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Global weak solutions for a generalized Dullin-Gottwald-Holm equation in the space $H^1(R)$

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Abstract

The existence of global weak solutions of the Cauchy problem for a generalized Dullin-Gottwald-Holm equation is established under the assumption that the initial value $u_0(x)$ merely lies in the space $H^1(R)$. The limit of the viscous approximation for the equation is used to prove the global existence in the space $C([0,\infty)\times R)\cap L^\infty([0,\infty);H^1(R))$. The elements in our study include a one-sided super bound estimate and a space-time higher-norm estimate on the first order derivative of the solution with respect to the space variable.

MSC: 35Q35; 35Q51

Keywords: global weak solution; the Dullin-Gottwald-Holm equation; viscous approximation

1 Introduction

Dullin, Gottwald and Holm [1] investigated the following equation for a unidirectional water wave:

$$u_t-\alpha^2u_{txx}+c_0u_x+3uu_x+\gamma u_{xxx}=\alpha^2(2u_xu_{xx}+uu_{xxx}),\tag{1}$$

where $u(t,x)$ is the fluid velocity, $x\in R$, $t\ge 0$, the constants α^2 and $\frac{\gamma}{c_0}$ are squares of length scales, and $c_0=\sqrt{gh}$ is the linear wave speed for undisturbed water resting at spatial infinity (see [2]). The Dullin, Gottwald and Holm equation (1) was derived through an asymptotic expansion from the Hamiltonian of Euler's equation in the shallow water regime. It possesses bi-Hamiltonian and has a Lax pair formulation [2, 3]. The equation is an integrable system and contains both the Korteweg-de Vries and Camassa-Holm equations [4, 5] as limiting cases.

Extensive research has been carried out to study various dynamic properties of the Dullin, Gottwald and Holm model (DGH). Tang and Yang [6] found general explicit expressions of the two wave solutions for (1) by using bifurcation phase portraits of the traveling wave system. Mustafa [7] studied the local existence and uniqueness of solutions for the DGH equation with continuously differentiable periodic initial data. Zhou [8] found the best constants for two convolution problems on the unit circle via a variational method, and then applied the best constants on a nonlinear integrable shallow water equation (the Dullin, Gottwald and Holm equation) and obtained sufficient conditions required on the initial data to guarantee a finite time singularity formation for the corresponding solutions. Zhou and Guo [9] investigated the persistence properties of the strong solutions and infinite propagation speed for the DGH model. The existence of global weak solutions to (1) is proved by Zhang and Yin [10] under certain conditions imposed on the initial value. In [11], Tian, Gui and Liu established the global well-posedness of strong solution $H^s(R)$ with $s > 3$ provided that the initial data u_0 satisfies certain positive conditions. The blow-up of solutions for the DGH equation was also discussed in [11] and it was established that, similarly to the Camassa-Holm equation, singularities can arise only in the form of wave breaking, namely, the solution remains bounded but its slope becomes unbounded in finite time [12–14]). Mustafa [15] used the mathematical transform $V(T,X) = u(t,x) + \frac{\gamma}{\alpha^2}$, $T = \alpha t$, $X = \alpha x$ to reduce DGH (1) to a classical Camassa-Holm equation. Namely, $V(T,X)$ satisfies the Camassa-Holm equation. Mustafa [15] applied the approaches in Bressan and Constantin [16] to establish the existence of global conservative solution with constant $H^1(R)$ energy of $V(T,X)$ provided that $V(0,x) \in H^1(R)$, and then obtained many meaningful conclusions for $V(t,x)$. As we know, $V(T,X) \in H^1(R)$ is not equal to $u(t,x) \in H^1(R)$ and $u(t,x) \in H^1(R)$ cannot derive $V(T,X) \in H^1(R)$. In this paper, we only assume $u(0,x) \in H^1(R)$ to establish the existence of global weak solutions to a generalized Dullin-Gottwald-Holm equation in the space $H^1(R)$.

In fact, we are interested in the Cauchy problem for the nonlinear model

$$u_t - \alpha^2 u_{txx} + \partial_x f(u) + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + u u_{xxx}), \tag{2}$$

where $\alpha > 0$, $\gamma \geq 0$, $f(u)$ is a polynomial with order n. When $f(u) = c_0 u + \frac{3}{2}u^2$, $\gamma = 0$, $\alpha = 1$, (2) is the classical Camassa-Holm equation [3]. When $f(u) = c_0 u + \frac{3}{2}u^2$, $\gamma \neq 0$, (2) becomes the Dullin-Gottwald-Holm equation (1).

To link with previous works in the field of study, we review here several works on the global weak solution for the Camassa-Holm and Degasperis-Procesi equations. The existence and uniqueness results for the global weak solutions of the Camassa-Holm model have been proved by Constantin and Escher [17] and Danchin [18, 19] by assuming that the initial data satisfy the sign condition. Xin and Zhang [20] established the global existence of the weak solution for the Camassa-Holm equation in the energy space $H^1(R)$ without imposing any sign conditions on the initial value, and the uniqueness of the weak solution was then obtained under certain conditions on the solution [21]. Coclite *et al.* [22] employed the analysis presented in [20, 21] and investigated the global weak solutions for a generalized hyperelastic rod wave equation or a generalized Camassa-Holm equation. The existence of a strongly continuous semigroup of global weak solutions for the generalized hyperelastic rod equation with the initial value in the space $H^1(R)$ was established in [22]. Under the sign condition for the initial value, Yin and Lai [23] proved the existence and uniqueness of a global weak solution for a nonlinear shallow water equation, which includes the Camassa-Holm and Degasperis-Procesi equations as special cases. The existence of global weak solutions for a weakly dissipative Camassa-Holm equation was established in Lai *et al.* [24].

The aim of this work is to study the existence of global weak solutions for the generalized Dullin-Gottwald-Holm equation (2) in the space $C([0,\infty) \times R) \cap L^\infty([0,\infty); H^1(R))$ under the assumption $u_0(x) \in H^1(R)$. The key elements in our analysis are that we establish a one-sided upper bound and space-time higher-norm estimates on the first order derivatives of the solution. The limit of viscous approximations for the equation is applied to establish the existence of the global weak solution. Here we should mention that our assumption $u_0 \in H^1(R)$ has never been used as a unique condition to prove the global existence of weak solutions for DGH equation (1) or the generalized Dullin-Gottwald-Holm equation (2) in the literature.

Here we state that the ideas to prove our main result come from those presented in [20] (also see [22]). We need to show that the derivative $q_\varepsilon = \frac{\partial u_\varepsilon(t,x)}{\partial x}$ (see (17)), which is only weakly compact, converges strongly. Namely, the strong convergence of q_ε is necessary to be established if we want to send ε to zero in the viscous problem (11). One of key factors, which is employed to prove that weak convergence is equal to strong convergence, is the higher integrability estimate (18) in Section 3. It means that the weak limit of q_ε^2 does not contain singular measures.

The rest of this paper is organized as follows. The main result is given in Section 2. In Section 3, we present the viscous problem and give a corresponding well-posedness result. An upper bound, a higher integrability estimate and some basic compactness properties for the viscous approximations are also established in Section 3. Strong compactness of the derivative of the viscous approximations is obtained in Section 4, where the main result for (2) is proved.

2 Main result

Consider the Cauchy problem for (2)

$$\begin{cases} u_t - \alpha^2 u_{txx} + \partial_x f(u) + \gamma u_{xxx} = \alpha^2(2u_x u_{xx} + u u_{xxx}), \\ u(0,x) = u_0(x), \end{cases} \tag{3}$$

which is equivalent to

$$\begin{cases} u_t + u u_x - \frac{\gamma}{\alpha^2} u_x + \frac{\partial P}{\partial x} = 0, \\ P = \Lambda^{-2}[\frac{1}{2}u^2 - \frac{\gamma}{\alpha^2}u - \frac{\alpha^2}{2}u_x^2 - f(u)], \\ u(0,x) = u_0(x), \end{cases} \tag{4}$$

where the operator $\Lambda^2 = 1 - \alpha^2 \frac{\partial^2}{\partial x^2}$. For any $g(x) \in L^2(R)$, we have

$$\begin{aligned} \Lambda^{-2} g(x) &= \frac{1}{2\alpha} \int_R e^{-\frac{|x-y|}{\alpha}} g(y)\,dy \\ &= \frac{1}{2\alpha} e^{-\frac{x}{\alpha}} \int_{-\infty}^{x} e^{\frac{y}{\alpha}} g(y)\,dy + \frac{1}{2\alpha} e^{\frac{x}{\alpha}} \int_x^{\infty} e^{-\frac{y}{\alpha}} g(y)\,dy. \end{aligned} \tag{5}$$

In fact, as proved in [11], problem (3) satisfies the following conservation law:

$$\int_R \left(u^2 + \alpha^2 u_x^2\right) dx = \int_R \left(u_0^2 + \alpha^2 \left(\frac{\partial u_0}{\partial x}\right)^2\right) dx. \tag{6}$$

For simplicity, throughout this article, we assume $u_0(x) \in H^1(R)$ and let c denote any positive constant which is independent of parameter ε.

Now we introduce the definition of a weak solution to the Cauchy problem (3) or (4) (see [20]).

Definition 2.1 A continuous function $u : [0,\infty) \times R \to R$ is said to be a global weak solution to the Cauchy problem (3) if

(i) $u \in C([0,\infty) \times R) \cap L^\infty([0,\infty); H^1(R))$;
(ii) $\|u(t,\cdot)\|_{H^1(R)} \le c\|u_0\|_{H^1(R)}$;
(iii) $u = u(t,x)$ satisfies (3) in the sense of distributions and takes on the initial value pointwise.

Now we illustrate the main result of this paper as follows.

Theorem 2.1 *Assume $u_0(x) \in H^1(R)$. Then the Cauchy problem* (3) *or* (4) *has a global weak solution $u(t,x)$ in the sense of Definition* 2.1. *Furthermore, the weak solution satisfies the following properties.*

(a) *There exists a positive constant c_0 depending on $\|u_0\|_{H^1(R)}$ and the coefficients of* (2) *such that the following one-sided L^∞ norm estimate on the first order spatial derivative holds*:

$$\frac{\partial u(t,x)}{\partial x} \le \frac{2}{t} + c_0, \quad \text{for } (t,x) \in [0,\infty) \times R. \tag{7}$$

(b) *Let $0 < \delta < 1$, $T > 0$, and $a, b \in R$, $a < b$. Then there exists a positive constant c_1 depending only on $\|u_0\|_{H^1(R)}$, δ, T, a, b, and the coefficients of* (2) *such that the following space higher integrability estimate holds*:

$$\int_0^t \int_a^b \left| \frac{\partial u(t,x)}{\partial x} \right|^{2+\delta} dx \le c_1. \tag{8}$$

3 Viscous approximations

Defining

$$\phi(x) = \begin{cases} e^{\frac{1}{x^2-1}}, & |x| < 1, \\ 0, & |x| \ge 1, \end{cases} \tag{9}$$

and setting the mollifier $\phi_\varepsilon(x) = \varepsilon^{-\frac{1}{4}}\phi(\varepsilon^{-\frac{1}{4}}x)$ with $0 < \varepsilon < \frac{1}{4}$ and $u_{\varepsilon,0} = \phi_\varepsilon \star u_0$, we know that $u_{\varepsilon,0} \in C^\infty$ for any $u_0 \in H^s$, $s > 0$ (see [25, 26]). In fact, we have

$$\|u_{\varepsilon,0}\|_{H^1(R)} \le c\|u_0\|_{H^1(R)} \quad \text{and} \quad u_{\varepsilon,0} \to u_0 \quad \text{in } H^1(R), \tag{10}$$

where c is independent of parameter ε.

The existence of a weak solution to the Cauchy problem (4) will be established by proving compactness of a sequence of smooth functions $\{u_\varepsilon\}_{\varepsilon>0}$ solving the following viscous problem:

$$\begin{cases} \frac{\partial u_\varepsilon}{\partial t} + u_\varepsilon \frac{\partial u_\varepsilon}{\partial x} - \frac{\gamma}{\alpha^2}\frac{\partial u_\varepsilon}{\partial x} + \frac{\partial P_\varepsilon}{\partial x} = \varepsilon u_{xx}, \\ P_\varepsilon = \Lambda^{-2}[\frac{1}{2}u_\varepsilon^2 - \frac{\gamma}{\alpha^2}u_\varepsilon - \frac{\alpha^2}{2}(\frac{\partial u_\varepsilon}{\partial x})^2 - f(u_\varepsilon)], \\ u_\varepsilon(0,x) = u_{\varepsilon,0}. \end{cases} \tag{11}$$

Now start our analysis by establishing the following well-posedness result for problem (11).

Lemma 3.1 *Provided that* $u_0 \in H^1(R)$. *Then for any* $\sigma \geq 3$, *there exists a unique solution* $u_\varepsilon \in C([0,\infty); H^\sigma(R))$ *to the Cauchy problem* (11). *Moreover, for any* $t > 0$, *we have*

$$\begin{aligned}&\int_R \left(u_\varepsilon^2 + \alpha^2\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2\right)dx + 2\varepsilon \int_0^t \int_R \left(\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2 + \left(\frac{\partial^2 u_\varepsilon}{\partial x^2}\right)^2\right)(s,x)\,dx\,ds \\ &\quad = \int_R \left(u_{\varepsilon,0}^2 + \alpha^2\left(\frac{\partial u_{\varepsilon,0}}{\partial x}\right)^2\right)dx, \end{aligned} \tag{12}$$

or

$$\left\|u_\varepsilon(t,\cdot)\right\|_{H^1(R)}^2 + 2\varepsilon \int_0^t \left\|\frac{\partial u_\varepsilon}{\partial x}(s,\cdot)\right\|_{H^1(R)}^2 ds \leq c\|u_0\|_{H^1(R)}, \tag{13}$$

where c is a constant independent of ε.

Proof For any $\sigma \geq 3$ and $u_0 \in H^1(R)$, we have $u_{\varepsilon,0} \in C([0,\infty); H^\sigma(R))$. From Theorem 2.1 in [22] or Theorem 2.3 in [27], we conclude that the problem (11) has a unique solution $u_\varepsilon \in C([0,\infty); H^\sigma(R))$ for an arbitrary $\sigma > 3$.

We know that the first equation in system (11) is equivalent to the form

$$\begin{aligned}&\frac{\partial u_\varepsilon}{\partial t} - \alpha^2 \frac{\partial^3 u_\varepsilon}{\partial x^2 t} + \partial_x f(u_\varepsilon) + \gamma \frac{\partial^3 u}{\partial x^3} \\ &\quad = 2\alpha^2 \frac{\partial u_\varepsilon}{\partial x}\frac{\partial^2 u_\varepsilon}{\partial x^2} + \alpha^2 u_\varepsilon \frac{\partial^3 u_\varepsilon}{\partial x^3} + \varepsilon\left(\frac{\partial^2 u_\varepsilon}{\partial x^2} - \frac{\partial^4 u_\varepsilon}{\partial x^4}\right), \end{aligned} \tag{14}$$

from which we derive

$$\frac{1}{2}\frac{d}{dt}\int_R \left(u_\varepsilon^2 + \alpha^2\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2\right)dx + \varepsilon \int_R \left(\left(\frac{\partial u_\varepsilon}{\partial x}\right)^2 + \left(\frac{\partial^2 u_\varepsilon}{\partial x^2}\right)^2\right)dx = 0, \tag{15}$$

which completes the proof. □

From Lemma 3.1, we have

$$\|u_\varepsilon\|_{L^\infty(R)} \leq c\|u_\varepsilon\|_{H^1(R)} \leq c\|u_{\varepsilon,0}\|_{H^1(R)} \leq c\|u_0\|_{H^1(R)}, \tag{16}$$

where c is a constant independent of ε.

Differentiating the first equation of problem (11) with respect to x and writing $\frac{\partial u_\varepsilon}{\partial x} = q_\varepsilon$, we obtain

$$\begin{aligned}&\frac{\partial q_\varepsilon}{\partial t} + \left(u_\varepsilon - \frac{\gamma}{\alpha^2}\right)\frac{\partial q_\varepsilon}{\partial x} - \varepsilon \frac{\partial^2 q_\varepsilon}{\partial x^2} + \frac{1}{2}q_\varepsilon^2 \\ &\quad = \frac{1}{\alpha^2}f(u_\varepsilon) - \frac{1}{2\alpha^2}u_\varepsilon^2 + \frac{\gamma}{\alpha^4}u_\varepsilon + \Lambda^{-2}\left[\frac{1}{2\alpha^2}u_\varepsilon^2 - \frac{\gamma}{\alpha^4}u_\varepsilon - \frac{1}{2}q_\varepsilon^2 - \frac{1}{\alpha^2}f(u_\varepsilon)\right] \\ &\quad = Q_\varepsilon(t,x). \end{aligned} \tag{17}$$

Lemma 3.2 *Let* $0 < \delta < 1$, $T > 0$, *and* $a, b \in R$, $a < b$. *Then there exists a positive constant* c_1 *depending only on* $\|u_0\|_{H^1(R)}$, γ, T, a, b, *and the coefficients of* (2), *but independent of* ε, *such that the space higher integrability estimate holds*

$$\int_0^T \int_a^b \left| \frac{\partial u_\varepsilon(t,x)}{\partial x} \right|^{2+\delta} dx \le c_1, \tag{18}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ *is the unique solution of problem* (11).

Proof The proof is a variant of the proof presented in Xin and Zhang [20] (also see Coclite *et al.* [22]). Let $\chi \in C^\infty(R)$ be a cut-off function such that $0 < \chi < 1$ and

$$\phi(x) = \begin{cases} 1, & \text{if } x \in [a,b], \\ 0, & \text{if } x \in (-\infty, a-1] \cup [b+1, \infty). \end{cases} \tag{19}$$

Considering the map $\theta(\xi) := \xi(1+|\xi|)^\delta$, $\xi \in R$, $0 < \delta < 1$, and observing that

$$\begin{aligned} \theta'(\xi) &= \bigl(1 + (1+\delta)|\xi|\bigr)\bigl(1+|\xi|\bigr)^{\delta-1}, \\ \theta''(\xi) &= \delta \operatorname{sign}(\xi)\bigl(1+|\xi|\bigr)^{\delta-2}\bigl(2+(1+\delta)|\xi|\bigr) \\ &= \delta(1+\delta)\operatorname{sign}(\xi)\bigl(1+|\xi|\bigr)^{\delta-1} + (1-\delta)\delta \operatorname{sign}(\xi)\bigl(1+|\xi|\bigr)^{\delta-2}, \end{aligned}$$

we have

$$\bigl|\theta(\xi)\bigr| \le |\xi| + |\xi|^{1+\delta}, \qquad \bigl|\theta'(\xi)\bigr| \le 1 + (1+\delta)|\xi|, \qquad \bigl|\theta''(\xi)\bigr| \le 2\delta \tag{20}$$

and

$$\begin{aligned} \xi\theta(\xi) - \frac{1}{2}\xi^2\theta'(\xi) &= \frac{1-\delta}{2}\xi^2\bigl(1+|\xi|\bigr)^\delta + \frac{\delta}{2}\xi^2\bigl(1+|\xi|\bigr)^{\delta-1} \\ &\ge \frac{1-\delta}{2}\xi^2\bigl(1+|\xi|\bigr)^\delta. \end{aligned} \tag{21}$$

Differentiating the first equation of problem (11) with respect to x and writing $u = u_\varepsilon$ and $\frac{\partial u_\varepsilon}{\partial x} = q_\varepsilon = q$ for simplicity, we obtain

$$\begin{aligned} &\frac{\partial q}{\partial t} + \left(u - \frac{\gamma}{\alpha^2}\right)\frac{\partial q}{\partial x} - \varepsilon\frac{\partial^2 q}{\partial x^2} + \frac{1}{2}q^2 \\ &\quad = \frac{1}{\alpha^2}f(u) - \frac{1}{2\alpha^2}u^2 + \frac{\gamma}{\alpha^4}u + \Lambda^{-2}\left[\frac{1}{2\alpha^2}u^2 - \frac{\gamma}{\alpha^4}u - \frac{1}{2}q^2 - \frac{1}{\alpha^2}f(u)\right] \\ &\quad = Q_\varepsilon(t,x). \end{aligned} \tag{22}$$

Multiplying (22) by $\chi\theta'(q)$, using the chain rule and integrating over $\Pi_T := [0,T] \times R$, we have

$$\begin{aligned} &\int_{\Pi_T} \chi(x)q\theta(q)\,dt\,dx - \frac{1}{2}\int_{\Pi_T} q^2\chi(x)\theta'(q)\,dt\,dx \\ &\quad = \int_R \chi(x)\bigl(\theta\bigl(q(T,x)\bigr) - \theta\bigl(q(0,x)\bigr)\bigr)\,dx - \int_{\Pi_T}\left(u - \frac{\gamma}{\alpha^2}\right)\chi'(x)\theta(q)\,dt\,dx \end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\int_{\Pi_T}\frac{\partial q}{\partial x}\chi'(x)\theta'(q)\,dt\,dx+\varepsilon\int_{\Pi_T}\left(\frac{\partial q}{\partial x}\right)^2\chi(x)\theta''(q)\,dt\,dx\\
&-\int_{\Pi_T}Q_\varepsilon(t,x)\chi(x)\theta'(q)\,dt\,dx.
\end{aligned}
\tag{23}
$$

From (21), we get

$$
\begin{aligned}
&\int_{\Pi_T}\chi(x)q\theta(q)\,dt\,dx-\frac{1}{2}\int_{\Pi_T}q^2\chi(x)\theta'(q)\,dt\,dx\\
&\quad=\int_{\Pi_T}\chi(x)\left(q\theta(q)-\frac{1}{2}q^2\theta'(q)\right)dt\,dx\\
&\quad\geq\frac{(1-\delta)}{2}\int_{\Pi_T}\chi(x)q^2\big(1+|q|\big)^\delta\,dt\,dx.
\end{aligned}
\tag{24}
$$

Using the Hölder inequality, (16), and (20) yields

$$
\begin{aligned}
\left|\int_R\chi(x)\theta(q)\,dx\right|&\leq\int_R\chi(x)\big(|q|^{1+\delta}+|q|\big)\,dx\\
&\leq\|\chi\|_{L^{\frac{2}{1-\delta}}(R)}\|q\|^{1+\delta}_{L^2(R)}+\|\chi\|_{L^2(R)}\|q\|_{L^2(R)}\\
&\leq c(b-a+2)^{\frac{1-\delta}{2}}\|u_0\|^{1+\delta}_{H^1(R)}+(b-a+2)^{\frac{1}{2}}\|u_0\|_{H^1(R)}
\end{aligned}
\tag{25}
$$

and

$$
\begin{aligned}
&\left|\int_{\Pi_T}\left(u-\frac{\gamma}{\alpha^2}\right)\chi'(x)\theta(q)\,dt\,dx\right|\\
&\quad\leq\int_{\Pi_T}\left(|u|+\frac{\gamma}{\alpha^2}\right)\big|\chi'(x)\big|\big(|q|^{1+\delta}+|q|\big)\,dt\,dx\\
&\quad\leq\left(\|u_0\|_{H^1(R)}+\frac{\gamma}{\alpha^2}\right)\int_0^T\big(\big\|\chi'\big\|_{L^{\frac{2}{1-\delta}}(R)}\|q\|^{1+\delta}_{L^2(R)}+\big\|\chi'\big\|_{L^2(R)}\|q\|_{L^2(R)}\big)\,dt\\
&\quad\leq Tc\big(\big\|\chi'\big\|_{L^{\frac{2}{1-\delta}}(R)}\|u_0\|^{1+\delta}_{H^1(R)}+\big\|\chi'\big\|_{L^2(R)}\|u_0\|_{H^1(R)}\big).
\end{aligned}
\tag{26}
$$

Integration by parts gives rise to

$$
\int_{\Pi_T}\frac{\partial q}{\partial x}\chi'(x)\theta'(q)\,dt\,dx=-\int_{\Pi_T}\theta(q)\chi''(x)\,dt\,dx.
\tag{27}
$$

From (20), (27), and the Hölder inequality, we have

$$
\begin{aligned}
\varepsilon\left|\int_{\Pi_T}\frac{\partial q}{\partial x}\chi'(x)\theta'(q)\,dt\,dx\right|&\leq\varepsilon\int_{\Pi_T}\big|\theta(q)\big|\big|\chi''(x)\big|\,dt\,dx\\
&\leq\varepsilon\int_{\Pi_T}\big|\chi''(x)\big|\big(|q|^{1+\delta}+|q|\big)\,dt\,dx\\
&\leq\varepsilon\int_0^T\big(\big\|\chi''\big\|_{L^{\frac{2}{1-\delta}}(R)}\|q\|^{1+\delta}_{L^2(R)}+\big\|\chi''\big\|_{L^2(R)}\|q\|_{L^2(R)}\big)\,dt\\
&\leq\varepsilon cT\big(\big\|\chi''\big\|_{L^{\frac{2}{1-\delta}}(R)}\|u_0\|^{1+\delta}_{H^1(R)}+\big\|\chi''\big\|_{L^2(R)}\|u_0\|_{H^1(R)}\big).
\end{aligned}
\tag{28}
$$

Using (20) and Lemma 3.1, we have

$$\varepsilon\left|\int_{\Pi_T}\left(\frac{\partial q}{\partial x}\right)^2\chi(x)\theta''(q)\,dt\,dx\right| \le 2\delta\varepsilon\int_{\Pi_T}\left(\frac{\partial q}{\partial x}\right)^2 dt\,dx \le \delta c\|u_0\|^2_{H^1(R)}. \tag{29}$$

It follows from (5) and (16) that

$$\begin{aligned}
&\left|\Lambda^{-2}\left[\frac{1}{2\alpha^2}u^2 - \frac{\gamma}{\alpha^4}u - \frac{1}{2}q^2 - \frac{1}{\alpha^2}f(u)\right]\right| \\
&\quad \le \left|\frac{1}{2\alpha}\int_R e^{-|\frac{x-y}{\alpha}|}\left[\frac{1}{2\alpha^2}u^2 - \frac{\gamma}{\alpha^4}u - \frac{1}{2}q^2 - \frac{1}{\alpha^2}f(u)\right]dy\right| \\
&\quad \le c\left(\|u_0\|^2_{H^1(R)} + \|u_0\|_{H^1(R)} + \|u_0\|^n_{H^1(R)}\right) \\
&\quad \le c
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
&\left|\frac{1}{\alpha^2}f(u) - \frac{1}{2\alpha^2}u^2 + \frac{\gamma}{\alpha^4}u\right| \\
&\quad \le c\left|\left\|f(u)\right\|_{L^\infty(R)} + \|u\|^2_{L^\infty(R)} + \|u\|_{L^\infty(R)}\right| \\
&\quad \le c\left(\|u_0\|^2_{H^1(R)} + \|u_0\|_{H^1(R)} + \|u_0\|^n_{H^1(R)}\right) \\
&\quad \le c.
\end{aligned} \tag{31}$$

From (30) and (31), we know that there exists a positive constant c depending on $\|u_0\|_{H^1(R)}$ and the coefficients of (2), but independent of ε, such that

$$\left\|Q_\varepsilon(t,x)\right\|_{L^\infty(R)} \le c, \tag{32}$$

which results in

$$\begin{aligned}
&\left|\int_{\Pi_T} Q_\varepsilon(t,x)\chi(x)\theta'(q)\,dt\,dx\right| \\
&\quad \le c\int_{\Pi_T}\left|\chi(x)\right|\left((1+\delta)|q|+1\right)dt\,dx \\
&\quad \le cT\left((1+\delta)\left\|\chi(x)\right\|_{L^2(R)}\|u_0\|_{H^1(R)} + \int_R\left|\chi(x)\right|dx\right).
\end{aligned} \tag{33}$$

By inequalities (23)-(29), and (33) we derive the desired result (15). □

Lemma 3.3 *There exists a positive constant c depending only on* $\|u_0\|_{H^1(R)}$ *and the coefficients of* (2) *such that*

$$\left\|Q_\varepsilon(t,\cdot)\right\|_{L^\infty(R)} \le c, \tag{34}$$

$$\left\|Q_\varepsilon(t,\cdot)\right\|_{L^2(R)} \le c, \tag{35}$$

$$\left\|P_\varepsilon(t,\cdot)\right\|_{L^\infty(R)} \le c, \tag{36}$$

$$\left\| \frac{\partial P_\varepsilon(t,\cdot)}{\partial x} \right\|_{L^\infty(R)} \le c, \tag{37}$$

$$\left\| \frac{\partial P_\varepsilon(t,\cdot)}{\partial x} \right\|_{L^1(R)} \le c, \tag{38}$$

$$\left\| \frac{\partial P_\varepsilon(t,\cdot)}{\partial x} \right\|_{L^2(R)} \le c, \tag{39}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ *is the unique solution of system* (11).

Proof For simplicity, setting $u(t,x) = u_\varepsilon(t,x)$, we have

$$\begin{aligned} Q_\varepsilon(t,x) = {}& \frac{1}{\alpha^2} f(u) - \frac{1}{2\alpha^2} u^2 + \frac{\gamma}{\alpha^4} u \\ & + \Lambda^{-2}\left[\frac{1}{2\alpha^2} u^2 - \frac{\gamma}{\alpha^4} u - \frac{1}{2} q^2 - \frac{1}{\alpha^2} f(u)\right] \end{aligned} \tag{40}$$

and

$$P_\varepsilon = \Lambda^{-2}\left[\frac{1}{2} u^2 - \frac{\gamma}{\alpha^2} u - \frac{\alpha^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 - f(u)\right]. \tag{41}$$

Inequality (34) is proved in Lemma 3.2 (see (32)). Now we prove (35). Using (16) and Lemma 3.1 yields

$$\begin{aligned} & \left\| \frac{1}{\alpha^2} f(u) - \frac{1}{2\alpha^2} u^2 + \frac{\gamma}{\alpha^4} u \right\|_{L^2(R)} \\ & \quad \le c\left(\|u\|_{L^2(R)} + \|u^2\|_{L^2(R)} + \|f(u)\|_{L^2(R)}\right) \\ & \quad \le c\left(\|u_0\|_{L^2(R)} + \|u\|_{L^\infty(R)} \|u\|_{L^2(R)} + \|u\|_{L^\infty}^{n-2} \|u\|_{L^2(R)}^2\right) \\ & \quad \le c\left(\|u_0\|_{H^1(R)} + \|u_0\|_{H^1(R)}^2 + \|u_0\|_{H^1(R)}^n\right) \\ & \quad \le c. \end{aligned} \tag{42}$$

Similar to the proof of (30), we have

$$\left\| \Lambda^{-2}\left(\frac{1}{2\alpha^2} u^2 - \frac{1}{2} q^2 - \frac{1}{\alpha^2} f(u)\right) \right\|_{L^\infty(R)} \le c. \tag{43}$$

The Parseval inequality shows that

$$\|\Lambda^{-2} u\|_{L^2(R)} \le c \left\| \frac{1}{1+\alpha^2\zeta^2} \widetilde{u}(\zeta) \right\|_{L^2(R)} \le c, \tag{44}$$

where $\widetilde{u}(\zeta)$ is the Fourier transform of $u(t,x)$ with respect to x.

It follows from (43), (44), and Lemma 3.1 that

$$\begin{aligned} & \int_R \left| \Lambda^{-2}\left[\frac{1}{2\alpha^2} u^2 - \frac{\gamma}{\alpha^4} u - \frac{1}{2} q^2 - \frac{1}{\alpha^2} f(u)\right] \right|^2 dx \\ & \quad \le c \int_R \left| \Lambda^{-2}\left[\frac{1}{2\alpha^2} u^2 - \frac{1}{2} q^2 - \frac{1}{\alpha^2} f(u)\right] \right| dx + c\|\Lambda^{-2} u\|_{L^2(R)}^2 \end{aligned}$$

$$
\begin{aligned}
&= c\int_R \left| \frac{1}{2\alpha}\int_R e^{-\frac{|x-y|}{\alpha}} \left[\frac{1}{2\alpha^2}u^2 - \frac{\gamma}{\alpha^4}u - \frac{1}{2}q^2 - \frac{1}{\alpha^2}f(u)\right] dy \right| dx + c \\
&\le c\int_R \left| \frac{1}{2\alpha^2}u^2 - \frac{1}{2}q^2 - \frac{1}{\alpha^2}f(u)\right| dy \int_R e^{-\frac{|x-y|}{\alpha}}\, dx + c \\
&\le c.
\end{aligned} \tag{45}
$$

Inequalities (42) and (45) result in (35).

Since

$$
\begin{aligned}
\frac{\partial P_\varepsilon}{\partial x} &= \Lambda^{-2}\partial_x\left[\frac{1}{2}u^2 - \frac{\gamma}{\alpha^2}u - \frac{\alpha^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 - f(u)\right] \\
&= \partial_x\left(\frac{1}{2\alpha}e^{-\frac{x}{\alpha}}\int_{-\infty}^x e^{\frac{y}{\alpha}}\left[\frac{1}{2}u^2 - \frac{\gamma}{\alpha^2}u - \frac{\alpha^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 - f(u)\right] dy \right.\\
&\quad \left. + \frac{1}{2\alpha}e^{\frac{x}{\alpha}}\int_x^{\infty} e^{-\frac{y}{\alpha}}\left[\frac{1}{2}u^2 - \frac{\gamma}{\alpha^2}u - \frac{\alpha^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 - f(u)\right] dy\right) \\
&= -\frac{1}{2\alpha^2}e^{-\frac{x}{\alpha}}\int_{-\infty}^x e^{\frac{y}{\alpha}}\left[\frac{1}{2}u^2 - \frac{\gamma}{\alpha^2}u - \frac{\alpha^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 - f(u)\right] dy \\
&\quad + \frac{1}{2\alpha^2}e^{\frac{x}{\alpha}}\int_x^{\infty} e^{-\frac{y}{\alpha}}\left[\frac{1}{2}u^2 - \frac{\gamma}{\alpha^2}u - \frac{\alpha^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 - f(u)\right] dy,
\end{aligned} \tag{46}
$$

from which we obtain

$$
\begin{aligned}
\left|\frac{\partial P_\varepsilon}{\partial x}\right| &\le c\int_{-\infty}^{\infty} e^{-\frac{|x-y|}{\alpha}}\left|\frac{1}{2}u^2 - \frac{\gamma}{\alpha^2}u - \frac{\alpha^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 - f(u)\right| dy \\
&\le c\left(\|u\|_{H^1(R)} + \|u\|_{H^1(R)}^2 + \|u\|_{H^1(R)}^n\right) \\
&\le c\left(\|u_0\|_{H^1(R)} + \|u_0\|_{H^1(R)}^2 + \|u_0\|_{H^1(R)}^n\right) \\
&\le c
\end{aligned} \tag{47}
$$

and

$$
\begin{aligned}
\int_R \left|\frac{\partial P_\varepsilon}{\partial x}\right| dx &\le c\int_R\int_{-\infty}^{\infty} e^{-\frac{|x-y|}{\alpha}}\left|\frac{1}{2}u^2 - \frac{\gamma}{\alpha^2}u - \frac{\alpha^2}{2}\left(\frac{\partial u}{\partial x}\right)^2 - f(u)\right| dy\, dx \\
&\le c\left(\|u\|_{H^1(R)} + \|u\|_{H^1(R)}^2 + \|u\|_{H^1(R)}^n\right) \\
&\le c.
\end{aligned} \tag{48}
$$

The above inequalities mean that (37) and (38) hold. The inequality

$$
\left\|\frac{\partial P_\varepsilon}{\partial x}\right\|_{L^2(R)}^2 \le \left\|\frac{\partial P_\varepsilon}{\partial x}\right\|_{L^\infty(R)}\left\|\frac{\partial P_\varepsilon}{\partial x}\right\|_{L^1(R)} \tag{49}
$$

together with (37) and (38) shows that we have (39). The proof is completed. □

Lemma 3.4 *Assume that $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution of* (11). *There exists a positive constant c depending only on $\|u_0\|_{H^1(R)}$ and the coefficients of* (2) *such that the following* *one-sided L^∞ norm estimate on the first order spatial derivative holds*:

$$\frac{\partial u_\varepsilon(t,x)}{\partial x} \le \frac{2}{t} + c, \quad \text{for } (t,x) \in [0,\infty) \times R. \tag{50}$$

Proof From (22) and Lemma 3.3, we know that there exists a positive constant c depending only on $\|u_0\|_{H^1(R)}$ and the coefficients of (2) such that $\|Q_\varepsilon(t,x)\|_{L^\infty(R)} \le c$. Therefore,

$$\frac{\partial q}{\partial t} + \left(u - \frac{\gamma}{\alpha^2}\right)\frac{\partial q}{\partial x} - \varepsilon\frac{\partial^2 q}{\partial x^2} + \frac{1}{2}q^2 = Q_\varepsilon(t,x) \le c. \tag{51}$$

Let $f = f(t)$ be a supersolution of (51) associated with the initial value $q_\varepsilon(0,x) = \frac{\partial u_{\varepsilon,0}}{\partial x}$ and satisfy

$$\frac{df}{dt} + \frac{1}{2}f^2 = c, \quad t > 0, f(0) = \left\|\frac{\partial u_{\varepsilon,0}}{\partial x}\right\|_{L^\infty(R)}. \tag{52}$$

From the comparison principle for parabolic equations, we get

$$q_\varepsilon(t,x) \le f(t). \tag{53}$$

Letting $F(t) = \frac{2}{t} + \sqrt{2c}$, we have $\frac{dF(t)}{dt} + \frac{1}{2}F^2(t) - c = \frac{2\sqrt{2c}}{t} > 0$. From the comparison principle for ordinary differential equations, we get $f(t) \le F(t)$ for all $t > 0$. Therefore, the estimate (50) is proved. □

Lemma 3.5 *There exist a sequence $\{\varepsilon_j\}_{j\in N}$ converging to zero and a function $u \in L^\infty([0,\infty); H^1(R)) \cap H^1([0,T] \times R)$ such that, for each $T \ge 0$, we have*

$$u_{\varepsilon_j} \rightharpoonup u \quad \text{in } H^1\big([0,T] \times R\big), \text{ for each } T \ge 0, \tag{54}$$

$$u_{\varepsilon_j} \to u \quad \text{in } L^\infty_{\text{loc}}\big([0,\infty) \times R\big), \tag{55}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution of (11).

Proof For fixed $T > 0$, using Lemmas 3.1 and 3.3 and

$$\frac{\partial u_\varepsilon}{\partial t} + \left(u_\varepsilon - \frac{\gamma}{\alpha^2}\right)\frac{\partial u_\varepsilon}{\partial x} + \frac{\partial P_\varepsilon}{\partial x} = \varepsilon\frac{\partial^2 u_\varepsilon}{\partial x^2},$$

we obtain

$$\left\|\frac{\partial u_\varepsilon}{\partial t}\right\|_{L^2([0,T]\times R)} \le c_1\big(1 + \sqrt{\varepsilon}\|u_0\|_{H^1(R)}\big), \tag{56}$$

where c_1 depends on T. Hence $\{u_\varepsilon\}$ is uniformly bounded in

$$L^\infty\big([0,\infty); H^1(R)\big) \cap H^1\big([0,T] \times R\big)$$

and (54) follows.

Observe that, for each $0 \leq s, t \leq T$,

$$\begin{aligned}\|u_\varepsilon(t,\cdot) - u_\varepsilon(s,\cdot)\|^2_{L^2(R)} &= \int_R \left(\int_t^s \frac{\partial u_\varepsilon}{\partial t}(\tau,x)\,d\tau \right)^2 dx \\ &\leq |t-s| \int_R \int_0^T \left(\frac{\partial u_\varepsilon}{\partial t}(\tau,x) \right)^2 d\tau\,dx. \end{aligned} \tag{57}$$

Moreover, $\{u_\varepsilon\}$ is uniformly bounded in $L^\infty([0,T];H^1(R))$ and $H^1(R) \subset L^\infty_{\mathrm{loc}} \subset L^2_{\mathrm{loc}}(R)$. Then (55) is valid. □

Lemma 3.6 *For an arbitrary $T > 0$, there exist a sequence $\{\varepsilon_j\}_{j\in N}$ converging to zero and a function $Q \in L^\infty([0,T] \times R)$ such that $Q_{\varepsilon_j} \to Q$ in $L^\infty([0,T] \times R)$ and for each $2 \leq p < \infty$*

$$Q_{\varepsilon_j} \to Q \quad \textit{strongly in } L^p_{\mathrm{loc}}\big([0,T] \times R\big). \tag{58}$$

Proof Using (11), (16), (40), and (56), we derive that $\|\frac{dQ_\varepsilon}{dt}\|_{L^2(R)}$ is bounded in $[0,T]$. Applying Corollary 8 on page 90 in Simon [28], we complete the proof. □

Throughout this paper we use overbars to denote weak limits (the space in which these weak limits are taken is $L^r([0,\infty) \times R)$ with $1 < r < \frac{3}{2}$).

Lemma 3.7 *There exist a sequence $\{\varepsilon_j\}_{j\in N}$ converging to zero and two functions $q \in L^p_{\mathrm{loc}}([0,\infty) \times R)$, $\overline{q^2} \in L^r_{\mathrm{loc}}([0,\infty) \times R)$ such that*

$$q_{\varepsilon_j} \rightharpoonup q \quad \textit{in } L^p_{\mathrm{loc}}\big([0,\infty) \times R\big), \qquad q_{\varepsilon_j} \overset{\star}{\rightharpoonup} q \quad \textit{in } L^\infty_{\mathrm{loc}}\big([0,\infty); L^2(R)\big), \tag{59}$$

$$q^2_{\varepsilon_j} \rightharpoonup \overline{q^2} \quad \textit{in } L^r_{\mathrm{loc}}\big([0,\infty) \times R\big), \tag{60}$$

for each $1 < p < 3$ and $1 < r < \frac{3}{2}$. Moreover,

$$q^2(t,x) \leq \overline{q^2}(t,x) \quad \textit{for almost every } (t,x) \in [0,\infty) \times R \tag{61}$$

and

$$\frac{\partial u}{\partial x} = q \quad \textit{in the sense of distributions on } [0,\infty) \times R. \tag{62}$$

Proof Equations (59) and (60) are direct consequences of Lemmas 3.1 and 3.2. Inequality (61) is valid because of the weak convergence in (60). Finally, (62) is a consequence of the definition of q_ε, Lemma 3.5, and (59). □

In the following, for notational convenience, we replace the sequence $\{u_{\varepsilon_j}\}_{j\in N}$, $\{q_{\varepsilon_j}\}_{j\in N}$ and $\{Q_{\varepsilon_j}\}_{j\in N}$ by $\{u_\varepsilon\}_{\varepsilon>0}$, $\{q_\varepsilon\}_{\varepsilon>0}$ and $\{Q_\varepsilon\}_{\varepsilon>0}$, respectively.

Using (59), we conclude that for any convex function $\eta \in C^1(R)$ with η' being bounded and Lipschitz continuous on R and for any $1 < p < 3$, we get

$$\eta(q_\varepsilon) \rightharpoonup \overline{\eta(q)} \quad \text{in } L^p_{\mathrm{loc}}\big([0,\infty) \times R\big), \tag{63}$$

$$\eta(q_\varepsilon) \overset{\star}{\rightharpoonup} \overline{\eta(q)} \quad \text{in } L^\infty_{\mathrm{loc}}\big([0,\infty); L^2(R)\big). \tag{64}$$

Multiplying (17) by $\eta'(q_\varepsilon)$ yields

$$\frac{\partial}{\partial t}\eta(q_\varepsilon) + \frac{\partial}{\partial x}\left[\left(u_\varepsilon - \frac{\gamma}{\alpha^2}\right)\eta(q_\varepsilon)\right] - \varepsilon\frac{\partial^2}{\partial x^2}\eta(q_\varepsilon) + \varepsilon\eta''(q_\varepsilon)\left(\frac{\partial q_\varepsilon}{\partial x}\right)^2 \\ = q_\varepsilon\eta(q_\varepsilon) - \frac{1}{2}\eta'(q_\varepsilon)q_\varepsilon^2 + Q_\varepsilon(t,x)\eta'(q_\varepsilon). \tag{65}$$

Lemma 3.8 *For any convex $\eta \in C^1(R)$ with η' being bounded and Lipschitz continuous on R, we have*

$$\frac{\overline{\partial\eta(q)}}{\partial t} + \frac{\partial}{\partial x}\left(\left(u - \frac{\gamma}{\alpha^2}\right)\overline{\eta(q)}\right) \le \overline{q\eta(q)} - \frac{1}{2}\overline{\eta'(q)q^2} + Q(t,x)\overline{\eta'(q)} \tag{66}$$

in the sense of distributions on $[0,\infty) \times R$. Here $\overline{q\eta(q)}$ and $\overline{\eta'(q)q^2}$ denote the weak limits of $q_\varepsilon\eta(q_\varepsilon)$ and $q_\varepsilon^2\eta'(q_\varepsilon)$ in $L^r_{\mathrm{loc}}([0,\infty) \times R)$, $1 < r < \frac{3}{2}$, respectively.

Proof In (65), by the convexity of η, (14), Lemmas 3.5, 3.6, and 3.7, taking the limit for $\varepsilon \to 0$ gives rise to the desired result. □

Remark 3.1 From (59) and (60), we know that

$$q = q_+ + q_- = \overline{q_+} + \overline{q_-}, \qquad q^2 = (q_+)^2 + (q_-)^2, \qquad \overline{q^2} = \overline{(q_+)^2} + \overline{(q_-)^2} \tag{67}$$

almost everywhere in $[0,\infty) \times R$, where $\xi_+ := \xi_{\chi[0,+\infty)}(\xi), \xi_- := \xi_{\chi(-\infty,0]}(\xi)$ for $\xi \in R$. From Lemma 3.4 and (59), we have

$$q_\varepsilon(t,x), q(t,x) \le \frac{2}{t} + c, \quad \text{for } t > 0, x \in R, \tag{68}$$

where c is a constant depending only on $\|u_0\|_{H^1(R)}$ and the coefficients of (2).

Lemma 3.9 *In the sense of distributions on $[0,\infty) \times R$, we have*

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left[\left(u - \frac{\gamma}{\alpha^2}\right)q\right] = \frac{1}{2}\overline{q^2} + Q(t,x). \tag{69}$$

Proof Using (17), Lemmas 3.5 and 3.6, (59), (60), and (62), the conclusion (69) holds by taking the limit for $\varepsilon \to 0$ in (17). □

The next lemma contains a generalized formulation of (69).

Lemma 3.10 *For any $\eta \in C^1(R)$ with $\eta \in L^\infty(R)$, we have*

$$\frac{\partial\eta(q)}{\partial t} + \frac{\partial}{\partial x}\left(\left(u - \frac{\gamma}{\alpha^2}\right)\eta(q)\right) = q\eta(q) + \left(\frac{1}{2}\overline{q^2} - q^2\right)\eta'(q) + Q(t,x)\eta'(q) \tag{70}$$

in the sense of distributions on $[0,\infty) \times R$.

Proof Let $\{\omega_\delta\}_\delta$ be a family of mollifiers defined on R. Denote $q_\delta(t,x) := (q(t,\cdot) \star \omega_\delta)(x)$ where the $\star$ is the convolution with respect to x variable. Multiplying (69) by $\eta'(q_\delta)$ yields

$$\begin{aligned}\frac{\partial \eta(q_\delta)}{\partial t} &= \eta'(q_\delta)\frac{\partial q_\delta}{\partial t}\\ &= \eta'(q_\delta)\left[\frac{1}{2}\overline{q^2} \star \omega_\delta + Q(t,x) \star \omega_\delta - q^2 \star \omega_\delta - \left(u - \frac{\gamma}{\alpha^2}\right)\frac{\partial q}{\partial x} \star \omega_\delta\right]\end{aligned} \tag{71}$$

and

$$\frac{\partial}{\partial x}\left(\left(u - \frac{\gamma}{\alpha^2}\right)\eta(q_\delta)\right) = q\eta(q_\delta) + \left(u - \frac{\gamma}{\alpha^2}\right)\eta'(q_\delta)\left(\frac{\partial q_\delta}{\partial x}\right). \tag{72}$$

Using the boundedness of η, η', and letting $\delta \to 0$ in the above two equations, we obtain (70). □

4 Strong convergence of q_ε and proof of main result

Following the work of [20] or [22], in this section we proceed to improve the weak convergence of q_ε in (59) to strong convergence, and then we establish a global existence result for problem (4). We will derive a 'transport equation' for the evolution of the defect measure $(\overline{q^2} - q^2) \geq 0$. Namely, we will prove that if the measure is zero initially, then it will continue to be zero at all later times $t > 0$.

Lemma 4.1 *Assume $u_0 \in H^1(R)$. We have*

$$\lim_{t\to 0}\int_R q^2(t,x)\,dx = \lim_{t\to 0}\int_R \overline{q^2}(t,x)\,dx = \int_R \left(\frac{\partial u_0}{\partial x}\right)^2 dx. \tag{73}$$

Lemma 4.2 *If $u_0 \in H^1(R)$, for each $M > 0$, we have*

$$\lim_{t\to 0}\int_R \left(\overline{\eta_M^{\pm}(q)}(t,x) - \eta_M^{\pm}\big(q(t,x)\big)\right) dx = 0, \tag{74}$$

where

$$\eta_M(\xi) := \begin{cases}\frac{1}{2}\xi^2, & \text{if } |\xi| \leq M,\\ M|\xi| - \frac{1}{2}M^2, & \text{if } |\xi| > M,\end{cases} \tag{75}$$

and $\eta_M^+(\xi) := \eta_M(\xi)\chi_{[0,+\infty)}(\xi)$, $\eta_M^-(\xi) := \eta_M(\xi)\chi_{(-\infty,0]}(\xi)$, $\xi \in R$.

Lemma 4.3 *Let $M > 0$. Then for each $\xi \in R$*

$$\begin{cases}\eta_M(\xi) = \frac{1}{2}\xi^2 - \frac{1}{2}(M - |\xi|)^2\chi_{(-\infty,-M)\cap(M,\infty)}(\xi),\\ \eta_M'(\xi)\xi = \xi + (M - |\xi|)\operatorname{sign}(\xi)\chi_{(-\infty,-M)\cap(M,\infty)}(\xi),\\ \eta_M^+(\xi) = \frac{1}{2}(\xi_+)^2 - \frac{1}{2}(M - \xi)^2\chi_{(M,\infty)}(\xi),\\ (\eta_M^+)'(\xi) = \xi_+ + (M - \xi)\chi_{(M,\infty)}(\xi),\\ \eta_M^-(\xi) = \frac{1}{2}(\xi_-)^2 - \frac{1}{2}(M + \xi)^2\chi_{(-\infty,-M)}(\xi),\\ (\eta_M^-)'(\xi) = \xi_- - (M + \xi)\chi_{(-\infty,-M)}(\xi).\end{cases}$$

The proofs of Lemmas 4.1-4.3 can be found in [20] or [22].

Lemma 4.4 *Assume $u_0 \in H^1(R)$. Then for almost all $t > 0$*

$$\frac{1}{2}\int_R \big(\overline{(q_+)^2} - q_+^2\big)(t,x)\,dx \le \int_0^t \int_R Q(s,x)\big[\overline{q_+}(s,x) - q_+(s,x)\big]\,ds\,dx. \tag{76}$$

Proof For an arbitrary $T > 0$ ($0 < t < T$), we let M be sufficiently large (see Lemma 3.4). Subtracting (70) from (66) and using the entropy η_M^+ (see Lemma 4.2) result in

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(\overline{\eta_M^+(q)} - \eta_M^+(q)\big) + \frac{\partial}{\partial x}\left(\left(u - \frac{\gamma}{\alpha^2}\right)\big[\overline{\eta_M^+(q)} - \eta_M^+(q)\big]\right) \\
&\quad \le \big(\overline{q\eta_M^+(q)} - q\eta_M^+(q)\big) - \frac{1}{2}\big(\overline{q^2\big(\eta_M^+\big)'(q)} - q^2\big(\eta_M^+\big)'(q)\big) \\
&\qquad - \frac{1}{2}\big(\overline{q^2} - q^2\big)\big(\eta_M^+\big)'(q) + Q(t,x)\big(\overline{\big(\eta_M^+\big)'(q)} - \big(\eta_M^+\big)'(q)\big).
\end{aligned} \tag{77}$$

By the increasing property of η_M^+, from (61), we have

$$-\frac{1}{2}\big(\overline{q^2} - q^2\big)\big(\eta_M^+\big)'(q) \le 0. \tag{78}$$

It follows from Lemma 4.3 that

$$\begin{aligned}
&q\eta_M^+(q) - \frac{1}{2}q^2\big(\eta_M^+\big)'(q) = -\frac{M}{2}q(M-q)\chi_{(M,\infty)}(q), \\
&\overline{q\eta_M^+(q)} - \frac{1}{2}\overline{q^2\big(\eta_M^+\big)'(q)} = -\frac{M}{2}\overline{q(M-q)\chi_{(M,\infty)}(q)}.
\end{aligned} \tag{79}$$

In view of Remark 3.1, let $\Omega_M = (\frac{2}{M-C}, \infty) \times R$. Applying (68) gives rise to

$$q\eta_M^+(q) - \frac{1}{2}q^2\big(\eta_M^+\big)'(q) = \overline{q\eta_M^+(q)} - \frac{1}{2}\overline{q^2\big(\eta_M^+\big)'(q)} = 0, \quad \text{in } \Omega_M. \tag{80}$$

In Ω_M, one has

$$\eta_M^+ = \frac{1}{2}(q_+)^2, \qquad \big(\eta_M^+\big)'(q) = q_+, \qquad \overline{\eta_M^+(q)} = \frac{1}{2}\overline{(q_+)^2}, \qquad \overline{\big(\eta_M^+\big)'(q)} = \overline{q_+}. \tag{81}$$

From (77)-(81), we find that the following inequality holds in Ω_M:

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(\overline{\eta_M^+(q)} - \eta_M^+(q)\big) + \frac{\partial}{\partial x}\left(\left(u - \frac{\gamma}{\alpha^2}\right)\big[\overline{\eta_M^+(q)} - \eta_M^+(q)\big]\right) \\
&\quad \le Q(t,x)\big(\overline{\big(\eta_M^+\big)'(q)} - \big(\eta_M^+\big)'(q)\big).
\end{aligned} \tag{82}$$

Integrating the resultant inequality over $(\frac{2}{M-C}, t) \times R$ yields

$$\begin{aligned}
\frac{1}{2}\int_R \big(\overline{(q_+)^2} - q_+^2(t,x)\big)\,dx &\le \lim_{t\to 0}\int_R \big[\overline{\eta_M^+(q)}(t,x) - \eta_M^+(q)(t,x)\big]\,dx \\
&\quad + \int_{\frac{2}{M-C}}^t \int_R Q(s,x)\big[\overline{q_+}(s,x) - q_+(s,x)\big]\,ds\,dx,
\end{aligned} \tag{83}$$

for almost all $t > \frac{4}{\alpha(M-C)}$. Letting $M \to \infty$ and using Lemma 4.2, we complete the proof. □

Lemma 4.5 *For any $t > 0$ and $M > 0$, we have*

$$\begin{aligned}
&\int_R \big(\overline{\eta_M^-(q)} - \eta_M^-(q)\big)(t,x)\,dx \\
&\quad\le \frac{M^2}{2}\int_0^t\int_R \overline{u(M+q)\chi_{(-\infty,-M)}(q)}\,ds\,dx - \frac{M^2}{2}\int_0^t\int_R u(M+q)\chi_{(-\infty,-M)}(q)\,ds\,dx \\
&\qquad + M\int_0^t\int_R u\big[\overline{\eta_M^-(q)} - \eta_M^-(q)\big]\,ds\,dx + \frac{M}{2}\int_0^t\int_R u\big(\overline{q_+^2} - q_+^2\big)\,ds\,dx \\
&\qquad + \int_0^t\int_R Q(s,x)\big(\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\big)\,ds\,dx.
\end{aligned} \tag{84}$$

Proof Let $M > 0$. Subtracting (70) from (66) and using the entropy η_M^-, we deduce

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(\overline{\eta_M^-(q)} - \eta_M^-(q)\big) + \frac{\partial}{\partial x}\left(\left(u - \frac{\gamma}{\alpha^2}\right)\big[\overline{\eta_M^-(q)} - \eta_M^-(q)\big]\right) \\
&\quad\le \big(\overline{q\eta_M^-(q)} - q\eta_M^-(q)\big) - \frac{1}{2}\big(\overline{q^2(\eta_M^-)'(q)} - q^2(\eta_M^-)'(q)\big) \\
&\qquad - \frac{1}{2}\big(\overline{q^2} - q^2\big)(\eta_M^-)'(q) + Q(t,x)\big(\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\big).
\end{aligned} \tag{85}$$

Since $-M \le (\eta_M^-)' \le 0$ and $\alpha \ge 0$, we get

$$-\frac{1}{2}\big(\overline{q^2} - q^2\big)(\eta_M^-)'(q) \le \frac{M}{2}\big(\overline{q^2} - q^2\big). \tag{86}$$

Using Remark 3.1 and Lemma 4.3 yields

$$q\eta_M^-(q) - \frac{1}{2}q^2(\eta_M^-)'(q) = -\frac{M}{2}q(M+q)\chi_{(-\infty,-M)}(q), \tag{87}$$

$$\overline{q\eta_M^-(q)} - \frac{1}{2}\overline{q^2(\eta_M^-)'(q)} = -\frac{M}{2}\overline{q(M+q)\chi_{(-\infty,-M)}(q)}. \tag{88}$$

Using (86) to (88), it follows from (85) that

$$\begin{aligned}
&\frac{\partial}{\partial t}\big(\overline{\eta_M^-(q)} - \eta_M^-(q)\big) + \frac{\partial}{\partial x}\left(\left(u - \frac{\gamma}{\alpha^2}\right)\big[\overline{\eta_M^-(q)} - \eta_M^-(q)\big]\right) \\
&\quad\le -\frac{M}{2}\overline{q(M+q)\chi_{(-\infty,-M)}(q)} + \frac{M}{2}q(M+q)\chi_{(-\infty,-M)}(q) \\
&\qquad + \frac{M}{2}\big(\overline{q^2} - q^2\big) + Q(t,x)\big(\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\big).
\end{aligned} \tag{89}$$

Integrating the above inequality over $(0,t) \times R$, we obtain

$$\begin{aligned}
&\int_R \big(\overline{\eta_M^-(q)} - \eta_M^-(q)\big)(t,x)\,dx \\
&\quad\le -\frac{M}{2}\int_0^t\int_R \overline{q(M+q)\chi_{(-\infty,-M)}(q)}\,ds\,dx + \frac{M}{2}\int_0^t\int_R q(M+q)\chi_{(-\infty,-M)}(q)\,ds\,dx \\
&\qquad + \frac{M}{2}\int_0^t\int_R \big(\overline{q^2} - q^2\big)\,ds\,dx + \int_0^t\int_R Q(t,x)\big(\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\big)\,ds\,dx.
\end{aligned} \tag{90}$$

It follows from Lemma 4.3 that

$$\begin{aligned}\overline{\eta_M^-(q)} - \eta_M^-(q) &= \frac{1}{2}\left(\overline{(q_-)^2} - (q_-)^2\right) + \frac{1}{2}(M+q)^2\chi_{(-\infty,-M)}(q) \\ &\quad - \frac{1}{2}\overline{(M+q)^2\chi_{(-\infty,-M)}(q)}.\end{aligned} \tag{91}$$

Using Remark 3.1 and (91), we have

$$\begin{aligned}&\int_R \left(\overline{\eta_M^-(q)} - \eta_M^-(q)\right)(t,x)\,dx \\ &\quad\le -\frac{\alpha M}{2}\int_0^t\int_R \overline{q(M+q)\chi_{(-\infty,-M)}(q)}\,ds\,dx + \frac{\alpha M}{2}\int_0^t\int_R q(M+q)\chi_{(-\infty,-M)}(q)\,ds\,dx \\ &\qquad + \alpha M\int_0^t\int_R \left[\overline{\eta_M^-(q)} - \eta_M^-(q)\right]ds\,dx + \frac{\alpha M}{2}\int_0^t\int_R \overline{(M+q)^2\chi_{(-\infty,-M)}(q)}\,ds\,dx \\ &\qquad - \frac{\alpha M}{2}\int_0^t\int_R (M+q)^2 u\chi_{(-\infty,-M)}(q)\,ds\,dx + \frac{\alpha M}{2}\int_0^t\int_R \left(\overline{q_+^2} - q_+^2\right)ds\,dx \\ &\qquad + \int_0^t\int_R Q(t,x)\left(\overline{\left(\eta_M^-\right)'(q)} - \left(\eta_M^-\right)'(q)\right)ds\,dx.\end{aligned} \tag{92}$$

Applying the identity $M(M+q)^2 - Mq(M+q) = M^2(M+q)$, we obtain (84). □

Lemma 4.6 *We have*

$$\overline{q^2} = q^2 \quad \textit{almost everywhere in } [0,\infty)\times(-\infty,\infty). \tag{93}$$

Proof Applying Lemmas 4.4 and 4.5 gives rise to

$$\begin{aligned}&\int_R \left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_M^-} - \eta_M^-\right]\right)(t,x)\,dx \\ &\quad\le \frac{M^2}{2}\left(\int_0^t\int_R \overline{(M+q)\chi_{(-\infty,-M)}(q)}\,ds\,dx - \frac{M^2}{2}\int_0^t\int_R (M+q)\chi_{(-\infty,-M)(q)}\,ds\,dx\right) \\ &\qquad + M\int_0^t\int_R \left[\overline{\eta_M^-} - \eta_M^-\right]ds\,dx + \frac{M}{2}\int_0^t\int_R \left[\overline{(q_+)^2} - (q_+)^2\right]ds\,dx \\ &\qquad + \int_0^t\int_R Q(s,x)\left(\left[\overline{q_+} - q_+\right] + \left[\overline{\left(\eta_M^-\right)'(q)} - \left(\eta_M^-\right)'(q)\right]\right)ds\,dx.\end{aligned} \tag{94}$$

From Lemma 3.6, we know that there exists a constant $L > 0$, depending only on $\|u_0\|_{H^1(R)}$, such that

$$\left\|Q(t,x)\right\|_{L^\infty([0,\infty)\times R)} \le L. \tag{95}$$

Using Remark 3.1 and Lemma 4.3 yields

$$\begin{aligned}&q_+ + \left(\eta_M^-\right)'(q) = q - (M+q)\chi_{(-\infty,-M)}, \\ &\overline{q_+} + \overline{\left(\eta_M^-\right)'(q)} = q - \overline{(M+q)\chi_{(-\infty,-M)}(q)}.\end{aligned} \tag{96}$$

Thus, by the convexity of the map $\xi \to \xi_+ + (\eta_M^-)'(\xi)$, we get

$$\begin{aligned} 0 &\le [\overline{q_+} - q_+] + \left[\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\right] \\ &= (M+q)\chi_{(-\infty,-M)} - \overline{(M+q)\chi_{(-\infty,-M)}(q)}. \end{aligned} \tag{97}$$

Using (95) one derives

$$\begin{aligned} &Q(s,x)\left([\overline{q_+} - q_+] + \left[\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\right]\right) \\ &\quad \le -L\left(\overline{(M+q)\chi_{(-\infty,-M)}(q)} - (M+q)\chi_{(-\infty,-M)}\right). \end{aligned} \tag{98}$$

Since $\xi \to (M+\xi)\chi_{(-\infty,-M)}$ is concave and choosing M large enough, we have

$$\begin{aligned} &\frac{M^2}{2}\left(\overline{(M+q)\chi_{(-\infty,-M)}(q)} - (M+q)\chi_{(-\infty,-M)}\right) \\ &\qquad + Q(s,x)\left([\overline{q_+} - q_+] + \left[\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\right]\right) \\ &\quad \le \left(\frac{M^2}{2} - L\right)\left(\overline{(M+q)\chi_{(-\infty,-M)}(q)} - (M+q)\chi_{(-\infty,-M)}\right) \le 0. \end{aligned} \tag{99}$$

Then, from (94) and (99), we have

$$\begin{aligned} 0 &\le \int_R \left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_M^-} - \eta_M^-\right]\right)(t,x)\,dx \\ &\le cM \int_0^t \int_R \left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_M^-} - \eta_M^-\right]\right)(t,x)\,ds\,dx. \end{aligned} \tag{100}$$

Using the Gronwall inequality and Lemmas 4.1 and 4.2, for each $t > 0$, we have

$$0 \le \int_R \left(\frac{1}{2}\left[\overline{(q_+)^2} - (q_+)^2\right] + \left[\overline{\eta_M^-} - \eta_M^-\right]\right)(t,x)\,dx = 0.$$

Applying the Fatou lemma, Remark 3.1, (61) and letting $M \to \infty$, we obtain

$$0 \le \int_R \left(\overline{q^2} - q^2\right)(t,x)\,dx = 0, \quad \text{for } t > 0, \tag{101}$$

which completes the proof. □

Proof of the main result Using Lemmas 3.1 and 3.5, we know that the conditions (i) and (ii) in Definition 2.1 are satisfied. We have to verify (iii). Due to Lemma 4.6, we have

$$q_\varepsilon \to q \quad \text{in } L^2_{\text{loc}}\left([0,\infty) \times R\right). \tag{102}$$

From Lemma 3.5, (58), and (102), we know that u is a distributional solution to problem (3). In addition, inequalities (7) and (8) are deduced from Lemmas 3.2 and 3.4. The proof of the main result is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The article is a joint work of two authors who contributed equally to the final version of the paper. All authors read and approved the final manuscript.

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