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A nonlinear boundary value problem for fourth-order elastic beam equations

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Abstract
By using an infinitely many critical points theorem, we study the existence of infinitely many solutions for a fourth-order nonlinear boundary value problem, depending on two real parameters. No symmetric condition on the nonlinear term is assumed. Some recent results are improved and extended.

1 Introduction

In this paper, we consider a beam equation with nonlinear boundary conditions of the type:

$$\begin{cases} u^{(4)} = \lambda f(x,u) + \mu h(x,u), \quad 0 < x < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = g(u(1)), \end{cases} \tag{1.1}$$

where λ, μ are two positive parameters, f, h are two L^1-Carathéodory functions, and $g \in C(\mathbb{R})$ is real function. This kind of problem arises in the study of deflections of elastic beams on nonlinear elastic foundations. The problem has the following physical description: a thin flexible elastic beam of length 1 is clamped at its left end $x = 0$ and resting on an elastic device at its right end $x = 1$, which is given by g. Then the problem models the static equilibrium of the beam under a load, along its length, characterized by f and h. The derivation of the model can be found in [1, 2].

Fourth-order boundary value problems modeling bending equilibria of elastic beams have been considered in several papers. Most of them are concerned with nonlinear equations with null boundary conditions; see [3–6]. When the boundary conditions are nonzero or nonlinear, fourth-order equations can model beams resting on elastic bearings located in their extremities; see for instance [1, 2, 7–11] and the references therein. More precisely, in [10], using variant fountain theorems, the author obtains the existence of infinitely many solutions for problem (1.1) with $\lambda = 1$ and $\mu = 0$ under the symmetric condition and some other suitable assumptions of the nonlinear term f.

Motivated by the above works, in the present paper we establish some multiplicity results for problem (1.1) under rather different assumptions on the functions f, h and g. It is worth noticing that in our results neither the symmetric nor the monotonic condition on the nonlinear term is assumed. We require that f has a suitable oscillating behavior either at infinity or at zero. In the first case, we obtain an unbounded sequence of solutions

(Theorem 3.1); in the second case, we obtain a sequence of nonzero solutions strongly converging at zero (Theorem 3.4), which improve and extend the results in [10].

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proofs of our main results.

2 Variational setting and preliminaries

We prove our results applying the following smooth version of Theorem 2.1 of Bonanno and Bisci [12], which is a more precise version of Ricceri's variational principle [13, Lemma 2.5].

Theorem 2.1 *Let E be a reflexive real Banach space, let $\Phi, \Psi : E \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_E \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r)} \Psi(v)) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty} \varphi(r) \quad \text{and} \quad \delta := \liminf_{r\to(\inf_E \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) *For every $r > \inf_E \Phi$ and every $\lambda \in (0, 1/\varphi(r))$; the restriction of the functional*

$$I_\lambda := \Phi - \lambda\Psi$$

to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (local minimum) of I_λ in E.

(b) *If $\gamma < +\infty$; then for each $\lambda \in (0, 1/\gamma)$, the following alternative holds: either*
  (b1) *I_λ possesses a global minimum, or*
  (b2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$; then for each $\lambda \in (0, 1/\delta)$, the following alternative holds: either*
  (c1) *there is a global minimum of Φ which is a local minimum of I_λ, or*
  (c2) *there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ that converges weakly to a global minimum of Φ.*

Let E be the Hilbert space

$$E = \{u \in H^2(0,1); u(0) = u'(0) = 0\}$$

with the inner product and norm

$$\langle u, v\rangle = \int_0^1 u''(x)v''(x)\,dx, \qquad \|u\| = \|u''\|_2, \tag{2.1}$$

where $H^2(0,1)$ is the Sobolev space of all functions $u : [0,1] \to \mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $L^2([0,1])$, and $\|\cdot\|_p$ denotes the standard L^p norm. In addition, E is compactly embedded in the spaces $L^2([0,1])$ and $C([0,1])$, and therefore, there exist immersion constants $S_2, \bar{S} > 0$, such that

$$\|u\|_2 \le S_2\|u\| \quad \text{and} \quad \|u\|_\infty \le \bar{S}\|u\|. \tag{2.2}$$

We recall that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if

(a) the mapping $x \mapsto f(x,u)$ is measurable for every $u \in \mathbb{R}$;
(b) the mapping $u \mapsto f(x,u)$ is continuous for almost every $x \in [0,1]$;
(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,1])$ such that

$$\sup_{|u|\le\rho} \bigl|f(x,u)\bigr| \le l_\rho(x),$$

for almost every $x \in [0,1]$.

Define the functions $F, H : [0,1] \times \mathbb{R} \to \mathbb{R}$ as follows:

$$F(x,u) = \int_0^u f(x,s)\,ds \quad \text{and} \quad H(x,u) = \int_0^u h(x,s)\,ds,$$

for all $(x,u) \in [0,1] \times \mathbb{R}$, and $G(t) = \int_0^t g(x)\,dx$. Thus we define the functional $I_{\lambda,\mu} \in C^1(E,\mathbb{R})$ by

$$I_{\lambda,\mu}(u) := \frac{1}{2}\|u\|^2 - \lambda \int_0^1 F(x,u)\,dx - \mu \int_0^1 H(x,u)\,dx + G\bigl(u(1)\bigr), \quad \text{for all } u \in E.$$

Definition 2.1 We say that a function $u \in E$ is a weak solution of problem (1.1) if

$$\int_0^1 u''(x)v''(x)\,dx - \lambda \int_0^1 f(x,u)v\,dx - \mu \int_0^1 h(x,u)v\,dx + g\bigl(u(1)\bigr)v(1) = 0$$

holds for any $v \in E$.

3 Main results

In this section we establish the main abstract results of this paper. Let

$$A := \liminf_{\xi\to+\infty} \frac{\int_0^1 \max_{|u|\le\xi} F(x,u)\,dx}{\xi^2},$$

$$B := \limsup_{\xi\to+\infty} \frac{\int_a^b \max_{|u|\le\xi} F(x,u)\,dx}{\xi^2},$$

$$\Lambda_c := \frac{\alpha}{c} + \frac{\beta}{\sigma+1}c^{\sigma-1}$$

and

$$\lambda_1 := \frac{\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx}{2B}, \qquad \lambda_2 := \frac{1}{2\bar{S}A},$$

where α, β are given by (A1), c is a positive constant, and $d(x)$, $e(x)$ are given by (A3).

Theorem 3.1 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function and $0 < a < b < 1$. Assume that*

(A1) *there exist constants $\alpha, \beta > 0$ and $\sigma \in [0,1)$ such that*

$$\left|g(u)\right| \leq \alpha + \beta|u|^{\sigma}, \quad \textit{for all } u \in \mathbb{R};$$

(A2) *$F(x,u) \geq 0$ for all $(x,u) \in ([0,a] \cup [b,1]) \times \mathbb{R}$;*

(A3) *there exist two functions $d \in C^2([0,a])$ and $e \in C^2([b,1])$ satisfying*

$$d(0) = d'(0) = 0, \qquad d(a) = e(b) = 1, \qquad d'(a) = e'(b) = 0$$

and

$$\int_0^a \left|d''\right|^2 dx + \int_b^1 \left|e''\right|^2 dx \neq 0,$$

such that

$$\bar{S}A\left[\int_0^a \left|d''\right|^2 dx + \int_b^1 \left|e''\right|^2 dx\right] < B. \tag{3.1}$$

Then, for every $\lambda \in (\lambda_1, \lambda_2)$ and for any L^1-Carathéodory function $h : [0,1] \times \mathbb{R} \to \mathbb{R}$, whose potential $H(x,u) = \int_0^u h(x,s)\,ds$ for all $(x,u) \in [0,1] \times \mathbb{R}$, is a nonnegative function satisfying the condition

$$H_\infty := \limsup_{\xi \to +\infty} \frac{\int_0^1 \max_{|u| \leq \xi} H(x,u)\,dx}{\xi^2} < +\infty, \tag{3.2}$$

if we put

$$\mu_{H,\lambda} := \frac{1}{2\bar{S}H_\infty}(1 - 2\bar{S}\lambda A),$$

where $\mu_{H,\lambda} = +\infty$ when $H_\infty = 0$, for every $\mu \in [0, \mu_{H,\lambda})$ problem (1.1) *has an unbounded sequence of weak solutions in E.*

Proof Obviously, it follows from (A3) that $\lambda_1 < \lambda_2$. Fix $\bar{\lambda} \in (\lambda_1, \lambda_2)$. Since $\bar{\lambda} < \lambda_2$, we have

$$\mu_{H,\bar{\lambda}} = \frac{1}{2\bar{S}H_\infty}(1 - 2\bar{S}\bar{\lambda}A) > 0.$$

Now fix $\bar{\mu} \in (0, \mu_{H,\bar{\lambda}})$ and set

$$J(x,u) := F(x,u) + \frac{\bar{\mu}}{\bar{\lambda}}H(x,u), \quad \text{for all } (x,u) \in [0,1] \times \mathbb{R}.$$

Let the functionals $\Phi, \Psi : E \to \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2}\|u\|^2,$$
$$\Psi(u) = \int_0^1 J(x,u)\,dx - \frac{1}{\bar{\lambda}}G\bigl(u(1)\bigr),$$

where $G(t) = \int_0^t g(x)\,dx$. Put

$$I_{\bar{\lambda},\bar{\mu}}(u) := \Phi(u) - \bar{\lambda}\Psi(u), \quad \text{for all } u \in E.$$

Using the property of f, h and the continuity of g, we obtain $\Phi, \Psi \in C^1(E,\mathbb{R})$ and for any $v \in E$, we have

$$\langle \Phi'(u), v\rangle = \int_0^1 u''(x)v''(x)\,dx$$

and

$$\langle \Psi'(u), v\rangle = \int_0^1 f\big(x,u(x)\big)v(x)\,dx + \frac{\bar{\mu}}{\bar{\lambda}}\int_0^1 h\big(x,u(x)\big)v(x)\,dx - \frac{1}{\bar{\lambda}}g\big(u(1)\big)v(1).$$

So, with standard arguments, we deduce that the critical points of the functional $I_{\bar{\lambda},\bar{\mu}}$ are the weak solutions of problem (1.1) and so they are classical. We first observe that the functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2.1.

First of all, we show that $\bar{\lambda} < \frac{1}{\gamma}$. Let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty}\xi_n = +\infty$ and

$$\lim_{n\to+\infty} \frac{\int_0^1 \max_{|u|\le\xi_n} F(x,u)\,dx}{\xi_n^2} = A.$$

Set $r_n := \frac{1}{2\bar{S}}\xi_n^2$ for all $n \in \mathbb{N}$. Then, for all $v \in E$ with $\Phi(v) < r_n$, taking (2.2) into account, one has $\|v\|_\infty < \xi_n$. Note that $\Phi(0) = \Psi(0) = 0$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r_n)}\Psi(v)) - \Psi(u)}{r_n - \Phi(u)} \\
&\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_n)}\Psi(u)}{r_n} \\
&\le \frac{\int_0^1 \max_{|u|\le\xi_n} J(x,u)\,dx + \frac{1}{\bar{\lambda}}(\alpha\xi_n + \frac{\beta}{\sigma+1}\xi_n^{\sigma+1})}{\frac{1}{2\bar{S}}\xi_n^2} \\
&\le 2\bar{S}\left[\frac{\int_0^1 \max_{|u|\le\xi_n} F(x,u)\,dx}{\xi_n^2} + \frac{\bar{\mu}}{\bar{\lambda}}\frac{\int_0^1 \max_{|u|\le\xi_n} H(x,u)\,dx}{\xi_n^2} + \frac{1}{\bar{\lambda}}\Lambda_{\xi_n}\right].
\end{aligned}$$

Since $\lim_{n\to+\infty}\Lambda_{\xi_n} = 0$, from the assumption (A3) and the condition (3.2), we have

$$\gamma < \liminf_{n\to+\infty}\varphi(r_n) \le 2\bar{S}\left(A + \frac{\bar{\mu}}{\bar{\lambda}}H_\infty\right) < +\infty,$$

and combining the assumption $\bar{\mu} \in (0, \mu_{G,\bar{\lambda}})$, we obtain

$$\gamma < \liminf_{n\to+\infty}\varphi(r_n) \le 2\bar{S}\left(A + \frac{\bar{\mu}}{\bar{\lambda}}H_\infty\right) < 2\bar{S}A + \frac{1-2\bar{S}\bar{\lambda}A}{\bar{\lambda}}.$$

This implies that

$$\bar{\lambda} < \frac{1}{\gamma}.$$

Let $\bar{\lambda}$ be fixed. We claim that the functional $I_{\bar{\lambda},\bar{\mu}}$ is unbounded from below. Since

$$\frac{1}{\bar{\lambda}} < \frac{2B}{\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx},$$

there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n\to+\infty} \eta_n = +\infty$ and

$$\frac{1}{\bar{\lambda}} < \tau < \frac{2\int_a^b F(x,\eta_n)\,dx}{\eta_n^2[\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx]},$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$ we define v_n by

$$v_n(x) := \begin{cases} d(x)\eta_n, & x \in [0,a], \\ \eta_n, & x \in (a,b], \\ e(x)\eta_n, & x \in (b,1]. \end{cases} \tag{3.3}$$

From the condition (A3), it is easy to verify that $v_n \in E$. For any $n \in \mathbb{N}$, one has

$$\Phi(v_n) = \frac{1}{2}\|v_n\|^2 = \frac{\eta_n^2}{2}\left[\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx\right]. \tag{3.4}$$

On the other hand, by (A2) and since H is nonnegative, from the definition of Ψ, we infer

$$\Psi(v_n) \geq \int_a^b F(x,\eta_n)\,dx - \frac{1}{\bar{\lambda}}\eta_n^2\Lambda_{(\bar{S}\eta_n)}.$$

By (3.3) and (3.4), we have

$$\begin{aligned} I_{\bar{\lambda},\bar{\mu}}(v_n) &\leq \frac{\eta_n^2}{2}\left[\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx\right] - \bar{\lambda}\int_a^b F(x,\eta_n)\,dx + \eta_n^2\Lambda_{(\bar{S}\eta_n)} \\ &< \frac{\eta_n^2}{2}\left[\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx\right](1 - \bar{\lambda}\tau) + \eta_n^2\Lambda_{(\bar{S}\eta_n)}, \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Since $\sigma < 1$, $\bar{\lambda}\tau > 1$ and $\lim_{n\to+\infty} \eta_n = +\infty$, we have

$$\lim_{n\to+\infty} I_{\bar{\lambda},\bar{\mu}}(v_n) = -\infty.$$

Then the functional $I_{\bar{\lambda},\bar{\mu}}$ is unbounded from below, and it follows that $I_{\bar{\lambda},\bar{\mu}}$ has no global minimum. Therefore, by Theorem 2.1(b), there exists a sequence $\{u_n\}$ of critical points of $I_{\bar{\lambda},\bar{\mu}}$ such that

$$\lim_{n\to+\infty} \|u_n\| = +\infty,$$

and the conclusion is achieved. □

Remark 3.1 Indeed, it is not difficult to find such functions $d(x)$ and $e(x)$ satisfying the condition (A3). For example, let $a = \frac{1}{3}$ and $b = \frac{2}{3}$. We can choose

$$d(x) = -54x^2\left(x - \frac{1}{2}\right), \quad x \in \left[0, \frac{1}{3}\right]$$

and

$$e(x) = -3x\left(\frac{3}{4}x - 1\right), \quad x \in \left[\frac{2}{3}, 1\right].$$

Remark 3.2 Under the conditions $A = 0$ and $B = +\infty$, from Theorem 3.1 we see that for every $\lambda > 0$ and for each $\mu \in [0, \frac{1}{2\bar{S}H_\infty})$, problem (1.1) admits a sequence of classical solutions which is unbounded in E. Moreover, if $H_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu > 0$.

Corollary 3.2 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be an* L^1*-Carathéodory function and* $0 < a < b < 1$. *Suppose that hypotheses* (A1)-(A2) *hold. Moreover, the condition* (A3) *is satisfied if* (3.1) *is replaced by*

$$\int_0^a \left|d''\right|^2 dx + \int_b^1 \left|e''\right|^2 dx < 2B, \qquad 2\bar{S}A < 1.$$

Then, for any L^1*-Carathéodory function* $h : [0,1] \times \mathbb{R} \to \mathbb{R}$, *whose potential* $H(x,u) := \int_0^u h(x,s)\,ds$ *for all* $(x,u) \in [0,1] \times \mathbb{R}$, *is a nonnegative function satisfying the condition* (3.2), *if we put*

$$\mu_H := \frac{1}{2\bar{S}H_\infty}(1 - 2\bar{S}A),$$

where $\mu_H = +\infty$ *when* $H_\infty = 0$, *the problem*

$$\begin{cases} u^{(4)} = f(x,u) + \mu h(x,u), & 0 < x < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = g(u(1)), \end{cases}$$

has an unbounded sequence of weak solutions for every $\mu \in [0, \mu_H)$ *in* E.

Corollary 3.3 *Under the assumptions of Corollary* 3.2, *for any nonnegative continuous function* $h : [0,1] \to \mathbb{R}$, *the problem*

$$\begin{cases} u^{(4)} = f(x,u) + h(x), & 0 < x < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = g(u(1)), \end{cases}$$

has infinitely many distinct weak solutions in E.

Now, let

$$\bar{A} := \liminf_{\xi \to 0^+} \frac{\int_0^1 \max_{|u| \le \xi} F(x,u)\,dx}{\xi^2},$$

$$\bar{B} := \limsup_{\xi \to 0^+} \frac{\int_a^b \max_{|u| \le \xi} F(x,u)\,dx}{\xi^2},$$

$$\Theta_c := \min_{|u| \le c} \int_0^{u(1)} g(x)\,dx, \quad \text{for all } c > 0$$

and

$$\bar{\lambda}_1 := \frac{\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx}{2\bar{B}}, \qquad \bar{\lambda}_2 := \frac{1}{2\bar{S}\bar{A}}.$$

Theorem 3.4 *Let* $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ *be an* L^1*-Carathéodory function and* $0 < a < b < 1$. *Moreover, assume that* (A2) *and*

(A1)′ $g(u) \le 0$ *for all* $u \in \mathbb{R}$ *and* $\lim_{u\to 0^+} \frac{\int_0^u g(s)\,ds}{u^2} = 0$;
(A3)′ *there exist two functions* $d \in C^2([0,a])$ *and* $e \in C^2([b,1])$ *satisfying*

$$d(0) = d'(0) = 0, \qquad d(a) = e(b) = 1, \qquad d'(a) = e'(b) = 0, \qquad e(1) > 0$$

and

$$\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx \neq 0,$$

such that

$$\bar{S}\bar{A}\left[\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx\right] < \bar{B},$$

are satisfied. Then, for every $\lambda \in (\bar{\lambda}_1, \bar{\lambda}_2)$ *and for any* L^1*-Carathéodory function* h : $[0,1] \times \mathbb{R} \to \mathbb{R}$, *whose potential* $H(x,u) := \int_0^u h(x,s)\,ds$ *for all* $(x,u) \in [0,1] \times \mathbb{R}$, *is a nonnegative function satisfying the condition*

$$H_0 := \limsup_{\xi\to 0^+} \frac{\int_0^1 \max_{|u|\le\xi} H(x,u)\,dx}{\xi^2} < +\infty,$$

if we put

$$\bar{\mu}_{H,\lambda} := \frac{1}{2\bar{S}H_0}(1 - 2\bar{S}\lambda\bar{A}),$$

where $\bar{\mu}_{H,\lambda} = +\infty$ *when* $H_0 = 0$, *for every* $\mu \in [0, \bar{\mu}_{H,\lambda})$ *problem* (1.1) *has a sequence of weak solutions, which strongly converges to zero in* E.

Proof It follows from (A3)′ that $\bar{\lambda}_1 < \bar{\lambda}_2$. Fix $\bar{\lambda} \in (\bar{\lambda}_1, \bar{\lambda}_2)$. Since $\bar{\lambda} < \bar{\lambda}_2$, we have

$$\mu_{H,\bar{\lambda}} = \frac{1}{2\bar{S}H_0}(1 - 2\bar{S}\bar{\lambda}\bar{A}) > 0.$$

Now fix $\bar{\mu} \in (0, \bar{\mu}_{H,\bar{\lambda}})$ and set

$$J(x,u) := F(x,u) + \frac{\bar{\mu}}{\bar{\lambda}} H(x,u), \quad \text{for all } (x,u) \in [0,1] \times \mathbb{R}.$$

We take Φ, Ψ, and $I_{\bar{\lambda},\bar{\mu}}$ as in the proof of Theorem 3.1. Now, as has been pointed out before, the functionals Φ and Ψ satisfy the regularity assumptions required in Theorem 2.1. As first step, we will prove that $\bar{\lambda} < 1/\delta$. Let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = 0$ and

$$\lim_{n\to+\infty} \frac{\int_0^1 \max_{|u|\le\xi_n} F(x,u)\,dx}{\xi_n^2} = \bar{A}.$$

By the fact that $\inf_{u\in E}\Phi(u) = 0$ and the definition of δ, we have $\delta = \liminf_{r\to 0^+}\varphi(r)$. Set $r_n := \frac{1}{2\bar{S}}\xi_n^2$ for all $n \in \mathbb{N}$. Then, for all $v \in E$ with $\Phi(v) < r_n$, taking (2.2) into account, one has $\|v\|_\infty < \xi_n$. Note that $\Phi(0) = \Psi(0) = 0$. Then, for all $n \in \mathbb{N}$,

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r_n)}\Psi(v)) - \Psi(u)}{r_n - \Phi(u)} \\
&\le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_n)}\Psi(u)}{r_n} \le \frac{\int_0^1 \max_{|u|\le\xi_n} J(x,u)\,dx - \frac{1}{\bar{\lambda}}\Theta_{\xi_n}}{\frac{1}{2\bar{S}}\xi_n^2} \\
&\le 2\bar{S}\left[\frac{\int_0^1 \max_{|u|\le\xi_n} F(x,u)\,dx}{\xi_n^2} + \frac{\bar{\mu}}{\bar{\lambda}}\frac{\int_0^1 \max_{|u|\le\xi_n} H(x,u)\,dx}{\xi_n^2} - \frac{1}{\bar{\lambda}}\frac{\Theta_{\xi_n}}{\xi_n^2}\right].
\end{aligned}$$

It follows from (A1)$'$ that $\lim_{n\to+\infty}\frac{\Theta_{\xi_n}}{\xi_n^2} = 0$. Then we have

$$\delta < \liminf_{n\to+\infty}\varphi(r_n) \le 2\bar{S}\left(\bar{A} + \frac{\bar{\mu}}{\bar{\lambda}}H_0\right) < +\infty.$$

From $\bar{\mu} \in (0, \mu_{G,\bar{\lambda}})$, we obtain

$$\delta \le 2\bar{S}\left(\bar{A} + \frac{\bar{\mu}}{\bar{\lambda}}H_0\right) < 2\bar{S}\bar{A} + \frac{1 - 2\bar{S}\bar{\lambda}\bar{A}}{\bar{\lambda}},$$

which implies that

$$\bar{\lambda} < \frac{1}{\delta}.$$

Let $\bar{\lambda}$ be fixed. We claim that the functional $I_{\bar{\lambda},\bar{\mu}}$ does not have a local minimum at zero. Since

$$\frac{1}{\bar{\lambda}} < \frac{2\bar{B}}{\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx},$$

there exist a sequence $\{\eta_n\}$ of positive numbers and $\tau > 0$ such that $\lim_{n\to+\infty}\eta_n = 0$ and

$$\frac{1}{\bar{\lambda}} < \tau < \frac{2\int_a^b F(x,\eta_n)\,dx}{\eta_n^2[\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx]},$$

for each $n \in \mathbb{N}$ large enough. For all $n \in \mathbb{N}$, let v_n be defined by (3.3) with the above η_n. Note that $\bar{\lambda}\tau > 1$. Then, since $g(u) \le 0$ for all $u \in \mathbb{R}$ and $e(1) > 0$, we obtain

$$\begin{aligned}
I_{\bar{\lambda},\bar{\mu}}(v_n) &\le \frac{\eta_n^2}{2}\left[\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx\right] - \bar{\lambda}\int_a^b F(x,\eta_n)\,dx + \int_0^{v_n(1)} g(x)\,dx \\
&< \frac{\eta_n^2}{2}\left[\int_0^a |d''|^2\,dx + \int_b^1 |e''|^2\,dx\right](1 - \bar{\lambda}\tau) < 0,
\end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Thus, since

$$\lim_{n\to+\infty} I_{\bar{\lambda},\bar{\mu}}(v_n) = I_{\bar{\lambda},\bar{\mu}}(0) = 0,$$

we see that zero is not a local minimum of $I_{\bar{\lambda},\bar{\mu}}$. This, together with the fact that zero is the only global minimum of Φ, we deduce that the energy functional $I_{\bar{\lambda},\bar{\mu}}$ does not have a local minimum at the unique global minimum of Φ. Therefore, by Theorem 2.1(c), there exists a sequence $\{u_n\}$ of critical points of $I_{\bar{\lambda},\bar{\mu}}$, which converges weakly to zero. In view of the fact that the embedding $E \hookrightarrow C([0,1])$ is compact, we know that the critical points converge strongly to zero, and the proof is complete. □

Remark 3.3 Applications similar to Corollaries 3.2 and 3.3 can also be made to Theorem 3.4. Now we give an example illustrating Theorem 3.4. Consider the problem

$$\begin{cases} u^{(4)} = \lambda f(x,u), \quad 0 < x < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \qquad u'''(1) = g(u(1)), \end{cases} \tag{3.5}$$

where $f(x,u) = |u|$. Obviously, $\bar{A} = \bar{B} = \frac{1}{2}$. Let $a = \frac{1}{3}$ and $b = \frac{2}{3}$, and choose

$$d(x) = -\frac{3x^2}{2\sqrt{S}}\left(x - \frac{1}{2}\right), \quad x \in \left[0, \frac{1}{3}\right]$$

and

$$e(x) = -\frac{x}{12\sqrt{S}}\left(\frac{3}{4}x - 1\right), \quad x \in \left[\frac{2}{3}, 1\right].$$

By calculating, we have $\int_0^{\frac{1}{3}} |d''|^2\,dx + \int_{\frac{2}{3}}^1 |e''|^2\,dx = \frac{1}{S}(\frac{1}{4} + \frac{1}{4\times 36^2})$. Thus, $\bar{\lambda}_1 = \frac{1}{S}(\frac{1}{4} + \frac{1}{4\times 36^2})$ and $\bar{\lambda}_2 = \frac{1}{S}$. Furthermore, the conditions (A2) and (A3)′ are satisfied. Let $g(u) = -u^2$. Then (A1)′ holds. Therefore, by Theorem 3.4, we find that problem (3.5) has a sequence of weak solutions which strongly converges to zero in E for all $\lambda \in (\bar{\lambda}_1, \bar{\lambda}_2)$.

Competing interests

The author declares that they have no competing interests.

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