Zeng *Boundary Value Problems* 2014, **2014**:202
http://www.boundaryvalueproblems.com/content/2014/1/202

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

The estimation and profile of the critical value for a Schrödinger equation

Jing Zeng[*]

[*]Correspondence:
zengjing@fjnu.edu.cn
School of Mathematics and Computer Sciences, Fujian Normal University, Fuzhou, 350007, P.R. China

Abstract

In this paper, we are concerned with the following Schrödinger problem: $-\Delta u + V(x)u = f(u)$, $u > 0$, in $\mathbb{R}^N$, where $f : \mathbb{R} \to \mathbb{R}$ is of class C^1. The estimation and profile of the critical value of the corresponding functional is proved, which entails the relationship between the critical value on the balls and the least-energy value on the whole space. Our results are also true for three cases of the potential function $V(x)$.

Keywords: estimation; critical value; Schrödinger equation; potential function; compactness and noncompactness condition

1 Introduction

The main subject of this paper is the following problem:

$$-\Delta u + V(x)u = f(u), \quad u > 0, \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $\Delta = \sum_{i=1}^{N} \partial^2/\partial x_i^2$ is the Laplace operator. Compactness and noncompactness assumptions posed on the potential function $V(x)$ are also discussed.

The nonlinear Schrödinger equation (1.1) serves as a model for various problems in physics. For the last 20 years, (1.1) has received considerable attention as its solutions seem both mathematically intriguing and scientifically useful. We would like to mention earlier results on the existence of entire solutions of Schrödinger type equations with or without potentials, which was studied in [1–3] (see references therein).

A more general form of nonlinearity, *i.e.*

$$-\Delta u + V(x)u = f(x,u), \quad u \in H^1(\mathbb{R}^N), \tag{1.2}$$

is also studied by many authors. Kryszewski-Szulkin [4] considered the existence of a nontrivial solution of (1.2) in a situation when $f(x,u)$, $V(x)$ are periodic in the x-variables, $f(x,u)$ is superlinear at $u = 0$ and $|u| = \infty$, and 0 lies in a spectral gap of $-\Delta + V$. In addition, if $f(x,u)$ is odd in u, they proved that (1.2) has infinitely many solutions. The result from Bartsch-Wang [5] suggested that (1.2) should have one sign changing solution. Bartsch-Liu-Weth [6] further proved the existence of sign changing solutions of (1.2) in $H^1(\mathbb{R})$ with superlinear and subcritical nonlinearity $f(x,u)$, and the number of nodal domains can be controlled. If $f(x,u)$ is odd, they obtained an unbounded sequence of sign changing solutions u_k $(k \geq 1)$, and they have at most $k+1$ nodal domains.

For the power nonlinearity $f(t) = t^p$ $(1 < p < \frac{N+2}{N-2})$, since $V(x) > 0$, it is well known that this problem has a positive solution which goes to zero at infinity. This solution is, besides, radially symmetric around some point and unique up to translations; see [7] and [8]. Moreover, the linearized equation around w is nondegenerate in the sense that the equation

$$\Delta u - V(x)u + pw^{p-1}u = 0, \quad \text{in } \mathbb{R}^N,$$

has linear combinations of the functions $\partial w/\partial x_i$ as its only solutions which go to zero at infinity [8, 9]. These facts are crucial in the formulation of a Lyapunov-Schmidt type procedure, which was first introduced by Floer-Weinstein [10] for the one-dimensional case, and then was extended by Oh [11, 12] to higher dimensions. Ni-Wei [13] studied the critical value of the energy functional

$$E(u) = \frac{1}{2}\int_\Omega \left(\varepsilon^2|\nabla u|^2 + u^2\right) - \frac{1}{p+1}\int_\Omega u^{p+1}, \quad u \in H^1(\Omega),$$

of the classical singular perturbation equation $-\varepsilon^2\Delta u + u = u^p$ $(u > 0)$ on Ω, where Ω is a bounded smooth domain in $\mathbb{R}^N$.

Throughout this paper the following hypotheses on $f \in C^1$ and $V(x)$ will be assumed.

- (f_1) $f(0) = 0, f(t) = o(|t|)$, as $|t| \to 0$, uniformly in x.
- (f_2) There exists a number $p > 1$, with $p < \frac{N+2}{N-2}$ if $N \geq 3$, such that $\lim_{|t|\to\infty} \frac{F(t)}{|t|^p} < +\infty$, where $F(t) = \int_0^t f(s)\,ds$.
- (f_3) There exists a number $\mu > 2$ such that $0 < \mu F(t) \leq tf(t)$, for all $|t| > 0$.
- (f_4) For any x, $\frac{f'(t)t-t}{f(t)-t}$ is nonincreasing in t.
- (V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\inf_{\mathbb{R}^N} V(x) > 0$.
- (V_2) $V(x) = V(|x|)$.

Theorem 1.1 *Under assumptions* $(f_1) \sim (f_3)$ *and* (V_1), *there exists a least-energy ground state solution of* (1.1).

Theorem 1.2 *Under assumptions* $(f_1) \sim (f_4)$ *and* (V_2), *there exists a least-energy ground state of* (1.1), *which is radially uniqueness solution, with the corresponding least-energy value* c_*.

Remark 1.3 The assumptions (V_1) and (V_2) are adopted in [14], which means that the potential functions possess certain compactness conditions. (V_2) indicates that the x dependence is radially symmetric. For this case, $\{u \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} V(x)u^2 < \infty, u \text{ is radial}\}$ is compactly embedded in $L^p(\mathbb{R}^N)$ $(2 \leq p < 2^* = 2N/(N-2))$.

Remark 1.4 By standard variational arguments, the assumptions $(f_1) \sim (f_3)$ and (V_1) guarantee the results of Theorem 1.1, which can be proved by the traditional Minmax Theory.

In fact, the critical value of the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) - \int_{\mathbb{R}^N} F(u), \quad u \in H^1\left(\mathbb{R}^N\right) \tag{1.3}$$

can be characterized as

$$c_* = \inf_{u \neq 0} \sup_{t>0} I(tu).$$

The associated critical point actually solves (1.1) and is called a least-energy solution. It decays exponentially at infinity.

Remark 1.5 As far as we know, the most general result of uniqueness of (1.1) type is obtained by Serrin-Tang [15], which would guarantee radial uniqueness in (1.1) if additionally one assumes (f_4). The proof of Theorem 1.2 is similar to the steps in [15]; we omit the details.

Now we define the energy functional of (1.1) on $B_\rho = \{x \in \mathbb{R}^N | |x| \leq \rho\}$:

$$I_\rho(u) = \frac{1}{2} \int_{B_\rho} \left(|\nabla u|^2 + V(x)u^2\right) - \int_{B_\rho} F(u). \tag{1.4}$$

The main result of this paper is the following.

Theorem 1.6 *Under assumptions* $(f_1) \sim (f_4)$ *and* (V_2), *the critical value* c_ρ *of the functional* I_ρ *satisfies*

$$c_\rho = c_* + \gamma^2 \exp\left(-2\rho\left(1 + o(1)\right)\right), \tag{1.5}$$

where c_* *is the least-energy value in Theorem* 1.2 *or Remark* 1.4, γ *is defined as*

$$\gamma = \frac{1}{2} \int_{\mathbb{R}^N} f(w)V > 0, \tag{1.6}$$

where w *is the unique solution in Theorem* 1.2.

Remark 1.7 The assumptions $(f_1) \sim (f_4)$ and (V_2) can guarantee the existence of c_ρ on the bounded domain B_ρ. In fact, it can be proved by the Minmax Theory as in Remark 1.4.

Remark 1.8 Assuming the conditions (V_3) or (V_4) on $V(x)$, we can get a similar equality as (1.5).

(V_3) There exists $r_0 > 0$ such that, for any $M > 0$,

$$\lim_{|y| \to \infty} m\left(\left\{x \in \mathbb{R}^N : |x - y| \leq r_0\right\} \cap \left\{x \in \mathbb{R}^N : V(x) \leq M\right\}\right) = 0,$$

where m denotes the Lebesgue measure on $\mathbb{R}^N$.

(V_4) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V(x) > 0$, and $V(x) \to \infty$ as $|x| \to \infty$.

The assumptions (V_3) and (V_4) are certain compactness conditions, listed in [14]. (V_3) is a more general condition, which gives a compact embedding. For (V_4), we have a compact embedding from $\{u \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} V(x)u^2 < \infty\}$ in $L^p(\mathbb{R}^N)$ for $2 \leq p < 2^*$.

Remark 1.9 There are two cases of noncompactness conditions that are posed on the potential functions $V(x)$ [16], and the assumption (V_6) is also adopted in [17]:

(V_5) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V(x) \geq V_0 > 0$. $V(x)$ is 1-periodic in each of $x_1, x_2, \ldots, x_N$.
(V_6) $0 < \inf_{\mathbb{R}^N} V(x) \leq \lim_{x\to\infty} V(x) = \sup_{\mathbb{R}^N} V(x) < \infty$.

Assumption (V_5) is periodic, *i.e.*, the x-dependence is periodic. (V_6) means that $V(x)$ has a bounded potential well in the sense that $\lim_{|x|\to\infty} V(x)$ exists and is equal to $\sup_{\mathbb{R}^N} V(x)$.

Remark 1.10 A particularly interesting case is whether one can come to the same conclusion as Theorem 1.6 under the noncompact assumptions (V_5) and (V_6).

Our theorems generalize the results in [18] to three cases of compactness potential function entailing a type of nonlinear Schrödinger equation. The existence of the least-energy ground state solution of (1.1) is essential. Our results show the relationship between the critical value on the balls and the least-energy value on the whole space. The estimation of the critical value can be used to locate the geometrical shape of the solution.

2 Preliminaries

In this section, we give some preliminary lemmas, which will be adopted in the proof of the theorems.

Lemma 2.1 *Assume w is a solution of* (1.1), *and w_ρ is a solution of*

$$\begin{cases} -\Delta u + V(x)u = f(u), \quad u > 0, \ \textit{in } B_\rho, \\ u = 0, \quad \textit{on } \partial B_\rho. \end{cases} \tag{2.1}$$

Then

$$w_\rho(\rho - 1) = \gamma \exp\bigl(-\rho\bigl(1 + o(1)\bigr)\bigr),$$
$$w(\rho - 1) = \gamma \exp\bigl(-\rho\bigl(1 + o(1)\bigr)\bigr),$$

where γ is defined in (1.6).

Proof The proof of the two equalities is similar, we only prove the latter. Let w be the unique positive solution of (1.1), then the function

$$\overline{w} = w\left(\frac{x}{\sqrt{V(x)}}\right) V(x)^{\frac{\ln u}{\ln(u/f(u))}}$$

satisfies the equation

$$-\Delta u + u = f(u), \quad u > 0, \ \text{in } \mathbb{R}^N. \tag{2.2}$$

Next we consider the solution of the equation

$$\begin{cases} u'' - (1 - \frac{\tau}{2})u = 0, \quad \text{in } (R, \rho), \\ u(R) = 1, \qquad u(\rho) = 0, \end{cases} \tag{2.3}$$

where $\rho > R$.

Note $s = \sqrt{1-\frac{\tau}{2}}$, $h = e^{sR} - e^{2s\rho - sR}$; then the solution of (2.3) is

$$\overline{u}(x) = \frac{-e^{2s\rho}}{h} e^{-sx} + \frac{1}{h} e^{sx}. \tag{2.4}$$

We have $s \to 1$, as $\tau \to 0$. For ρ big enough,

$$\frac{1}{h}\left(e^{s\rho - s} - e^{s\rho + s}\right) \le e^{-(1-\tau)\rho},$$

so

$$-\frac{e^{2s\rho}}{h} e^{-s(\rho-1)} + \frac{1}{h} e^{s(\rho-1)} \le e^{-(1-\tau)\rho},$$

that is,

$$\overline{u}(\rho - 1) \le e^{-(1-\tau)\rho}.$$

The $\overline{u}$ in (2.4) is the supersolution of (2.2) on $[R, \rho)$. So on $[R, \rho)$, $\overline{w} \le \overline{u}$, we deduce that

$$w\left(\frac{x}{\sqrt{V(x)}}\right) V(x)^{\frac{\ln u}{\ln(u/f(u))}} \le e^{-(1-\tau)\rho},$$

so

$$\lim_{r\to\infty} w(r) e^{(1-\tau)r} = \gamma > 0,$$

where γ is defined in (1.6); then $w(\rho - 1) \le \gamma e^{-(1-\tau)\rho}$.

For the lower boundary estimation, given $R > 0$, consider the equation

$$\begin{cases} u'' + \frac{N-1}{R} u' - u = 0, & \text{in } (R, \rho), \\ u(R) = w_\rho(R), & u(\rho) = 0. \end{cases}$$

Similarly to the computation of (2.3), we get, for ρ big enough, $\underline{u}(\rho - 1) \ge e^{-(1-\tau)\rho}$, and $\underline{u}$ is a subsolution of the above equation. So $\overline{w} \ge \underline{u}$. Therefore, for ρ big enough, $w(\frac{x}{\sqrt{V(x)}}) V(x)^{\frac{\ln u}{\ln(u/f(u))}} \ge e^{-(1-\tau)\rho}$, so $w(\rho - 1) \ge \gamma e^{-\rho(1+o(1))}$. We conclude that $w(\rho - 1) = \gamma e^{-\rho(1+o(1))}$. □

Lemma 2.2 *Let u be a solution of*

$$\begin{cases} u'' - u = 0, & \textit{in } (\rho - 1, \infty), \\ u(\rho - 1) = 1, & u(+\infty) = 0. \end{cases} \tag{2.5}$$

Let v be a solution of

$$\begin{cases} v'' - h(\rho) v = 0, & \textit{in } (\rho - 1, \rho), \\ v(\rho - 1) = 1, & v(\rho) = 0, \end{cases} \tag{2.6}$$

where $h(\rho)$ is a function of ρ. Then $v'(\rho - 1) - u'(\rho - 1) \ge 1$.

Proof By computation $u = e^{\rho-1} \cdot e^{-x}$ is a solution of (2.5), and $u'(\rho-1) = -1$. Similarly, it can be checked that

$$v = \frac{-e^{\sqrt{h(\rho)}\cdot\rho}}{e^{-\sqrt{h(\rho)}} - e^{\sqrt{h(\rho)}}} \cdot e^{-\sqrt{h(\rho)}\cdot x} + \frac{1}{e^{\sqrt{h(\rho)}\cdot\rho}(e^{-\sqrt{h(\rho)}} - e^{\sqrt{h(\rho)}})} \cdot e^{\sqrt{h(\rho)}\cdot x}$$

is a solution of (2.6), and

$$v' = \frac{e^{\sqrt{h(\rho)}\cdot\rho}\sqrt{h(\rho)}}{e^{-\sqrt{h(\rho)}} - e^{\sqrt{h(\rho)}}} \cdot e^{-\sqrt{h(\rho)}\cdot x} + \frac{\sqrt{h(\rho)}}{e^{\sqrt{h(\rho)}\cdot\rho}(e^{-\sqrt{h(\rho)}} - e^{\sqrt{h(\rho)}})} \cdot e^{\sqrt{h(\rho)}\cdot x}.$$

So $v'(\rho-1) \geq 0 = u'(\rho-1)+1$, *i.e.* $v'(\rho-1) - u'(\rho-1) \geq 1$. □

3 The estimation of the critical value

This section is devoted to the proof of Theorem 1.6.

Proof of Theorem 1.6 c_ρ is the critical value of the functional I_ρ, c_* is the least-energy value of $I(u)$.

First we find the upper bound of c_ρ. Let v_ρ be the solution of the equation

$$\begin{cases} -\Delta u + V(x)u = 0, & \text{in } B_\rho \backslash B_{\rho-1}, \\ u(\rho-1) = w(\rho-1), & u(\rho) = 0, \end{cases}$$

where w is the solution of (1.1) in Theorem 1.2, then

$$\int_{B_\rho\backslash B_{\rho-1}} V(x)v_\rho^2 = \int_{B_\rho\backslash B_{\rho-1}} \Delta v_\rho \cdot v_\rho = \nabla v_\rho \cdot v_\rho|_{\partial B_{\rho-1}} - \int_{B_\rho\backslash B_{\rho-1}} |\nabla v_\rho|^2,$$

so

$$\int_{B_\rho\backslash B_{\rho-1}} |\nabla v_\rho|^2 + \int_{B_\rho\backslash B_{\rho-1}} V(x)v_\rho^2 = \nabla v_\rho \cdot v_\rho|_{\partial B_{\rho-1}}. \tag{3.1}$$

Define

$$\overline{w}_\rho(r) = \begin{cases} w(r), & 0 \leq r \leq \rho-1, \\ v_\rho(r), & \rho-1 \leq r \leq \rho. \end{cases}$$

We have

$$c_\rho = I_\rho(w_\rho) = \max_{t\geq 0} I_\rho(tw_\rho) \leq \max_{t\geq 0} I_\rho(t\overline{w}_\rho) = I_\rho(t_\rho \overline{w}_\rho), \tag{3.2}$$

and

$$t_\rho \to 1, \quad \text{as } \rho \to \infty. \tag{3.3}$$

Moreover, by the definition of $\overline{w}_\rho$,

$$\begin{aligned} I_\rho(t_\rho\overline{w}_\rho) &= \frac{1}{2}\int_{B_\rho} t_\rho^2\bigl(|\nabla\overline{w}_\rho|^2 + V(x)\overline{w}_\rho^2\bigr) - \int_{B_\rho} F(t_\rho\overline{w}_\rho) \\ &= \frac{1}{2}\int_{B_{\rho-1}} t_\rho^2\bigl(|\nabla\overline{w}_\rho|^2 + V(x)\overline{w}_\rho^2\bigr) - \int_{B_{\rho-1}} F(t_\rho\overline{w}_\rho) \\ &\quad + \left(\frac{1}{2}\int_{B_\rho\backslash B_{\rho-1}} t_\rho^2\bigl(|\nabla\overline{w}_\rho|^2 + V(x)\overline{w}_\rho^2\bigr) - \int_{B_\rho\backslash B_{\rho-1}} F(t_\rho\overline{w}_\rho)\right) \\ &\le \frac{1}{2}\int_{B_{\rho-1}} t_\rho^2\bigl(|\nabla w|^2 + V(x)w^2\bigr) - \int_{B_{\rho-1}} F(t_\rho w) \\ &\quad + \frac{1}{2}\int_{B_\rho\backslash B_{\rho-1}} t_\rho^2\bigl(|\nabla v_\rho|^2 + V(x)v_\rho^2\bigr) \\ &\le c_* + t_\rho^2\cdot\nabla v_\rho\cdot v_\rho|_{\partial B_{\rho-1}} \\ &\le c_* + t_\rho^2\cdot v_\rho'(r)\cdot v_\rho(r)|_{r=\rho-1}, \end{aligned} \tag{3.4}$$

where (3.1) is used in the second inequality.

Similar to the computation in Lemma 2.1, we have

$$v_\rho'(\rho-1) \le \gamma\exp\bigl(-\rho\bigl(1+o(1)\bigr)\bigr), \tag{3.5}$$

$$v_\rho(\rho-1) \le \gamma\exp\bigl(-\rho\bigl(1+o(1)\bigr)\bigr). \tag{3.6}$$

Combine (3.2), (3.3), (3.4), (3.5), and (3.6), then

$$c_\rho \le c_* + \gamma^2\exp\bigl(-2\rho\bigl(1+o(1)\bigr)\bigr).$$

Next we find the lower bound of c_ρ. Let $\overline{v}_\rho$ be the solution of

$$\begin{cases} -\Delta u + V(r)u = 0, & \text{in } \mathbb{R}^N\backslash B_{\rho-1}, \\ u(\rho-1) = w_\rho(\rho-1), & u(+\infty) = 0, \end{cases}$$

then

$$\int_{\mathbb{R}^N\backslash B_{\rho-1}} V\overline{v}_\rho^2 = \int_{\mathbb{R}^N\backslash B_{\rho-1}} \Delta\overline{v}_\rho\cdot\overline{v}_\rho = \nabla\overline{v}_\rho\cdot\overline{v}_\rho|_{\partial B_{\rho-1}} - \int_{\mathbb{R}^N\backslash B_{\rho-1}} |\nabla\overline{v}_\rho|^2, \tag{3.7}$$

so

$$\int_{\mathbb{R}^N\backslash B_{\rho-1}} |\nabla\overline{v}_\rho|^2 + V\overline{v}_\rho^2 = \nabla\overline{v}_\rho\cdot\overline{v}_\rho|_{\partial B_{\rho-1}}.$$

Define

$$\hat{w}_\rho(r) = \begin{cases} w_\rho(r), & 0 \le r \le \rho-1, \\ \overline{v}_\rho(r), & \rho-1 \le r \le \infty. \end{cases}$$

For $t \geq 0$,

$$\begin{aligned}
I_\rho(w_\rho) &= \frac{1}{2}\int_{B_\rho}\left(|\nabla w_\rho|^2 + Vw_\rho^2\right) - \int_{B_\rho} F(w_\rho) \\
&= \left(\frac{1}{2}\int_{B_{\rho-1}}\left(|\nabla w_\rho|^2 + Vw_\rho^2\right) - \int_{B_{\rho-1}} F(w_\rho)\right) \\
&\quad + \left(\frac{1}{2}\int_{B_\rho\setminus B_{\rho-1}}\left(|\nabla w_\rho|^2 + Vw_\rho^2\right) - \int_{B_\rho\setminus B_{\rho-1}} F(w_\rho)\right). \qquad (3.8)
\end{aligned}$$

For the second part in (3.8), by $I_\rho(w_\rho) = \max_{t>0} I_\rho(tw_\rho)$, for $t > 0$,

$$\begin{aligned}
&\frac{1}{2}\int_{B_\rho\setminus B_{\rho-1}}\left(|\nabla w_\rho|^2 + Vw_\rho^2\right) - \int_{B_\rho\setminus B_{\rho-1}} F(w_\rho) \\
&\quad \geq \frac{t^2}{2}\int_{B_\rho\setminus B_{\rho-1}}\left(|\nabla w_\rho|^2 + Vw_\rho^2\right) - \int_{B_\rho\setminus B_{\rho-1}} F(tw_\rho) \\
&\quad \geq \frac{1}{2}\int_{B_\rho\setminus B_{\rho-1}}\left(t^2|\nabla w_\rho|^2 + \max_{\rho-1\leq r\leq\rho}\left\{V - \frac{F(tw_\rho)}{(tw_\rho)^2}\right\}t^2w_\rho^2\right).
\end{aligned}$$

So

$$\begin{aligned}
I_\rho(tw_\rho) &= \frac{t^2}{2}\int_{B_\rho}\left(|\nabla w_\rho|^2 + Vw_\rho^2\right) - \int_{B_\rho} F(tw_\rho) \\
&\geq \frac{t^2}{2}\int_{B_{\rho-1}}\left(|\nabla w_\rho|^2 + Vw_\rho^2\right) - \int_{B_{\rho-1}} F(tw_\rho) \\
&\quad + \frac{1}{2}\int_{B_\rho\setminus B_{\rho-1}}\left(t^2|\nabla w_\rho|^2 + \max_{\rho-1\leq r\leq\rho}\left\{V - \frac{F(tw_\rho)}{(tw_\rho)^2}\right\}t^2w_\rho^2\right) \\
&= \frac{t^2}{2}\int_{B_{\rho-1}}\left(|\nabla w_\rho|^2 + Vw_\rho^2\right) - \int_{B_{\rho-1}} F(tw_\rho) \\
&\quad + \frac{t^2}{2}\int_{\mathbb{R}^N\setminus B_{\rho-1}}\left(|\nabla \bar{v}_\rho|^2 + V\bar{v}_\rho^2\right) - \frac{t^2}{2}\int_{\mathbb{R}^N\setminus B_{\rho-1}}\left(|\nabla \bar{v}_\rho|^2 + V\bar{v}_\rho^2\right) \\
&\quad + \frac{1}{2}\int_{B_\rho\setminus B_{\rho-1}} t^2|\nabla w_\rho|^2 + \frac{1}{2}\int_{B_\rho\setminus B_{\rho-1}}\max_{\rho-1\leq r\leq\rho}\left\{V - \frac{F(tw_\rho)}{(tw_\rho)^2}\right\}t^2w_\rho^2 \\
&\geq \frac{t^2}{2}\int_{B_{\rho-1}}\left(|\nabla w_\rho|^2 + Vw_\rho^2\right) - \int_{B_{\rho-1}} F(tw_\rho) + \frac{t^2}{2}\int_{\mathbb{R}^N\setminus B_{\rho-1}}\left(|\nabla \bar{v}_\rho|^2 + V\bar{v}_\rho^2\right) \\
&\quad - \int_{\mathbb{R}^N\setminus B_{\rho-1}} F(t\bar{v}_\rho) - \frac{t^2}{2}\int_{\mathbb{R}^N\setminus B_{\rho-1}}\left(|\nabla \bar{v}_\rho|^2 + V\bar{v}_\rho^2\right) \\
&\quad + \frac{1}{2}\int_{B_\rho\setminus B_{\rho-1}} t^2|\nabla w_\rho|^2 + \frac{1}{2}\int_{B_\rho\setminus B_{\rho-1}}\max_{\rho-1\leq r\leq\rho}\left\{V - \frac{F(tw_\rho)}{(tw_\rho)^2}\right\}t^2w_\rho^2 \\
&= I(t\hat{w}_\rho) - \frac{t^2}{2}\int_{\mathbb{R}^N\setminus B_{\rho-1}}\left(|\nabla \bar{v}_\rho|^2 + V\bar{v}_\rho^2\right) + \frac{1}{2}\int_{B_\rho\setminus B_{\rho-1}} t^2|\nabla w_\rho|^2 \\
&\quad + \frac{1}{2}\int_{B_\rho\setminus B_{\rho-1}}\max_{\rho-1\leq r\leq\rho}\left\{V - \frac{F(tw_\rho)}{(tw_\rho)^2}\right\}t^2w_\rho^2,
\end{aligned}$$

where I is defined in (1.3). Take (3.7) in the last equality of the above, then

$$\begin{aligned} I_\rho(tw_\rho) &\geq I(t\hat{w}_\rho) - \frac{t^2}{2}\nabla\overline{v}_\rho \cdot \overline{v}_\rho|_{\partial B_{\rho-1}} + \frac{1}{2}\int_{B_\rho\setminus B_{\rho-1}} t^2|\nabla w_\rho|^2 \\ &\quad + \frac{1}{2}\int_{B_\rho\setminus B_{\rho-1}} \max_{\rho-1\leq r\leq\rho}\left\{V - \frac{F(tw_\rho)}{(tw_\rho)^2}\right\} t^2 w_\rho^2. \end{aligned} \tag{3.9}$$

Choose t_ρ such that $I(t_\rho\hat{w}_\rho) \geq c_*$, and let $w_\rho \to w$ in H^1, then as $\rho \to \infty$, $t_\rho \to 1$. Moreover, $\max_{\rho-1\leq r\leq\rho}\{V - \frac{F(t_\rho w_\rho)}{(t_\rho w_\rho)^2}\} \to \max_{\rho-1\leq r\leq\rho} V$, as $\rho \to \infty$.

Next we consider $\hat{v}_\rho$, which gives the solution of

$$\begin{cases} -\Delta u + \max\{V - \frac{F(t_\rho w_\rho)}{(t_\rho w_\rho)^2}\}u = 0, & \text{in } B_\rho\setminus B_{\rho-1}, \\ u(\rho-1) = w_\rho(\rho-1), & u(\rho) = 0. \end{cases}$$

Here $\hat{v}_\rho$ works as the comparison function. Similarly to the computation of (3.1) and (3.7),

$$\begin{aligned} &\int_{B_\rho\setminus B_{\rho-1}} \max_{\rho-1\leq r\leq\rho}\left\{V - \frac{F(t_\rho w_\rho)}{(t_\rho w_\rho)^2}\right\}\hat{v}_\rho^2 \\ &\quad = \int_{B_\rho\setminus B_{\rho-1}} \Delta\hat{v}_\rho \cdot \hat{v}_\rho \\ &\quad = \nabla\hat{v}_\rho \cdot \hat{v}_\rho|_{\partial B_{\rho-1}} - \int_{B_\rho\setminus B_{\rho-1}} |\nabla\hat{v}_\rho|^2, \end{aligned}$$

that is,

$$\int_{B_\rho\setminus B_{\rho-1}} \left(|\nabla\hat{v}_\rho|^2 + \max_{\rho-1\leq r\leq\rho}\left\{V - \frac{F(t_\rho w_\rho)}{(t_\rho w_\rho)^2}\right\}\hat{v}_\rho^2\right) = \nabla\hat{v}_\rho \cdot \hat{v}_\rho|_{\partial B_{\rho-1}}. \tag{3.10}$$

Take (3.10) in (3.9); by the definition of $\overline{v}_\rho$ and $\hat{v}_\rho$, then

$$\begin{aligned} I_\rho(tw_\rho) &\geq I(t_\rho\hat{w}_\rho) - \frac{t_\rho^2}{2}\nabla\overline{v}_\rho \cdot \overline{v}_\rho|_{\partial B_{\rho-1}} + \frac{t_\rho^2}{2}\int_{B_\rho\setminus B_{\rho-1}} |\nabla\hat{v}_\rho|^2 \\ &\quad + \frac{t_\rho^2}{2}\int_{B_\rho\setminus B_{\rho-1}} \max_{\rho-1\leq r\leq\rho}\left\{V - \frac{F(t_\rho w_\rho)}{(t_\rho w_\rho)^2}\right\}\hat{v}_\rho^2 \\ &= I(t_\rho\hat{w}_\rho) - \frac{t_\rho^2}{2}\nabla\overline{v}_\rho \cdot \overline{v}_\rho|_{\partial B_{\rho-1}} + \frac{t_\rho^2}{2}\nabla\hat{v}_\rho \cdot \hat{v}_\rho|_{\partial B_{\rho-1}} \\ &\geq c_* + \frac{t_\rho^2}{2} w_\rho(\rho-1)\big(\hat{v}_\rho'(\rho-1) - \overline{v}_\rho'(\rho-1)\big). \end{aligned} \tag{3.11}$$

By Lemma 2.2, $\hat{v}_\rho'(\rho-1) - \overline{v}_\rho'(\rho-1) \geq w_\rho(\rho-1)$. And by Lemma 2.1 and (3.11),

$$c_\rho = I_\rho(w_\rho) = \max_{t>0} I_\rho(tw_\rho) \geq c_* + \gamma^2\exp\big(-2\rho\big(1+o(1)\big)\big).$$

So we conclude

$$c_\rho = c_* + \gamma^2\exp\big(-2\rho\big(1+o(1)\big)\big). \qquad \square$$

Competing interests
The author declares to have no competing interests.

Author's contributions
The work was carried out by the author.

Acknowledgements
The author is supported by China Postdoctoral Science Foundation (2014M551830) and the Nonlinear Analysis Innovation Team (IRTL1206) funded by Fujian Normal University.

Received: 30 July 2014 Accepted: 14 August 2014 Published online: 25 September 2014

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doi:10.1186/s13661-014-0202-7
Cite this article as: Zeng: The estimation and profile of the critical value for a Schrödinger equation. *Boundary Value Problems* 2014 **2014**:202.