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Blow-up and infinite propagation speed for a two-component b-family system

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Abstract

In this paper, we study the Cauchy problem of a two-component b-family system which arises in shallow water theory. We first derive the precise blow-up scenario and present a blow-up result. Then we investigate the infinite propagation speed in the sense that the corresponding solution with compact supported initial datum does not have compact spatial support any longer in its lifespan.

MSC: 35G25; 35L05; 35Q58

Keywords: a two-component b-family system; blow-up; infinite propagation speed

1 Introduction

In this paper, we consider the following nonlinear system:

$$\begin{cases} u_t - u_{txx} + (b+1)uu_x - bu_xu_{xx} - uu_{xxx} + \sigma\rho\rho_x = 0, & t > 0, x \in R, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in R, \\ u(0,x) = u_0(x), & x \in R, \\ \rho(0,x) = \rho_0(x), & x \in R, \end{cases} \quad (1.1)$$

where b is an arbitrary real constant and $\sigma = \pm 1$. The system (1.1) was recently derived in the context of shallow water theory. $u(t,x)$ represents the fluid velocity, the constant b is a balance or bifurcation parameter for nonlinear solution behavior, while ρ has a connection with the horizontal deviation of the surface from equilibrium, all measured in dimensionless units, and σ is the downward constant acceleration of gravity in application to shallow waves.

Using the notation $m := u - u_{xx}$, we can rewrite the system (1.1) as follows:

$$\begin{cases} m_t + um_x + bu_xm + \sigma\rho\rho_x = 0, & t > 0, x \in R, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in R, \\ m(0,x) = m_0(x), & x \in R, \\ \rho(0,x) = \rho_0(x), & x \in R. \end{cases} \quad (1.2)$$

Obviously, if $\rho \equiv 0$, the system (1.2) reduces to the following b-family equation, which was extensively studied in [1–5]:

$$\begin{cases} m_t + um_x + bu_xm = 0, & t > 0, x \in R, \\ u(0,x) = u_0(x), & x \in R. \end{cases} \quad (1.3)$$

Equation (1.3) can be derived as the family of asymptotically equivalent shallow water wave equations that emerge at quadratic order accuracy for any $b \neq 1$ by an appropriate Kodama transformation; for the case $b = -1$, the corresponding Kodama transformation is singular and the asymptotic ordering is violated; see [6, 7]. When $b = 2$, (1.3) becomes the Camassa-Holm equation, modeling the unidirectional propagation of shallow water waves over a flat bottom. Here u stands for the fluid velocity at time t in the spatial x direction [8–10]. It has a bi-Hamiltonian structure and is completely integrable [11, 12]. Its solitary waves are peaked, capturing a feature of the water waves of great height [13–16]. Moreover, the shape of some peakons is stable under small perturbations, making these waves recognizable physically [17, 18]. The Cauchy problem of the Camassa-Holm equation has been the subject of a number of studies, for example [12, 19]. When $b = 3$, we find the Degasperis-Procesi equation [20] from (1.3), which is regarded as a model for nonlinear shallow water dynamics. There are also many papers involving the Degasperis-Procesi equation; see [21, 22]. Both the Camassa-Holm equation and the Degasperis-Procesi equation have peakon solitons and model wave breaking (by wave breaking we understand that the wave remains bounded while its slop becomes unbounded in finite time [23]) [8, 24]. In [2, 4], the authors studied (1.3) on the line and on the circle, and established the local well-posedness, described the precise blow-up scenario, proved that the equation has strong solutions which exist globally in time and blow up in finite time.

For $\rho \not\equiv 0$, if $b = 2$, the system (1.2) becomes the two-component Camassa-Holm system

$$\begin{cases} m_t + um_x + 2u_x m + \sigma\rho\rho_x = 0, & t > 0, x \in R, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in R, \\ m(0, x) = m_0(x), & x \in R, \\ \rho(0, x) = \rho_0(x), & x \in R, \end{cases} \tag{1.4}$$

where $m = u - u_{xx}$, $\sigma = \pm 1$ was derived by Constantin and Ivanov [25] in the context of shallow water theory. Here $u(x, t)$ describes the horizontal velocity of the fluid and $\rho(x, t)$ describes the horizontal deviation of the surface from equilibrium, all measured in dimensionless units. This system (1.4) is the first negative flow of the AKNS hierarchy and possesses the peakon and multi-kink solutions and possesses the bi-Hamiltonian structure [26, 27]. Moreover, this model is connected with the energy dependent Schrödinger spectral problem [26]. Recently, the extended $N = 2$ super-symmetric Camassa-Holm system was presented recently by Popowicz in [28]. The mathematical properties of the two-component Camassa-Holm system have been studied in many works; see [25–27, 29–32]. One has established the local well-posedness for the two types of 2-component Camassa-Holm shallow water systems [25, 29], derived precise blow-up scenarios [29], and proved that the systems had strong solutions which blow up in finite time [25, 29, 30].

For $\rho \not\equiv 0$, if $b = 3$, the system (1.3) becomes the two-component Degasperis-Procesi shallow water system

$$\begin{cases} m_t + um_x + 3u_x m + \sigma\rho\rho_x = 0, & t > 0, x \in R, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in R, \\ m(0, x) = m_0(x), & x \in R, \\ \rho(0, x) = \rho_0(x), & x \in R. \end{cases} \tag{1.5}$$

This system first appeared in [33]. The author presented one Hamiltonian extension of the Degasperis-Procesi equation to this system by the Hamiltonian operator which is a Dirac reduced operator of the generalized but degenerated second Hamiltonian operator of the Boussinesq equation. The interest in (1.4) and in (1.5) lies in that model equations presenting breaking waves as well as peaked traveling waves are of great importance in hydrodynamics [23], and the traveling wave solutions of large amplitude to the governing equations for water waves are peaked waves [13]. Recently, Jin and Guo [34] considered the system (1.5) and analyzed some aspects of blow-up mechanism, traveling waves solution and the persistence properties.

For $\rho \not\equiv 0$ and general $b \in R$, the Cauchy problem of the system (1.1) has been studied in [35], authors first established the local well-posedness for a two-component b-family system by Kato's semigroup theory, then derived the precise blow-up scenario for strong solutions to the system and presented several blow-up results for strong solutions to the system. The aim of this paper is to present a blow-up result of solutions to (1.1) with the case of $\sigma = 1$ and to examine the propagation behavior of compactly supported solutions to (1.1) with $\sigma = 1$, namely whether solutions which are initially compactly supported will retain this property throughout their time of evolution.

The rest of this paper is organized as follows. In Section 2, we briefly give some needed results including the local well-posedness of system (1.1). In Section 3, we derive the precise blow-up scenario and present a blow-up result. The propagation behavior will be analyzed in Section 4.

2 Preliminaries

In this section, we recall some elementary results. For completeness, we list them and skip their proof for conciseness.

For convenience to show our results, we rewrite system (1.2). Let $G(x) := \frac{1}{2}e^{-|x|}$, $x \in R$. Then $(1 - \partial_x^2)^{-1}f = G * f$ for all $f \in L^2$ and $G * m = u$. Here we denote by $*$ the convolution. By a direct calculation, we can rewrite (1.2) with $\sigma = 1$ as follows:

$$\begin{cases} u_t + u u_x = -\partial_x G * (\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2 + \frac{1}{2}\rho^2), & t > 0, x \in R, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in R, \\ u(0,x) = u_0(x), & x \in R, \\ \rho(0,x) = \rho_0(x), & x \in R, \end{cases} \tag{2.1}$$

or the equivalent form

$$\begin{cases} u_t + u u_x = -\partial_x(1 - \partial_x^2)^{-1}(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2 + \frac{1}{2}\rho^2), & t > 0, x \in R, \\ \rho_t + (u\rho)_x = 0, & t > 0, x \in R, \\ u(0,x) = u_0(x), & x \in R, \\ \rho(0,x) = \rho_0(x), & x \in R. \end{cases} \tag{2.2}$$

Local well-posedness for system (1.1) can be obtained by Kato's semigroup theory [36]. In [35], the authors gave a detailed description on well-posedness theorem.

Theorem 2.1 *Given* $X_0 = (u_0, \rho_0)^T \in H^s \times H^{s-1}$, $s \geq 2$, *there exists a maximal* $T = T(\|X_0\|_{H^s \times H^{s-1}}) > 0$, *and a unique solution* $X = (u, \rho)^T$ *to system* (1.1) *such that*

$$X = X(\cdot, X_0) \in C([0,T); H^s \times H^{s-1}) \cap C^1([0,T); H^{s-1} \times H^{s-2}).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$X \to X(\cdot, X_0) : H^s \times H^{s-1} \to C\big([0, T); H^s \times H^{s-1}\big) \cap C^1\big([0, T); H^{s-1} \times H^{s-2}\big)$$

is continuous.

We also need to introduce the standard particle trajectory methods for later use. Consider now the following initial value problem:

$$\begin{cases} q_t = u(t, q), & 0 \le t < T, \\ q(0, x) = x, & x \in R, \end{cases} \tag{2.3}$$

where u denotes the first component of the solution X to the system (2.1). Applying classical results in the theory of ordinary differential equations, we can obtain the following two results on q.

Lemma 2.1 (See [25, 29]) *Let $u \in C([0, T); H^s) \cap C^1([0, T); H^{s-1})$, $s \ge 2$. Then (2.3) has a unique solution $q \in C^1([0, T) \times R; R)$. Moreover, we know the map $q(t, \cdot)$ is an increasing diffeomorphism of R with*

$$q_x(t, x) = \exp\bigg(\int_0^t u_x(s, q) \, ds \bigg) > 0, \quad (t, x) \in [0, T) \times R. \tag{2.4}$$

Lemma 2.2 *Let $X_0 = (u_0, \rho_0)^T \in H^s \times H^{s-1}$, $s \ge 2$, and T is assumed to be the maximal existence time of the solution $X = (u, \rho)^T$ to system (2.1) corresponding to the initial data X_0. Then for all $(t, x) \in [0, T) \times R$, we have*

$$\rho\big(t, q(t, x)\big) q_x(t, x) = \rho_0(x).$$

Remark 2.1 This lemma tell us that ρ always keeps sign with its initial value because of the positivity of $q_x(t, x)$ in (2.4). Actually this invariance result is due to the geometric underlying structure; see the discussion in [37, 38].

Proof Using (2.3) and the second equation in system (2.1), we obtain

$$\frac{d}{dt}\big[\rho(t, q) q_x(t, x)\big] = \big(\rho_t(t, q) + \rho_x(t, q) q_t(t, x)\big) q_x(t, x) + \rho(t, q) q_{xt}(t, x)$$

$$= \big[\rho_t(t, q) + \rho_x(t, q) u(t, q) + \rho(t, q) u_x(t, q)\big] q_x(t, x)$$

$$= 0.$$

This means that $\rho(t, q) q_x(t, x)$ is independent on time t. We may choose $t = 0$, due to (2.4) we know $q_x(0, x) = 1$. Therefore the lemma is easily proved. $\qquad\square$

3 Blow-up

In this section we are interesting in the formation of singularities for strong solutions to system (2.1) and establish a sufficient condition on the initial data to guarantee blow-up.

Theorem 3.1 *Let $X_0 = (u_0, \rho_0)^T \in H^s \times H^{s-1}$, $s > \frac{5}{2}$ be given and assume that $1 < b \leq 3$, suppose X_0 is odd and $u_0'(0) \leq 0$. Then the corresponding solution to system (2.1) with the initial data X_0 blows up in finite time. If $u_0'(0) < 0$, then the lifespan T can be estimated by $-\frac{2}{(b-1)u_0'(0)}$. In addition, the following inequalities hold*:

$$\rho_x(t, 0) \geq \rho_0'(0)e^{-2u_0'(0)t} \quad \text{for } \rho_0'(0) \geq 0; \qquad \rho_x(t, 0) \leq \rho_0'(0)e^{-2u_0'(0)t} \quad \text{for } \rho_0'(0) \leq 0.$$

Proof Let $X = (u, \rho)^T$ be the corresponding solution to system (2.1) and T be the maximal existence time of the solution $u(t, x)$. Differentiating system (2.1) with respect to x, we obtain

$$u_{xt} = -\frac{b-1}{2}u_x^2 - uu_{xx} + \frac{b}{2}u^2 + \frac{\rho^2}{2} - G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2 + \frac{\rho^2}{2}\right) \tag{3.1}$$

and

$$\rho_{xt} = -\rho_{xx}u - 2\rho_x u_x - \rho u_{xx}. \tag{3.2}$$

Note that the system (2.1) is invariant under the transformation $(X, x) \rightarrow (-X, -x)$. Thus we deduce that if $X_0(x)$ is odd, then $X(t, x)$ is odd with respect to x for any $t \in [0, T)$. Therefore

$$X(t, 0) = X_{xx}(t, 0) = 0, \quad \forall t \in [0, T).$$

Hence, in view of (3.1), we get

$$u_{xt}(t, 0) = -\frac{b-1}{2}u_x^2(t, 0) - G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2 + \frac{\rho^2}{2}\right)(t, 0) \leq -\frac{b-1}{2}u_x^2(t, 0). \tag{3.3}$$

If $u_0'(0) < 0$, solving the above inequality directly yields

$$u_x(t, 0) \leq \frac{u_0'(0)}{1 + \frac{b-1}{2}u_0'(0)t},$$

which tends to $-\infty$ as t goes to $-\frac{2}{(b-1)u_0'(0)}$.

If $u_0'(0) = 0$, we have

$$u_{xt}(t, 0) \leq -G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2 + \frac{\rho^2}{2}\right)(t, 0) < 0.$$

This means $u_x(t, 0)$ is decreasing, and $u_x(t, 0) < 0$ for all $t > 0$. So by (3.1) we can choose a proper time $t_0 > 0$ such that

$$u_{xt}(t, 0) \leq -\frac{b-1}{2}u_x^2(t, 0)$$

for $t > t_0$, and $u_x(t_0, 0) < 0$. We will have the following inequality by integration from t_0 to t:

$$\frac{1}{u_x(t_0, 0)} + \frac{b-1}{2}(t - t_0) \leq \frac{1}{u_x(t, 0)} < 0.$$

Consequently, we obtain $T < t_0 - \frac{2}{(b-1)u_x(t_0,0)}$. Similarly, for the equation in (3.2) we have

$$\rho_{xt}(t,0) = -2\rho_x(t,0)u_x(t,0), \quad t \in [0,T). \tag{3.4}$$

Equation (3.3) implies that $u_x(t,0)$ is decreasing with $u_0'(0) \le 0$, so we easily get $-u_x(t,0) \ge -u_0'(0) \ge 0$. This inequality in combination with (3.4) yields

$$\rho_x(t,0) \ge \rho_0'(0)e^{-2u_0'(0)t} \quad \text{for } \rho_0'(0) \ge 0; \qquad \rho_x(t,0) \le \rho_0'(0)e^{-2u_0'(0)t} \quad \text{for } \rho_0'(0) \le 0.$$

This completes the proof. □

4 Infinite propagation speed

In this section we examine whether classical solutions u, m, ρ of the two-component b-family system (2.1) which are initially compactly supported will retain this property throughout their evolution. Such compactly supported solutions represent localized perturbations or disturbances of the system. What we will see is that given ρ_0 compactly supported, then the unique solution ρ will remain compactly supported for all $t \in [0,T)$ regardless of the form of the initial data u_0, m_0; whereas if m_0 has compact support then m remains compactly supported, for all $t \in [0,T)$, only if ρ is also initially compactly supported. The situation is completely different for our solution u, however, since, as we will see, given u_0 compactly supported, then the only possible way the ensuing solution u can remain compactly supported for any further time is if $u(t,\cdot) \equiv 0$ for all $t \in [0,T)$.

Theorem 4.1 *Let $0 \le b \le 3$. Assume that the initial datum $0 \not\equiv X_0 = (u_0, \rho_0)^T \in H^s \times H^{s-1}$ with $s > \frac{5}{2}$ is compactly supported in $[\alpha, \beta]$ with $u_0 \not\equiv 0$, then the corresponding solution $X = (u, \rho)^T$ to the system (2.1) has the following property: for $0 < t < T$, $\rho(t,x)$, $m(t,x)$ are compactly supported in $[q(t,\alpha), q(t,\beta)]$ in its lifespan and*

$$u(x,t) = E_+e^{-x} \quad \text{for } x > q(t,\beta); \qquad u(x,t) = E_-e^x \quad \text{for } x < q(t,\alpha),$$

with $E_+(t) > 0$ and $E_-(t) < 0$ for $t \in (0,T)$, respectively, where $q(t,x)$ is defined by (2.3) and T is its lifespan. Furthermore, $E_+(t)$ and $E_-(t)$ denote continuous nonvanishing functions, with $E_+(t)$ being a strictly increasing function, while $E_-(t)$ being strictly decreasing.

Proof First, since X_0 has compact support, so do u_0, m_0, and ρ_0, we know from Lemma 2.2 that ρ is compactly supported in $[q(t,\alpha), q(t,\beta)]$ in its lifespan, *i.e.* $\rho(t,x) = 0$ for $x > q(t,\beta)$ or $x < q(t,\alpha)$.

Applying particle trajectory method to the first equation in (1.2), we obtain

$$\frac{d}{dt}\big(m\big(t,q(t,x)\big)q_x^b(t,x)\big) = \big(m_t + um_x + bu_xm\big)\big(t,q(t,x)\big)q_x^b(t,x)$$

$$= -\rho\big(t,q(t,x)\big)\rho_x\big(t,q(t,x)\big)q_x^b(t,x).$$

Thus we know when $x > \beta$ or $x < \alpha$

$$\frac{d}{dt}\big(m\big(t,q(t,x)\big)q_x^b(t,x)\big) = 0.$$

Thus $m(t, q(t,x))q_x^b(t,x)$ is independent on time t over the interval $(-\infty, \alpha) \cap (\beta, \infty)$. We will get by taking $t = 0$ without loss of generality,

$$m\big(t, q(t,x)\big)q_x^b(t,x) = m_0(x) = 0 \quad \text{for } x \in (-\infty, \alpha) \cap (\beta, \infty).$$

This implies that $m(t, q(t,x)) = 0$ when $x \in (-\infty, \alpha) \cap (\beta, \infty)$, *i.e.* $m(t,x)$ is compactly supported in $[q(t,\alpha), q(t,\beta)]$ in its lifespan. Hence the following functions are well defined:

$$F(t) = \int_R e^x m(t,x)\,dx \quad \text{and} \quad f(t) = \int_R e^{-x} m(t,x)\,dx. \tag{4.1}$$

By integration by parts, we have

$$F(0) = \int_R e^x m_0(x)\,dx = \int_R e^x u_0(x)\,dx - \int_R e^x u_{0xx}(x)\,dx = 0,$$

$$f(0) = \int_R e^{-x} m_0(x)\,dx = \int_R e^{-x} u_0(x)\,dx - \int_R e^{-x} u_{0xx}(x)\,dx = 0.$$

Then for $x > q(t, \beta)$, we get

$$u(t,x) = G * m(t,x) = \frac{1}{2}e^{-x}\int_{q(t,a)}^{q(t,\beta)} e^{\xi} m(t,\xi)\,d\xi = \frac{1}{2}e^{-x}F(t), \tag{4.2}$$

where (4.1) is used.

Similarly, when $x < q(t, \alpha)$, we have

$$u(t,x) = G * m(t,x) = \frac{1}{2}e^{x}\int_{q(t,a)}^{q(t,\beta)} e^{-\xi} m(t,\xi)\,d\xi = \frac{1}{2}e^{x}f(t). \tag{4.3}$$

Because $m(t,x)$ has compact support in the interval $[q(t,\alpha), q(t,\beta)]$ for any $t \in [0, T)$, we get $m(t,x) = u(t,x) - u_{xx}(t,x) = 0$ for $x > q(t,\beta)$ or $x < q(t,\alpha)$. Hence, as consequences of (4.2) and (4.3), we obtain

$$u(t,x) = -u_x(t,x) = u_{xx}(t,x) = \frac{1}{2}e^{-x}F(t) \quad \text{for } x > q(t, \beta) \tag{4.4}$$

and

$$u(t,x) = u_x(t,x) = u_{xx}(t,x) = \frac{1}{2}e^{x}f(t) \quad \text{for } x < q(t, \alpha). \tag{4.5}$$

On the other hand,

$$\frac{dF(t)}{dt} = \int_R e^x m_t(t,x)\,dx.$$

Differentiating the first equation in (2.1) twice, we get

$$0 = u_{xxt} + (uu_x)_{xx} + \partial_x \partial_x^2 G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2 + \frac{1}{2}\rho^2 \right)$$

$$= u_{xxt} + (uu_x)_{xx} - \partial_x(1 - \partial_x^2)G * \left(\frac{b}{2}u^2 + \frac{3-b}{2}u_x^2 + \frac{1}{2}\rho^2 \right)$$

$$+ \partial_x G * \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + \frac{1}{2} \rho^2 \right)$$

$$= u_{xxt} + (uu_x)_{xx} - \partial_x \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + \frac{1}{2} \rho^2 \right)$$

$$+ \partial_x G * \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + \frac{1}{2} \rho^2 \right). \tag{4.6}$$

Combining the first equation in (2.1) and (4.6), we obtain

$$m_t = \left(1 - \partial_x^2 \right) u_t = -uu_x + (uu_x)_{xx} - \partial_x \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + \frac{1}{2} \rho^2 \right). \tag{4.7}$$

Substituting the identity (4.7) into $\frac{dF(t)}{dt}$ and using (4.4) and (4.5), we have

$$\frac{dF(t)}{dt} = - \int_R e^x uu_x(t,x)\,dx + \int_R e^x (uu_x)_{xx}(t,x)\,dx$$

$$- \int_R e^x \partial_x \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + \frac{1}{2} \rho^2 \right)(t,x)\,dx$$

$$= e^x \left((uu_x)_x - uu_x \right)\big|_{-\infty}^{\infty} - e^x \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + \frac{1}{2} \rho^2 \right)\Big|_{-\infty}^{\infty}$$

$$+ \int_R e^x \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + \frac{1}{2} \rho^2 \right)(t,x)\,dx$$

$$= \int_R e^x \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + \frac{1}{2} \rho^2 \right)(t,x)\,dx.$$

Therefore, in the lifespan of the solution, we find that $F(t)$ is an increasing function with $F(0) = 0$, thus it follows that $F(t) > 0$ for $t \in (0, T]$, *i.e.*,

$$F(t) = \int_0^t \int_R e^x \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + \frac{1}{2} \rho^2 \right)(\tau,x)\,dx\,d\tau > 0.$$

By a similar argument, we can check that the following identity for $f(t)$ is true:

$$f(t) = - \int_0^t \int_R e^{-x} \left(\frac{b}{2} u^2 + \frac{3-b}{2} u_x^2 + \frac{1}{2} \rho^2 \right)(\tau,x)\,dx\,d\tau < 0.$$

In order to finish the proof, it is sufficient to let $E_+(t) = \frac{1}{2} F(t)$ and $E_-(t) = \frac{1}{2} f(t)$, respectively. □

It is really a very nice property for the two-component b-family system (2.1). No matter what the profile of the compactly supported initial datum $u_0(x)$ is (no matter whether it is positive or negative), for any $t > 0$ in its lifespan, the solution $u(x,t)$ is positive at infinity and negative at negative infinity. Moreover, we have the following unique continuation properties for the strong solution. The proofs are quite similar to that for the two-component Camassa-Holm system [39], so they are omitted to make the paper concise.

Theorem 4.2 *Assume that for $s > \frac{5}{2}$, $X(t,x) = (u(t,x), \rho(t,x))^T \in C([0, T]; H^s(R) \times H^{s-1}(R))$ is a strong solution of the initial value problem associated with system (2.1), and that*

$X_0(x) = (u_0(x), \rho_0(x))^T$ *satisfies for some* $\theta \in (0, 1)$

$$\left| X_0(x) \right|, \left| X_{0x}(x) \right| \sim O\big(x^{-\theta}\big) \quad \text{as } x \to \infty.$$

Then

$$\left| X(t, x) \right|, \left| X_x(t, x) \right| \sim O\big(x^{-\theta}\big) \quad \text{as } x \to \infty$$

uniformly in the time interval $[0, T]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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