beta > \gamma \quad \text{and} \quad \left(\frac{b}{a}\right)^{\frac{1}{2}} > \frac{(3\beta^2 - 2\beta\gamma + 8\beta + 4 - \gamma^2)}{8(\beta - \gamma)}.$$

Consider the set $D(I)$ in $L^2(I)$ consisting of all functions $y \in L^2(I)$ such that $y^{[i]}$ ($i = 0,\dots,3$) is locally absolutely continuous function on I and $y^{[4]} \in L^2(I)$. Then for arbitrary two functions y and χ in $D(I)$ we have the Green's formula

$$\int_{x_1}^{x_2} y^{[4]}\chi\,dx - \int_{x_1}^{x_2} y\chi^{[4]}\,dx = [y,\chi](x_2) - [y,\chi](x_1),$$

where $0 \leq x_1 < x_2 \leq \infty$ and $[y,\chi](x) = y^{[0]}(x)\chi^{[3]}(x) - y^{[3]}(x)\chi^{[0]}(x) + y^{[1]}(x)\chi^{[2]}(x) - y^{[2]}(x)\chi^{[1]}(x)$. Green's formula implies that for two functions $y,\chi \in D(I)$, the limit $[y,\chi](\infty) = \lim_{x\to\infty}[y,\chi](x)$ exists and is finite.

Let $\varphi_1(x)$, $\varphi_2(x)$, $\psi_1(x)$, and $\psi_2(x)$ be the solutions of the equation

$$y^{[4]} = 0, \quad x \in I,$$

satisfying the conditions [13]

$$\begin{cases} \varphi_1^{[0]}(0) = \alpha_2, & \varphi_1^{[3]}(0) = -\alpha_1, & \varphi_1^{[1]}(0) = \varphi_1^{[2]}(0) = 0, \\ \varphi_2^{[1]}(0) = \beta_2, & \varphi_2^{[2]}(0) = -\beta_1, & \varphi_2^{[0]}(0) = \varphi_2^{[3]}(0) = 0, \\ \psi_1^{[0]}(0) = \gamma_1, & \psi_1^{[3]}(0) = -\gamma_2, & \psi_1^{[1]}(0) = \psi_1^{[2]}(0) = 0, \\ \psi_2^{[1]}(0) = \theta_1, & \psi_2^{[2]}(0) = -\theta_2, & \psi_2^{[0]}(0) = \psi_2^{[3]}(0) = 0, \end{cases}$$

where α_i, β_i, γ_i, and θ_i ($i = 1,2$) are real numbers satisfying

$$\alpha_1\gamma_1 - \alpha_2\gamma_2 = 1$$

and

$$\beta_1\theta_1 - \beta_2\theta_2 = 1.$$

Since the lim-4 case holds for $y^{[4]} = 0$, $x \in I$, all the solutions $\varphi_i(x)$ and $\psi_i(x)$, $i = 1,2$, belong to $L^2(I)$ and $D(I)$. It is clear that

$$[\varphi_r,\psi_s](a) = \delta_{rs}$$

and

$$[\varphi_r,\varphi_s](a) = 0, \qquad [\psi_r,\psi_s](a) = 0,$$

where δ_{rs} is the Kronecker delta and $1 \leq r,s \leq 2$. This means that for arbitrary $y \in D(I)$, the values $[y,\varphi_1](\infty)$, $[y,\varphi_2](\infty)$, $[y,\psi_1](\infty)$ and $[y,\psi_2](\infty)$ exist and are finite.

Let $\{y_j(x); 1 \leq j \leq r\}$ be any r ($1 \leq r \leq 4$) solutions of $y^{[4]} = 0$. The notation $W\{y_1,y_2,\dots,y_r\}$ denotes the Wronskian of order r of this set of functions. The following relation is well known (see [16, 17, 23]):

$$\begin{aligned} q_2^2(x)W\{y_1,y_2,y_3,y_4\}(x) = {}& -[y_1,y_2](x)[y_3,y_4](x) + [y_1,y_3](x)[y_2,y_4](x) \\ & - [y_1,y_4](x)[y_2,y_3](x). \end{aligned} \tag{12}$$

Using (12) we get

$$q_2^2(x)W\{\varphi_1,\varphi_2,\psi_1,\psi_2\}(x) = 1.$$

For $y \in D(I)$, let us consider the following boundary conditions:

$$\alpha_1 y^{[0]}(0) + \alpha_2 y^{[3]}(0) = 0, \tag{13}$$

$$\beta_1 y^{[1]}(0) + \beta_2 y^{[2]}(0) = 0, \tag{14}$$

$$[y,\psi_1](\infty) - k_1[y,\varphi_1](\infty) = 0, \tag{15}$$

$$[y,\psi_2](\infty) - k_2[y,\varphi_2](\infty) = 0, \tag{16}$$

where α_i and β_i $(i = 1,2)$ are defined as above and k_1 and k_2 are some real numbers.

It should be noted that for any solutions $y(x)$ of $y^{[4]} = 0$, the conditions (13) and (14), respectively, can be written as

$$[y,\varphi_1](0) = 0, \tag{17}$$

$$[y,\varphi_2](0) = 0. \tag{18}$$

Note that the conditions (17) and (18) are called Kodaira conditions [24].

3 Green's function

For $y \in D(I)$, let us consider the following differential equation:

$$y^{[4]} = h(x), \quad x \in I, \tag{19}$$

where $h \in L^2(I)$, subject to the boundary conditions (13)-(16).

Now consider the solutions $\varphi_1(x)$, $\varphi_2(x)$, $\phi_1(x)$, and $\phi_2(x)$, where $\phi_1(x) = \psi_1(x) - k_1\varphi_1(x)$ and $\phi_2(x) = \psi_2(x) - k_2\varphi_2(x)$. $\varphi_1(x)$ and $\varphi_2(x)$ satisfy the conditions (13) and (14), respectively, and $\phi_1(x)$ and $\phi_2(x)$ satisfy the conditions (15) and (16), respectively.

Using Everitt's method (see [16, 23]) we find the solution of the boundary value problem (19), (13)-(16) as

$$y(x) = \int_0^\infty G(x,t)h(t)\,dt, \quad x \in I,$$

where

$$G(x,t) = \begin{cases} \phi^T(t)\varphi(x), & 0 \le x < t, \\ \phi^T(x)\varphi(t), & t < x \le \infty, \end{cases} \tag{20}$$

y^T denotes the transpose of the vector y, $\phi(x) = \psi(x) - k\varphi(x)$ and

$$\varphi(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}, \qquad \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}, \qquad k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}.$$

Consequently we obtain in $L^2(I)$ that the nonlinear boundary value problem (10), (13)-(16) is equivalent to the nonlinear integral equation

$$y(x) = \int_0^\infty G(x,t)f\bigl(t,y(t)\bigr)\,dt, \quad x \in I, \tag{21}$$

where $G(x,t)$ is defined by (20). Note that since the lim-4 case holds for $y^{[4]} = 0$, one finds that $G(x,t)$ is a Hilbert-Schmidt kernel.

Using (11) and (20) we can define an operator $\mathcal{L} : L^2(I) \to L^2(I)$ as follows:

$$\mathcal{L}y = \int_0^\infty G(x,t)f\bigl(t,y(t)\bigr)\,dt, \quad x \in I, \tag{22}$$

where $y \in L^2(I)$. Hence (21) can be rewritten as

$$y = \mathcal{L}y. \tag{23}$$

Therefore solving (23) in $L^2(I)$ is equivalent to find the fixed points of $\mathcal{L}$.

Now we recall the well-known fixed point theorem.

Banach fixed point theorem *Let B be a Banach space and S a nonempty closed subset of B. Assume $A : S \to S$ is a contraction, i.e., there is a λ, $0 < \lambda < 1$, such that $\|Au - Av\| \le \lambda\|u - v\|$ for all u, v in S. Then A has a unique fixed point in S.*

Theorem 3.1 *Let $f(x,t)$ satisfies the condition* (11) *and the following Lipschitz condition: there is a constant $C > 0$ so that*

$$\int_0^\infty \bigl|f\bigl(x,y(x)\bigr) - f\bigl(x,\chi(x)\bigr)\bigr|^2\,dx \le C^2 \int_0^\infty \bigl|y(x) - \chi(x)\bigr|^2\,dx \tag{24}$$

for all $y, \chi \in L^2(I)$. If

$$C\left(\int_0^\infty \int_0^\infty \bigl|G(x,t)\bigr|^2\,dx\,dt\right)^{\frac{1}{2}} < 1 \tag{25}$$

then the problem (10), (13)-(16) *has a unique solution in $L^2(I)$.*

Proof For arbitrary $y, \chi \in L^2(I)$ using (24) and (25) we have

$$\begin{aligned}
|\mathcal{L}y - \mathcal{L}\chi|^2 &= \left|\int_0^\infty G(x,t)\bigl[f\bigl(t,y(t)\bigr) - f\bigl(t,\chi(t)\bigr)\bigr]\,dt\right|^2 \\
&\le \int_0^\infty \bigl|G(x,t)\bigr|^2\,dt \int_0^\infty \bigl|f\bigl(t,y(t)\bigr) - f\bigl(t,\chi(t)\bigr)\bigr|^2\,dt \\
&\le C^2 \int_0^\infty \bigl|y(t) - \chi(t)\bigr|^2\,dt \int_0^\infty \bigl|G(x,t)\bigr|^2\,dt \\
&= C^2\|y - \chi\|^2 \int_0^\infty \bigl|G(x,t)\bigr|^2\,dt.
\end{aligned} \tag{26}$$

If we take the value λ as

$$\lambda = C\left(\int_0^\infty \int_0^\infty \bigl|G(x,t)\bigr|^2\,dx\,dt\right)^{\frac{1}{2}} < 1$$

then we obtain from (26)

$$\|\mathcal{L}y - \mathcal{L}\chi\| \leq \lambda \|y - \chi\|.$$

Therefore $\mathcal{L}$ is a contraction mapping. Hence from the Banach fixed point theorem the proof is completed. □

Now consider the set K in $L^2(I)$ as follows:

$$K = \{y \in L^2(I) : \|y\| \leq R\}.$$

Theorem 3.2 *Let $f(x,t)$ satisfies the condition* (11). *Further let us assume that for all $y, \chi \in K$ the following inequality holds*:

$$\int_0^\infty |f(x,y(x)) - f(x,\chi(x))|^2 \, dx \leq C^2 \int_0^\infty |y(x) - \chi(x)|^2 \, dx, \tag{27}$$

where $C > 0$ is a constant and may depend on R. If

$$\left(\int_0^\infty \int_0^\infty |G(x,t)|^2 \, dx \, dt\right)^{\frac{1}{2}} \left(\sup_{y \in K} \int_0^\infty |f(t,y(t))|^2 \, dt\right)^{\frac{1}{2}} \leq R \tag{28}$$

and

$$C\left(\int_0^\infty \int_0^\infty |G(x,t)|^2 \, dx \, dt\right)^{\frac{1}{2}} < 1 \tag{29}$$

then the boundary value problem (10), (13)-(16) *has a unique solution $y \in L^2(I)$ satisfying*

$$\int_0^\infty |y(x)|^2 \, dx \leq R^2.$$

Proof Equations (27) and (29) show that the operator $\mathcal{L}$ is a contraction in K. Now let $y \in K$. Then using (28) one obtains that

$$\begin{aligned} \|\mathcal{L}y\| &= \left\| \int_0^\infty G(x,t) f(t,y(t)) \, dt \right\| \\ &\leq \left(\int_0^\infty \int_0^\infty |G(x,t)|^2 \, dx \, dt\right)^{\frac{1}{2}} \left(\int_0^\infty |f(t,y(t))|^2 \, dt\right)^{\frac{1}{2}} \leq R. \end{aligned} \tag{30}$$

This implies that $\mathcal{L} : K \to K$. Since K is a closed subset of $L^2(I)$, the Banach fixed point theorem can be applied to obtain a unique solution of (21) in K. This completes the proof. □

4 Fixed points on Banach space

Nonlinear boundary value problems may have solutions without uniqueness. To show that the boundary value problem (10), (13)-(16) have solutions may be without uniqueness, we recall the following well-known theorems.

Schauder fixed point theorem *Let B be a Banach space and K a nonempty bounded, convex, and closed subset of B. Assume $L : B \to B$ is a completely continuous operator. Then L has at least one fixed point in K provided that $L(K) \subset K$.*

Theorem 4.1 *A set $K \subset L^2(I)$ is relatively compact if and only if for every $\epsilon > 0$, K is bounded, there exists a $\delta > 0$ such that $\int_0^\infty |y(x+h) - y(x)|^2\,dx < \epsilon$ for all $y \in K$ and all $h \geq 0$ with $h < \delta$, there exists a number $M > 0$ such that $\int_M^\infty |y(x)|^2\,dx < \epsilon$ for all $y \in K$.*

Now we can state the following theorem.

Theorem 4.2 *Let $f(x,t)$ satisfies the condition* (11). *Further we assume that there exists a number $R > 0$ such that*

$$\left(\int_0^\infty \int_0^\infty \left|G(x,t)\right|^2 dx\,dt\right)^{\frac{1}{2}} \left(\sup_{y\in K} \int_0^\infty \left|f\left(t,y(t)\right)\right|^2 dt\right)^{\frac{1}{2}} \leq R,$$

where

$$K = \left\{y \in L^2(I) : \|y\| \leq R\right\}.$$

Then the boundary value problem (10), (13)-(16) *has at least one solution $y \in L^2(I)$ with*

$$\int_0^\infty \left|y(x)\right|^2 dx \leq R^2.$$

Proof It is clear that K is bounded, convex, and closed. Further one can see that $\mathcal{L}$ maps K into itself. Hence the proof of Theorem 4.2 will be completed with the next lemma. □

Lemma 4.3 *Let $f(x,t)$ satisfies the condition* (11). *The operator $\mathcal{L}$ defined by* (22) *is completely continuous, i.e., $\mathcal{L}$ is continuous and maps bounded sets into relatively compact sets.*

Proof Let $y \in L^2(I)$. Then for $\delta > 0$ and $\chi \in L^2(I)$ we have

$$|\mathcal{L}y - \mathcal{L}\chi|^2 \leq \int_0^\infty \left|G(x,t)\right|^2 dt \int_0^\infty \left|f\left(t,y(t)\right) - f\left(t,\chi(t)\right)\right|^2 dt.$$

Since $G(x,t)$ is a Hilbert-Schmidt kernel, we take $S = \int_0^\infty \int_0^\infty |G(x,t)|^2\,dx\,dt$. Therefore, one immediately gets

$$\|\mathcal{L}y - \mathcal{L}\chi\|^2 \leq S \int_0^\infty \left|f\left(t,y(t)\right) - f\left(t,\chi(t)\right)\right|^2 dt.$$

The condition (11) implies that (see [11, 25]) the operator A defined by $Ay = f(x,y(x))$ is continuous in $L^2(I)$. Therefore for $\epsilon > 0$ we can find a $\delta > 0$ such that $\|y - \chi\| < \delta$ implies that

$$\int_0^\infty \left|f\left(t,y(t)\right) - f\left(t,\chi(t)\right)\right|^2 dt < \frac{\epsilon^2}{S}.$$

Hence we have obtained for $y \in L^2(I)$ and $\epsilon > 0$: there exists a $\delta > 0$ such that $\|y - \chi\| < \delta$ implies $\|\mathcal{L}y - \mathcal{L}\chi\| < \epsilon$ for $\chi \in L^2(I)$. This implies that $\mathcal{L}$ is continuous.

Now consider the bounded set

$$\Omega = \{y \in L^2(I) : \|y\| \leq \gamma\}.$$

Taking into account (30) we get for all $y \in \Omega$

$$\|\mathcal{L}y\| \leq \left(M \int_0^\infty \left|f\bigl(t, y(t)\bigr)\right|^2 dt\right)^{\frac{1}{2}}.$$

On the other side from (11) we get

$$\begin{aligned} \int_0^\infty \left|f\bigl(t, y(t)\bigr)\right|^2 dt &\leq \int_0^\infty \bigl(g(t) + a\left|y(t)\right|\bigr)^2 dt \\ &\leq 2\int_0^\infty \bigl(g^2(t) + a^2\left|y(t)\right|^2\bigr)\, dt = 2\bigl(\|g\|^2 + a^2\|y\|^2\bigr) \\ &\leq 2\bigl(\|g\|^2 + a^2\gamma^2\bigr). \end{aligned}$$

This implies that

$$\|\mathcal{L}y\| \leq \bigl[2M\bigl(\|g\|^2 + a^2\gamma^2\bigr)\bigr]^{\frac{1}{2}}$$

for all $y \in \Omega$. Therefore $L(\Omega)$ is bounded in $L^2(I)$.

Now for $y \in \Omega$ let us consider the inequality

$$\begin{aligned} &\int_0^\infty \left|\mathcal{L}y(x+h) - \mathcal{L}y(x)\right|^2 dx \\ &\quad= \int_0^\infty \left|\int_0^\infty \bigl[G(x+h,t) - G(x,t)\bigr] f\bigl(t, y(t)\bigr)\, dt\right|^2 dx \\ &\quad\leq 2\bigl(\|g\|^2 + a^2\|y\|^2\bigr) \int_0^\infty \int_0^\infty \left|G(x+h,t) - G(x,t)\right|^2 dx\, dt. \end{aligned}$$

Since $G(x,t)$ is a Hilbert-Schmidt kernel for $\epsilon > 0$ there exists a $\delta > 0$ ($\delta = \delta(\epsilon)$) such that

$$\int_0^\infty \left|\mathcal{L}y(x+h) - \mathcal{L}y(x)\right|^2 dx < \epsilon^2$$

for all $y \in \Omega$ and all $h \geq 0$ with $h < \delta$.

Moreover, for $y \in \Omega$ we get

$$\int_0^\infty \left|\mathcal{L}y(x)\right|^2 dx \leq 2\bigl(\|g\|^2 + a^2\|y\|^2\bigr) \int_0^\infty \int_0^\infty \left|G(x,t)\right|^2 dx\, dt.$$

This implies that for $\epsilon > 0$ there exists $N > 0$ ($N = N(\epsilon)$) such that $\int_N^\infty |\mathcal{L}y(x)|^2\, dx < \epsilon^2$. Consequently $L(\Omega)$ is a relatively compact set in $L^2(I)$. This completes the proof of Lemma 4.3 and therefore Theorem 4.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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