Wang and Tian *Boundary Value Problems* 2014, **2014**:192
http://www.boundaryvalueproblems.com/content/2014/1/192

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Generalized monotone iterative method for nonlinear boundary value problems with causal operators

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Abstract
This paper discusses nonlinear boundary value problems for causal differential equations where the right-hand side is the sum of two monotone functions. We develop the monotone iterative technique and establish the existence results of the extremal solutions. The results obtained include several special cases and extend previous results; two examples satisfying the assumptions are also presented.
MSC: 34B15; 39B12

Keywords: generalized monotone iterative method; nonlinear boundary value problem; upper and lower solutions; causal operator

1 Introduction

As we all know, the monotone iterative technique is an effective and a flexible method, and it provides a useful mechanism to prove existence results for nonlinear differential equations, for detail see for monographs [1], papers [2–11], and the references therein. The basic idea of this method is that using the upper and lower solutions as an initial iteration, one can construct monotone sequences from a corresponding linear equations, and these sequences converge monotonically to the minimal and maximal solutions of the nonlinear equations.

In 2004, West [12] developed this method, considered the generalized monotone iterative method for initial value problems, obtained the existence of extremal solutions for differential equations where the forcing function is the sum of two monotone functions, one of which is monotone non-decreasing and the other is non-increasing.

Recently, this method has been extended to causal differential equations. Its theory has the powerful quality of unifying ordinary differential equations, integro differential equations, differential equations with finite or infinite delay, Volterra integral equations and neutral equations. We refer to the monograph by Lakshmikantham [13] and papers [14, 15].

In 2009, Lakshmikantham discussed in [13] the following problem with causal operators: $x'(t) = (Px)(t) + (Qx)(t)$, $x(0) = x_0$, where $t \in J = [0, T]$, $P, Q : E \to E = C(J, \mathbb{R})$ are causal operators. However, we notice that the results are only valid for initial value problems. Motivated by the above excellent work, we extend the notion of casual operators to nonlinear boundary value problems and develop the monotone iterative technique.

In this paper, we deal with the following causal differential equation:

$$\begin{cases} u'(t) = (Qu)(t) + (Su)(t), \quad t \in J = [0,T], \\ g(u(0), u(T)) = 0, \end{cases} \tag{1.1}$$

where $Q, S : E \to E = C(J, \mathbb{R})$ are causal operators.

Note that the nonlinear boundary value problem (1.1) reduce to periodic boundary value problems for $g(u(0), u(T)) = u(0) - u(T)$, initial value problems for $g(u(0), u(T)) = u(0) - u_0$ which has been studied in [13] and other general conditions such as $g(u(0), u(T)) = h(u(0)) - u(T)$. Thus problem (1.1) can be regarded as a generalization of the boundary value problems mentioned above.

The rest of this paper is organized as follows. In Section 2, we develop the monotone technique for (1.1); four theorems and several special cases are given. In Section 3, we give two examples to illustrate the results obtained. Finally, a brief summary is given in Section 4.

2 Main results

Let $J = [0, T]$, $Q, S : E \to E = C(J, \mathbb{R})$. In order to prove general results, we need the following definitions.

Definition 2.1 For (1.1), the functions $y_0, z_0 \in C^1[J, \mathbb{R}]$ are said to be

(1) natural lower and upper solutions if

$$\begin{cases} y_0'(t) \le (Qy_0)(t) + (Sy_0)(t), \quad g(y_0(0), y_0(T)) \le 0, \\ z_0'(t) \ge (Qz_0)(t) + (Sz_0)(t), \quad g(z_0(0), z_0(T)) \ge 0; \end{cases}$$

(2) coupled lower and upper solutions of type I, if

$$\begin{cases} y_0'(t) \le (Qy_0)(t) + (Sz_0)(t), \quad g(y_0(0), y_0(T)) \le 0, \\ z_0'(t) \ge (Qz_0)(t) + (Sy_0)(t), \quad g(z_0(0), z_0(T)) \ge 0; \end{cases}$$

(3) coupled lower and upper solutions of type II, if

$$\begin{cases} y_0'(t) \le (Qz_0)(t) + (Sy_0)(t), \quad g(y_0(0), y_0(T)) \le 0, \\ z_0'(t) \ge (Qy_0)(t) + (Sz_0)(t), \quad g(z_0(0), z_0(T)) \ge 0; \end{cases}$$

(4) coupled lower and upper solutions of type III, if

$$\begin{cases} y_0'(t) \le (Qz_0)(t) + (Sz_0)(t), \quad g(y_0(0), y_0(T)) \le 0, \\ z_0'(t) \ge (Qy_0)(t) + (Sy_0)(t), \quad g(z_0(0), z_0(T)) \ge 0. \end{cases}$$

If we set $y_0(t) \le z_0(t)$, $t \in J$, (Qu) is non-decreasing and (Su) is non-increasing, then the natural lower and upper solutions and the coupled lower and upper solutions of type III satisfy type II. Thus, we only need to consider the case of the coupled lower and upper solutions of type I and II for (1.1).

Definition 2.2 Relative to the causal differential equation (1.1):

(1) A function $U \in C^1[J,\mathbb{R}]$ is said to be a natural solution if it satisfies (1.1).

(2) $U, V \in C^1[J,\mathbb{R}]$ are said to be coupled solutions of type I, if

$$\begin{cases} U'(t) = (QU)(t) + (SV)(t), & g(U(0),U(T)) = 0, \\ V'(t) = (QV)(t) + (SU)(t), & g(V(0),V(T)) = 0. \end{cases}$$

(3) U, V are said to be coupled solutions of type II, if

$$\begin{cases} U'(t) = (QV)(t) + (SU)(t), & g(U(0),U(T)) = 0, \\ V'(t) = (QU)(t) + (SV)(t), & g(V(0),V(T)) = 0. \end{cases}$$

(4) U, V are said to be coupled solutions of type III, if

$$\begin{cases} U'(t) = (QV)(t) + (SV)(t), & g(U(0),U(T)) = 0, \\ V'(t) = (QU)(t) + (SU)(t), & g(V(0),V(T)) = 0. \end{cases}$$

Definition 2.3 Coupled solutions $\rho, r \in C^1[J,\mathbb{R}]$, are said to be couple minimal and maximal solutions of (1.1), if for any coupled solutions U, V, we have $\rho \leq U, V \leq r$.

Theorem 2.1 *We suppose that the following hypotheses hold*:

H_1 $y_0, z_0 \in C^1(J,\mathbb{R})$ *are the coupled lower and upper solutions of type I for* (1.1) *with* $y_0(t) \leq z_0(t)$ *on* J;

H_2 *the operators* Q, S *in* (1.1) *are such that* $Q, S : E \to E$, (Qu) *is non-decreasing in* u *and* (Su) *is non-increasing in* u;

H_3 *the function* $g(u,v) \in C(\mathbb{R}^2,\mathbb{R})$ *is non-increasing in the second variable and there exists a constant* $M > 0$ *such that*

$$g(u_1,v) - g(u_2,v) \leq M(u_1 - u_2),$$

for $y_0(0) \leq u_2 \leq u_1 \leq z_0(0)$, $y_0(T) \leq v \leq z_0(T)$.

Then there exist two monotone sequences $\{y_n(t)\}$, $\{z_n(t)\}$ such that $\lim_{n\to\infty} y_n(t) = \rho(t)$, $\lim_{n\to\infty} z_n(t) = r(t)$ uniformly and monotonically on J and that ρ, r are coupled minimal and maximal solutions of type I for (1.1). Furthermore, if u is any natural solution of (1.1) such that $y_0 \leq u \leq z_0$ on J, then $\rho \leq u \leq r$ on J.

Proof We consider the following linear problem:

$$\begin{cases} y'_{n+1}(t) = (Qy_n)(t) + (Sz_n)(t), \\ y_{n+1}(0) = y_n(0) - \frac{1}{M} g(y_n(0), y_n(T)), \end{cases} \tag{2.1}$$

$$\begin{cases} z'_{n+1}(t) = (Qz_n)(t) + (Sy_n)(t), \\ z_{n+1}(0) = z_n(0) - \frac{1}{M} g(z_n(0), z_n(T)). \end{cases} \tag{2.2}$$

This is an adequate definition since by general results on the initial value problem of causal differential equations [13] the existence and uniqueness of solution for (2.1) and (2.2) are guaranteed.

First, we show that $y_0 \leq y_1 \leq z_1 \leq z_0$, putting $n = 0$ in (2.1) and setting $p = y_0 - y_1$, we acquire

$$p(0) = y_0(0) - y_0(0) + \frac{1}{M} g\big(y_0(0), y_0(T)\big) \leq 0,$$
$$p' = y_0' - y_1' \leq (Qy_0)(t) + (Sz_0)(t) - (Qy_0)(t) - (Sz_0)(t) = 0.$$

It follows that $p(t) \leq 0$ on J, which implies $y_0 \leq y_1$ on J. Similarly, we may obtain $z_1 \leq z_0$ on J.

Next, take $p = y_1 - z_1$, then from the hypotheses H_2, H_3, and the fact $y_0 \leq z_0$, one attains

$$\begin{aligned} p(0) &= y_1(0) - z_1(0) \\ &= y_0(0) - z_0(0) + \frac{1}{M}\big(g\big(z_0(0), z_0(T)\big) - g\big(y_0(0), y_0(T)\big)\big) \\ &\leq 0, \end{aligned}$$
$$p' = y_1' - z_1' = (Qy_0)(t) + (Sz_0)(t) - (Qz_0)(t) - (Sy_0)(t) \leq 0.$$

This implies that $p(t) \leq 0$ on J, and $y_1 \leq z_1$.

In the following, we shall show that y_1, z_1 are the coupled lower and upper solutions of type I for (1.1). Following H_2 and $y_0 \leq y_1$, $z_1 \leq z_0$, we obtain

$$y_1'(t) = (Qy_0)(t) + (Sz_0)(t) \leq (Qy_1)(t) + (Sz_1)(t), \tag{2.3}$$

and by means of the facts that $y_0 \leq y_1$, $z_1 \leq z_0$, H_1, and H_3, we have

$$\begin{aligned} g\big(y_1(0), y_1(T)\big) &= g\big(y_1(0), y_1(T)\big) - g\big(y_0(0), y_0(T)\big) - My_1(0) + My_0(0) \\ &\leq M\big(y_1(0) - y_0(0)\big) - M\big(y_1(0) - y_0(0)\big) = 0. \end{aligned} \tag{2.4}$$

Similarly, we can get

$$z_1'(t) \geq (Qz_1)(t) + (Sy_1)(t), \qquad g\big(z_1(0), z_1(T)\big) \geq 0, \tag{2.5}$$

from (2.3)-(2.5), we show that y_1, z_1 are coupled lower and upper solutions of type I for (1.1).

Now employing the mathematical induction, assume that, for some integer $k > 1$,

$$y_{k-1} \leq y_k \leq z_k \leq z_{k-1} \quad \text{on } J.$$

We need to show that

$$y_k \leq y_{k+1} \leq z_{k+1} \leq z_k \quad \text{on } J.$$

For this purpose, let $p = y_k - y_{k+1}$ and use H_2, H_3; we note that

$$p(0) = y_k(0) - y_{k+1}(0) = y_k(0) - y_k(0) + \frac{1}{M} g\bigl(y_k(0), y_k(T)\bigr) \le 0,$$

$$p' = y_k' - y_{k+1}' = (Qy_{k-1})(t) + (Sz_{k-1})(t) - (Qy_k)(t) - (Sz_k)(t) \le 0.$$

This implies $y_k \le y_{k+1}$ on J. Similarly, we can prove that $z_{k+1} \le z_k$ on J by using (2.2) and H_2, H_3. To prove $y_{k+1} \le z_{k+1}$, set $p = y_{k+1} - z_{k+1}$, then by using H_2, H_3, and the fact that $y_k \le z_k$, we get

$$\begin{aligned} p(0) &= y_{k+1}(0) - z_{k+1}(0) \\ &= y_k(0) - z_k(0) + \frac{1}{M}\Bigl(g\bigl(z_k(0), z_k(T)\bigr) - g\bigl(y_k(0), y_k(T)\bigr)\Bigr) \\ &\le 0, \\ p' &= y_{k+1}' - z_{k+1}' = (Qy_k)(t) + (Sz_k)(t) - (Qz_k)(t) - (Sy_k)(t) \le 0. \end{aligned}$$

That yields $p(t) \le 0$, *i.e.* $y_{k+1} \le z_{k+1}$. From the above discussion, we have

$$y_0 \le y_1 \le \cdots \le y_n \le z_n \le \cdots \le z_1 \le z_0.$$

Obviously, the constructed sequences $\{y_n\}$, $\{z_n\}$ are equicontinuous and uniform bounded. Thus, by the Ascoli-Arzela theorem, we have $\{y_n\} \to \rho$, $\{z_n\} \to r$ on J. Since the sequences $\{y_n\}$, $\{z_n\}$ are monotone, the entire sequences converge uniformly and monotonically to ρ, r on J, respectively.

Using the definition of (2.1), (2.2), and passing to the limit when $n \to \infty$, we obtain the result that ρ, r are coupled solutions of type I for (1.1).

It remains to show that ρ, r are coupled minimal and maximal solutions of type I for (1.1). Let $u_1, u_2 \in [y_0, z_0]$ be any coupled solutions of type I for (1.1). Assume that there exists a positive integer k such that $y_k \le u_1, u_2 \le z_k$ on J. Then, putting $p = y_{k+1} - u_1$, and employing H_2 and H_3, we arrive at

$$\begin{aligned} p(0) &= y_{k+1}(0) - u_1(0) \\ &= y_k(0) - u_1(0) + \frac{1}{M}\Bigl(g\bigl(u_1(0), u_1(T)\bigr) - g\bigl(y_k(0), y_k(T)\bigr)\Bigr) \\ &\le 0, \\ p' &= y_{k+1}' - u_1' = (Qy_k)(t) + (Sz_k)(t) - (Qu_1)(t) - (Su_2)(t) \le 0. \end{aligned}$$

That implies $p(t) \le 0$, which proves $y_{k+1} \le u_1$ on J. Using similar arguments we can conclude $y_{k+1} \le u_1, u_2 \le z_{k+1}$ on J. Since $y_0 \le u_1, u_2 \le z_0$, by the principle of induction, $y_n \le u_1, u_2 \le z_n$ holds for all n. Taking the limit as $n \to \infty$, we have $\rho \le u_1, u_2 \le r$ on J proving ρ, r are coupled minimal and maximal solutions of type I for (1.1). Since any natural solution u of (1.1) can be considered as (u, u) coupled solutions of type I, we also have $\rho \le u \le r$ on J. This completes the proof. □

Theorem 2.2 *Let the hypotheses of Theorem* 2.1 *hold. Then, for any natural solution u of* (1.1) *with $y_0 \leq u \leq z_0$, there exist alternating sequences $\{y_{2n}, z_{2n+1}\} \to \rho$, $\{z_{2n}, y_{2n+1}\} \to r$ uniformly on J with $y_0 \leq z_1 \leq \cdots \leq y_{2n} \leq z_{2n+1} \leq u \leq y_{2n+1} \leq z_{2n} \leq \cdots \leq y_1 \leq z_0$. Here ρ, r are the coupled minimal and maximal solutions of type I for* (1.1). *Also, $\rho \leq u \leq r$ on J.*

Proof Consider the following initial problem:

$$\left.\begin{aligned} y'_{n+1}(t) &= (Qz_n)(t) + (Sy_n)(t), \qquad & y_{n+1}(0) &= z_n(0) - \tfrac{1}{M}g(z_n(0), z_n(T)), \\ z'_{n+1}(t) &= (Qy_n)(t) + (Sz_n)(t), \qquad & z_{n+1}(0) &= y_n(0) - \tfrac{1}{M}g(y_n(0), y_n(T)). \end{aligned}\right\} \tag{2.6}$$

First we show that $y_0 \leq y_1$ and $z_1 \leq z_0$.

Set $p = y_0 - y_1$, following (2.6) and the hypotheses of Theorem 2.1, we obtain

$$\begin{aligned} p(0) = y_0(0) - y_1(0) &\leq y_0(0) - z_0(0) + \frac{1}{M}g\bigl(z_0(0), z_0(T)\bigr) \\ &\leq \frac{1}{M}g\bigl(y_0(0), y_0(T)\bigr) \leq 0, \\ p' = y'_0 - y'_1 &= (Qy_0)(t) + (Sz_0)(t) - (Qz_0)(t) - (Sy_0)(t) \leq 0. \end{aligned}$$

Thus we prove $p(t) \leq 0$, which gives $y_0 \leq y_1$ on J. Similarly, we can show $z_1 \leq z_0$.

Now we wish to prove that

$$y_0 \leq z_1 \leq y_2 \leq z_3 \leq u \leq y_3 \leq z_2 \leq y_1 \leq z_0. \tag{2.7}$$

Setting $p = u - y_1$, we get

$$p' = u' - y'_1 = (Qu)(t) + (Su)(t) - (Qz_0)(t) - (Sy_0)(t) \leq 0,$$

using the monotone nature of the operators Q, S, and the fact $y_0 \leq u \leq z_0$ on J, u being any natural solution of (1.1). Also, by using H_3, we have

$$\begin{aligned} p(0) &= u(0) - y_1(0) \\ &= u(0) - z_0(0) + \frac{1}{M}\bigl(g\bigl(z_0(0), z_0(T)\bigr) - g\bigl(u(0), u(T)\bigr)\bigr) \\ &\leq 0. \end{aligned}$$

Hence $p(t) \leq 0$ on J, *i.e.*, $u \leq y_1$ on J. A similar argument yields $z_1 \leq u$. In order to avoid repetition, we can prove each of the following: $y_2 \leq u$, $u \leq z_2$, $u \leq y_3$, and $z_3 \leq u$.

Now we shall show that $y_0 \leq z_1 \leq y_2 \leq z_3$ and $y_3 \leq z_2 \leq y_1 \leq z_0$. Take $p = y_0 - z_1$, and one attains

$$\begin{aligned} p' &= y'_0 - z'_1 \leq (Qy_0)(t) + (Sz_0)(t) - (Qy_0)(t) - (Sz_0)(t) = 0, \\ p(0) &= y_0(0) - z_1(0) = y_0(0) - y_0(0) + \frac{1}{M}g\bigl(y_0(0), y_0(T)\bigr) \leq 0. \end{aligned}$$

Then $p(t) \leq 0$ on J, thus $y_0 \leq z_1$. Similarly, we can obtain $y_1 \leq z_0$ on J.

Next, take $p = z_1 - y_2$, by H_2 and H_3, we derive

$$p' = z_1' - y_2' = (Qy_0)(t) + (Sz_0)(t) - (Qz_1)(t) - (Sy_1)(t) \leq 0,$$
$$p(0) = z_1(0) - y_2(0) = y_0(0) - z_1(0) + \frac{1}{M}\big(g\big(z_1(0), z_1(T)\big) - g\big(y_0(0), y_0(T)\big)\big) \leq 0.$$

This implies that $p(t) \leq 0$ on J, *i.e.*, $z_1 \leq y_2$ on J. Using similar arguments we can show $y_2 \leq z_3, y_3 \leq z_2, z_2 \leq y_1$. Combining all these arguments, we now have the desired relations (2.7).

Now, suppose there exists an integer $k \geq 2$ such that $z_{2k-1} \leq y_{2k} \leq z_{2k+1} \leq u \leq y_{2k+1} \leq z_{2k} \leq y_{2k-1}$ on J. Thus, we need to show

$$z_{2k+1} \leq y_{2k+2} \leq z_{2k+3} \leq u \leq y_{2k+3} \leq z_{2k+2} \leq y_{2k+1} \quad \text{on } J.$$

Setting $p = z_{2k+1} - y_{2k+2}$, then for $t \in J$ and utilizing the hypotheses $y_{2k} \leq z_{2k+1}$ and H_2, H_3, we may get

$$\begin{aligned} p(0) &= z_{2k+1}(0) - y_{2k+2}(0) \\ &= y_{2k}(0) - z_{2k+1}(0) + \frac{1}{M}\big(g\big(z_{2k+1}(0), z_{2k+1}(T)\big) - g\big(y_{2k}(0), y_{2k}(T)\big)\big) \\ &\leq 0, \end{aligned}$$
$$p' = z_{2k+1}' - y_{2k+2}' = (Qy_{2k})(t) + (Sz_{2k})(t) - (Qz_{2k+1})(t) - (Sy_{2k+1})(t) \leq 0.$$

This implies that $p \leq 0$ and $z_{2k+1} \leq y_{2k+2}$. Similarly, we obtain $z_{2k+2} \leq y_{2k+1}$, $y_{2k+2} \leq z_{2k+3}$, and $y_{2k+3} \leq z_{2k+2}$.

To prove $y_{2k+2} \leq u$, by using H_2, H_3, and the inequalities $z_{2k+1} \leq u$, consider the relations

$$p' = y_{2k+2}' - u' = (Qz_{2k+1})(t) + (Sy_{2k+1})(t) - (Qu)(t) - (Su)(t) \leq 0$$

and

$$\begin{aligned} p(0) &= y_{2k+2}(0) - u(0) \\ &= z_{2k+1}(0) - u(0) + \frac{1}{M}\big(g\big(u(0), u(T)\big) - g\big(z_{2k+1}(0), z_{2k+1}(T)\big)\big) \\ &\leq 0, \end{aligned}$$

one attains $p \leq 0$ on J, *i.e.* $y_{2k+2} \leq u$. Hence, as before, we can conclude that $u \leq z_{2k+2}$, $z_{2k+3} \leq u$, and $u \leq y_{2k+3}$ on J. Now, with the principle of induction, we have

$$y_0 \leq z_1 \leq \cdots \leq y_{2n} \leq z_{2n+1} \leq u \leq y_{2n+1} \leq z_{2n} \leq \cdots \leq y_1 \leq z_0.$$

By employing a reasoning similar to that of Theorem 2.1, we get the sequences $\{y_{2n}, z_{2n+1}\}$, $\{z_{2n}, y_{2n+1}\}$ which converge uniformly and monotonically to ρ, r on J, respectively. Thus, ρ, r are coupled solutions of type I for (1.1).

Finally, to prove that ρ, r are coupled minimal and maximal solutions of (1.1), let $u_1, u_2 \in [y_0, z_0]$ be any coupled solutions of type I for (1.1). Similar to the proof of the above, if

$z_{2k-1} \le y_{2k} \le z_{2k+1} \le u_1, u_2 \le y_{2k+1} \le z_{2k} \le y_{2k-1}$ for some positive integer k, we can easily see that $z_{2k+1} \le y_{2k+2} \le z_{2k+3} \le u_1, u_2 \le y_{2k+3} \le z_{2k+2} \le y_{2k+1}$ on J, by the induction, one has $y_{2n} \le z_{2n+1} \le u_1, u_2 \le y_{2n+1} \le z_{2n}$ holds on J for all n. Taking the limit as $n \to \infty$, we have $\rho \le u_1, u_2 \le r$ on J proving ρ, r are coupled minimal and maximal solutions of type I for (1.1). Since we have already shown that $y_{2n} \le z_{2n+1} \le u \le y_{2n+1} \le z_{2n}$ holds on J for all n. Now taking the limit as $n \to \infty$, we get $\rho \le u \le r$ on J. This completes the proof. □

Corollary 2.1 *Under the hypotheses of Theorem* 2.1, *let* $(Su) \equiv 0$, *then* y_0, z_0 *are natural lower and upper solutions of* (1.1), *and we can get the following results*:

(i) *There exist two monotone sequences* $\{y_n\}$, $\{z_n\}$ *such that* $y_0 \le y_1 \le \cdots \le y_n \le z_n \le \cdots \le z_1 \le z_0$ *which converge uniformly to the minimal and maximal solutions of* (1.1), *respectively.*

(ii) *There exist alternating sequences* $\{y_{2n}, z_{2n+1}\}$, $\{z_{2n}, y_{2n+1}\}$ *such that* $y_0 \le z_1 \le \cdots \le y_{2n} \le z_{2n+1} \le y_{2n+1} \le z_{2n} \le \cdots \le y_1 \le z_0$ *which converge uniformly to the minimal and maximal solutions of* (1.1), *respectively.*

Corollary 2.2 *If* $(Su) \equiv 0$ *and* (Qu) *is not non-decreasing, then* $(\widetilde{Q}u) = (Qu) + Lu$ *is non-decreasing for some* $L > 0$, *and we can consider the following problem*:

$$u'(t) = (Qu)(t) = (\widetilde{Q}u)(t) - Lu(t), \qquad g\bigl(u(0), u(T)\bigr) = 0,$$

we see that it can be seen as (1.1) *with* (Qu) *replaced by* $\widetilde{Q}u$ *and* (Su) *replaced by* $-Lu$. *Thus we get the same conclusions as for Theorem* 2.1 *and Theorem* 2.2.

Corollary 2.3 *Under the hypotheses of Theorem* 2.1, *let* $(Qu) \equiv 0$, *if* u *is any natural solution of* (1.1) *such that* $y_0 \le u \le z_0$ *on* J, *then we have*

(i) *there exist the monotone sequences* $\{y_n\}$, $\{z_n\}$ *converging to* ρ *and* r, *where* (ρ, r) *are coupled minimal and maximal solutions of* (1.1), *respectively, satisfying*

$$\begin{cases} \rho'(t) = (Sr)(t), & g(\rho(0), \rho(T)) = 0, \\ r'(t) = (S\rho)(t), & g(r(0), r(T)) = 0, \end{cases}$$

for $t \in J$ *and* $\rho \le u \le r$;

(ii) *there exist alternating sequences* $\{y_{2n}, z_{2n+1}\}$ *converging to* ρ, $\{z_{2n}, y_{2n+1}\}$ *converging to* r, *where* (ρ, r) *are coupled minimal and maximal solutions of* (1.1), *respectively, satisfying*

$$\begin{cases} \rho'(t) = (Sr)(t), & g(\rho(0), \rho(T)) = 0, \\ r'(t) = (S\rho)(t), & g(r(0), r(T)) = 0, \end{cases}$$

for $t \in J$ *and* $\rho \le u \le r$.

Corollary 2.4 *If* $(Qu) \equiv 0$ *and* (Su) *is not non-increasing, then* $(\widetilde{S}u) = (Su) - Nu$ *is non-increasing for some* $N > 0$, *we can consider the following problem*:

$$u'(t) = (Su)(t) = (\widetilde{S}u)(t) + Nu(t), \qquad g\bigl(u(0), u(T)\bigr) = 0,$$

which is the same as (1.1) *with* (Qu) *replaced by* Nu *and* (Su) *replaced by* $(\widetilde{S}u)$. *Hence the conclusions of Theorem* 2.1 *and Theorem* 2.2 *remain valid.*

Corollary 2.5 *Suppose* (Qu) *is non-decreasing but* (Su) *is not non-increasing, then* $(\widetilde{Q}u) = (Qu) + Nu$ *is non-decreasing and* $(\widetilde{S}u) = (Su) - Nu$ *is non-increasing for some* $N > 0$, *we can consider the following problem*:

$$u'(t) = (Qu) + (Su)(t) = (\widetilde{Q}u)(t) + (\widetilde{S}u)(t), \qquad g\bigl(u(0), u(T)\bigr) = 0,$$

which is the same as (1.1) *with* (Qu) *replaced by* $(\widetilde{Q}u)$ *and* (Su) *replaced by* $(\widetilde{S}u)$ *and the conclusions of Theorem* 2.1 *and Theorem* 2.2 *hold.*

Corollary 2.6 *If* (Qu) *is not non-decreasing but* (Su) *is non-increasing, then* $(\widetilde{Q}u) = (Qu) + Lu$ *is non-decreasing and* $(\widetilde{S}u) = (Su) - Lu$ *is non-increasing for some* $L > 0$, *we can consider the following problem*:

$$u'(t) = (Qu) + (Su)(t) = (\widetilde{Q}u)(t) + (\widetilde{S}u)(t), \qquad g\bigl(u(0), u(T)\bigr) = 0,$$

which is the same as (1.1) *with* (Qu) *replaced by* $(\widetilde{Q}u)$ *and* (Su) *replaced by* $(\widetilde{S}u)$ *and the conclusions of Theorem* 2.1 *and Theorem* 2.2 *hold.*

Corollary 2.7 *If* (Qu) *is not non-decreasing and* (Su) *is not non-increasing, then for some* $L > 0$, *such that* $(\widetilde{Q}u) = (Qu) + Lu$ *is non-decreasing and* $(\widetilde{S}u) = (Su) - Lu$ *is non-increasing, we can get the conclusions of Theorem* 2.1 *and Theorem* 2.1 *with* (Qu) *replaced by* $(\widetilde{Q}u)$ *and* (Su) *replaced by* $(\widetilde{S}u)$.

We can always construct coupled upper and lower solutions of type II as in this paper. To avoid repetition, we will merely state the next two theorems without proof since it follows along the same lines as Theorems 2.1 and 2.2.

Theorem 2.3 *Assume conditions* H_2, H_3 *of Theorem* 2.1 *hold, let* $y_0, z_0 \in C^1(J, \mathbb{R})$ *be the coupled lower and upper solutions of type II with* $y_0(t) \le z_0(t)$ *on* J, *we have the iterates* $\{y_n\}$, $\{z_n\}$ *satisfying*

$$y_0 \le y_1 \le \cdots \le y_n \le z_n \le z_{n-1} \le \cdots \le z_1 \le z_0,$$

provided $y_0 \le y_1$ *and* $z_1 \le z_0$ *on* J, *where the iterates are developed by*

$$y'_{n+1}(t) = (Qy_n)(t) + (Sz_n)(t), \qquad y_{n+1}(0) = y_n(0) - \frac{1}{M} g\bigl(y_n(0), y_n(T)\bigr),$$
$$z'_{n+1}(t) = (Qz_n)(t) + (Sy_n)(t), \qquad z_{n+1}(0) = z_n(0) - \frac{1}{M} g\bigl(z_n(0), z_n(T)\bigr).$$

Moreover, the monotone sequences $\{y_n\}$, $\{z_n\}$ *converge uniformly to* ρ *and* r, *respectively, where* (ρ, r) *are coupled minimal and maximal solutions of type II for* (1.1), *they satisfy the coupled system*

$$\begin{cases} \rho'(t) = (Q\rho)(t) + (Sr)(t), & g(\rho(0), \rho(T)) = 0, \\ r'(t) = (Qr)(t) + (S\rho)(t), & g(r(0), r(T)) = 0, \end{cases}$$

for $t \in J$ and if u is any natural solution of (1.1) *such that $y_0 \leq u \leq z_0$ on J, then $\rho \leq u \leq r$ on J.*

Theorem 2.4 *Assume the hypotheses* H_2, H_3 *of Theorem* 2.1 *to hold, and let $y_0, z_0 \in C^1(J, \mathbb{R})$ be the coupled lower and upper solutions of type II with $y_0(t) \leq z_0(t)$ on J, then for any natural solution u of* (1.1) *with $y_0 \leq u \leq z_0$ on J, we get the alternating sequences $\{y_{2n}, z_{2n+1}\}$, $\{z_{2n}, y_{2n+1}\}$ satisfying*

$$y_0 \leq z_1 \leq \cdots \leq y_{2n} \leq z_{2n+1} \leq u \leq y_{2n+1} \leq z_{2n} \leq \cdots \leq y_1 \leq z_0,$$

provided $y_0 \leq y_1$ and $z_1 \leq z_0$ on J, for every $n \geq 1$, where the iterative schemes are constructed by

$$y'_{n+1}(t) = (Qz_n)(t) + (Sy_n)(t), \qquad y_{n+1}(0) = z_n(0) - \frac{1}{M} g\big(z_n(0), z_n(T)\big),$$
$$z'_{n+1}(t) = (Qy_n)(t) + (Sz_n)(t), \qquad z_{n+1}(0) = y_n(0) - \frac{1}{M} g\big(y_n(0), y_n(T)\big).$$

Moreover, the monotone sequences $\{y_{2n}, z_{2n+1}\}$ converge to ρ and $\{z_{2n}, y_{2n+1}\}$ converge to r on J, where (ρ, r) are coupled minimal and maximal solutions of type I for (1.1), *respectively, satisfying the coupled system*

$$\begin{cases} \rho'(t) = (Q\rho)(t) + (Sr)(t), & g(\rho(0), \rho(T)) = 0, \\ r'(t) = (Qr)(t) + (S\rho)(t), & g(r(0), r(T)) = 0, \end{cases}$$

for $t \in J$ and $\rho \leq u \leq r$ on J.

3 Examples

In this section, we give two simple but illustrative examples, thereby validating the proposed theorems.

Example 3.1 Consider the following problem:

$$\begin{cases} u'(t) = (\sin t) u(t) + \frac{1}{15} e^{-u(t)}, \quad t \in J = [0, 1], \\ g(u(0), u(1)) = u(0) - \frac{1}{100} u(1) - \frac{1}{20e} = 0. \end{cases} \tag{3.1}$$

We construct a pair of coupled upper and lower solutions of type I for (3.1),

$$y_0(t) = 0, \qquad z_0(t) = t + \frac{1}{10}, \quad t \in J.$$

Obviously, $y_0(t) \leq z_0(t)$, and

$$\begin{cases} y'_0(t) = 0 \leq (\sin t) y_0(t) + \frac{1}{15} e^{-z_0(t)} = \frac{1}{15} e^{-t - \frac{1}{10}}, \\ g(y_0(0), y_0(1)) = -\frac{1}{20e} < 0, \end{cases}$$

$$\begin{cases} z'_0(t) = 1 \geq (\sin t) z_0(t) + \frac{1}{15} e^{-y_0(t)} = (\sin t)(t + \frac{1}{10}) + \frac{1}{15}, \\ g(z_0(0), z_0(1)) = \frac{1}{10} - \frac{1}{100}(1 + \frac{1}{10}) - \frac{1}{20e} > 0. \end{cases}$$

It proves that $y_0(t)$, $z_0(t)$ are coupled lower and upper solutions of problem (3.1). Then assumptions H_1, H_2, and H_3 hold with $M = 1$. By Theorem 2.1, we obtain the existence of monotone sequences that approximate the extremal solutions of (3.1) in the sector $[0, t + \frac{1}{10}]$. By Theorem 2.2, we obtain the existence of alternating sequences that also converge to the extremal solutions.

Example 3.2 Consider the following problem:

$$\begin{cases} u'(t) = u^3(t) - t^2u(t), \quad t \in J = [0,1], \\ g(u(0), u(1)) = u(0) - u^2(1) + \frac{1}{4} = 0. \end{cases} \tag{3.2}$$

Put $y_0(t) = -1$, $z_0(t) = 1$, $t \in J$. Obviously, $y_0(t) \leq z_0(t)$, then

$$\begin{cases} y_0'(t) = 0 \leq z_0^3(t) - t^2y_0(t) = 1 + t^2, \\ g(y_0(0), y_0(1)) = -1 - (-1)^2 + 1/4 < 0, \end{cases}$$

$$\begin{cases} z_0'(t) = 0 \geq y_0^3(t) - t^2z_0(t) = -1 - t^2, \\ g(z_0(0), z_0(1)) = 1/4 > 0. \end{cases}$$

Functions $y_0(t)$, $z_0(t)$ are coupled lower and upper solutions of type II for problem (3.2). It is easy to see that H_1, H_2, and H_3 hold with $M = 1$. Consider the function $y_1(t) = -\frac{t}{2}$, $z_1(t) = \frac{t}{2}$ satisfy $y_0 \leq y_1$, $z_1 \leq z_0$, respectively. Therefore, applying Theorem 2.3, by using (2.1) and (2.2), we obtain the existence of monotone sequences that approximate the extremal solutions of (3.2) in the sector $[-1, 1]$. From Theorem 2.4, we obtain the existence of alternating sequences that also converge to the extremal solutions.

4 Conclusions

In this paper, we have developed monotone iterative method for casual differential equations with nonlinear boundary conditions. The method is based on the new concepts of lower and upper solutions (resp. coupled lower and upper solutions). We have constructed monotone sequences and alternating sequences from a corresponding linear equations. It was proven that these sequences converge uniformly to the coupled minimal and maximal solutions of the problems.

The condition on the function $g(u, v)$ was necessary to carry out the proofs. A future research direction would be to consider this method for nonlinear conditions of $B(u(0), u) = 0$ or $B(u, u(T)) = 0$. Also, since the theory of causal differential equations has recently gained more attention, much work can be done on its theoretical research.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Acknowledgements
The authors would like to thank the reviewers and the editors for their valuable suggestions and comments. This work was supported by the NNSF of China (No. 61073121), the Fundamental Research Funds for the Central Universities (No. 13ZD19), and the Higher School Science Research of Hebei Province of China (No. Z2013038).

Received: 17 March 2014 Accepted: 28 July 2014 Published online: 25 September 2014

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doi:10.1186/s13661-014-0192-5
Cite this article as: Wang and Tian: **Generalized monotone iterative method for nonlinear boundary value problems with causal operators.** *Boundary Value Problems* 2014 **2014**:192.