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Uniqueness of an inverse problem for an integro-differential equation related to the Basset problem

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Abstract
This paper concerns an inverse problem for an integro-differential equation related to the Basset problem. The inverse problem aims to determine a weakly singular term from the time trace at a fixed point $x_0 \in \Omega$. We use the maximum principle for an integro-differential operator to derive the uniqueness of the inverse problem. Additionally, we prove the existence and uniqueness of the direct Basset problem with a general kernel function.
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Keywords: inverse problems; Basset problem; uniqueness

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ ($N = 1,2,3$) be a bounded domain with smooth boundary $\partial\Omega := \Gamma$. We further set $Q_{0,T} := \Omega \times (0,T)$, $\Sigma_{0,T} = \Gamma \times (0,T)$. Then we consider the following integro-differential equation [1–3]:

$$L_{0,t}[u](x,t) := u_t(x,t) - A[u](x,t) + \int_0^t k(t-s)u_t(x,s)\,\mathrm{d}s = f(x,t), \quad (x,t) \in Q_{0,T} \tag{1.1}$$

with the initial and boundary conditions:

$$\begin{cases} u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = g(x,t), & (x,t) \in \Sigma_{0,T}. \end{cases} \tag{1.2}$$

Here the operator A is uniformly elliptic on $\overline{\Omega}$, defined by

$$A[u](x,t) = \frac{\partial}{\partial x}\left(a(x)\frac{\partial}{\partial x}u(x,t)\right), \quad (x,t) \in Q_{0,T},$$

with $a(x) \in C^1(\overline{\Omega})$ satisfying $a(x) \geq a_0 > 0$ for $x \in \overline{\Omega}$, the function k with some power singularity is unknown and has to be determined by the following measurement data at a fixed point $x_0 \in \Omega$:

$$u(x_0,t) = h(t), \quad t \in [0,T]. \tag{1.3}$$

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For the direct problem (1.1) and (1.2), Ashyralyev [1] proved the well-posedness for the following form of k:

$$k(t) = \frac{1}{\Gamma(1-\alpha)} t^{-1/2}.$$

In this case, (1.1) can be written as a fractional parabolic equation

$$u_t(x,t) - Au(x,t) + \partial_t^{\frac{1}{2}} u(x,t) = f(x,t). \tag{1.4}$$

For details on the fractional derivative, we refer the readers to [4] or [5]. The system (1.1) and (1.2) is called the Basset problem, which describes a classical problem in fluid dynamics with the unsteady motion of a particle accelerates in a viscous fluid due to the force of gravity [6]. For the recent results on the Basset problem, we refer the readers to [7–12]. In Appendix A, we will prove the existence and uniqueness of the direct problem (1.1) and (1.2) for general k, which extends the results in [1] and [2]. (References [1] and [2] were concerned the direct Basset problem with the fractional order 1/2, *i.e.* the problem (1.4) and (1.2).) For the other models related to the integro-differential equation, we refer the readers to [13–15].

As for the inverse kernel problems related to the integro-differential equation, in [16–19] efficient strategies to prove the existence and uniqueness of inverse memory kernel problems were given. In particular, Colombo and Guidetti [16] showed that a semilinear integro-differential parabolic inverse problem had a unique solution global in time under suitable growth conditions for the nonlinearity involved in the evolution equation. Lorenzi and Rocca [18] studied an inverse two memory kernels problem in a hyperbolic phase-field model. But the additional measurements used in these papers are imposed on the whole or the part of Ω, which can be expressed in the following integral form:

$$\int_\Omega \phi(x)u(x,t)\,\mathrm{d}t = g(t), \quad t \in [0,T],$$

with a known function ϕ. Comparing with these papers, our study only needs the measurements at a fixed point x_0. Another difference is that the kernel function discussed in our current paper has power singularity $t^{-\gamma}$. Finally, our method in discussing the uniqueness of our inverse problem is different from those inverse memory kernel problems, in which the methods on the basis of the analytic semigroup theory were used. It is worth noting that in [20] the Fourier method was applied to prove the existence and uniqueness of determining a weakly singular kernel in a linear heat conduction model.

For some $\gamma \in (0,1)$, we use the notation $C_\gamma[0,T]$ to denote the following Banach space: $C_\gamma[0,T] := \{k \mid t^\gamma k \in C[0,T]\}$, endowed with the norm $\|k\|_{C_\gamma[0,T]} = \|t^\gamma k\|_{C[0,T]}$. Furthermore, we introduce

$$K = \{k \in C^1(0,T] \mid k(t) > 0, k'(t) < 0, t \in (0,T], k \in C_\gamma[0,T]\}. \tag{1.5}$$

We assume that $u_0 \in C^2(\overline{\Omega}), f \in C^1(\overline{Q}_{0,T})$ and $g_t \in C^{2,1}(\overline{Q}_{0,T})$ satisfy

$$\begin{cases} A[u_0](x) + f(x,0) = 0, & x \in \Omega, \\ f_t(x,t) > 0, & (x,t) \in Q_{0,T}, \\ g_t(x,t) \geq 0, & (x,t) \in \Sigma_{0,T}. \end{cases} \tag{1.6}$$

Now we state our main result in this paper.

Theorem 1.1 *Let $k \in K$ and* (1.6) *be held. Then the solution (u,k) of the inverse problem* (1.1)-(1.3) *is unique.*

Remark 1.1 Since k is continuous with power singularity $t^{-\gamma}$, we will discuss the uniqueness of k in the sense of $t^\gamma k_1(t) = t^\gamma k_2(t)$ for all $t \in [0,T]$, when $u(x_0,t;k_1) = u(x_0,t;k_2)$.

2 Proof of Theorem 1.1

In this section, the notations $Q_{\tau,T}$, $\Sigma_{\tau,T}$ and $L_{\tau,T}$ are similar to $Q_{0,T}$, $\Sigma_{0,T}$ and $L_{0,T}$ in Section 1, namely, $Q_{\tau,T} := \Omega \times (\tau,T)$, $\Sigma_{\tau,T} = \Gamma \times (\tau,T)$ and $L_{\tau,T}[u] := u_t(x,t) - A[u](x,t) + \int_0^t k(t-s)u_t(x,s)\,\mathrm{d}s$ in $Q_{\tau,T}$. In order to prove Theorem 1.1, we first give the following lemmas.

Lemma 2.1 *Let $k \in K$ and $u \in C^{2,1}(\overline{Q}_{0,T})$. If there exist $x_1 \in \Omega$ and $t_1 \in (0,T]$ such that u attains the minimum value at (x_1,t_1) on $\overline{Q}_{0,T}$, then we have*

$$\int_\tau^{t_1} k(t_1-s)u_t(x_1,s)\,\mathrm{d}s \le 0, \quad \tau \in [0,t_1). \tag{2.1}$$

Proof Here we borrow the ideas used in dealing with the extremum principle of the Caputo derivative in [21]. Integration by parts yields

$$\begin{aligned}
&\int_\tau^{t_1} k(t_1-s)u_t(x_1,s)\,\mathrm{d}s \\
&\quad= \int_\tau^{t_1} k(t_1-s)\,\mathrm{d}\big[u(x_1,s)-u(x_1,t_1)\big] \\
&\quad= k(t_1-s)\big[u(x_1,s)-u(x_1,t_1)\big]\big|_{t=\tau}^{t=t_1} - \int_\tau^{t_1} \big(k(t_1-s)\big)'_s\big[u(x_1,s)-u(x_1,t_1)\big]\,\mathrm{d}s \\
&\quad= \lim_{s\to t_1} k(t_1-s)\big[u(x_1,s)-u(x_1,t_1)\big] - k(t_1-\tau)\big[u(x_1,\tau)-u(x_1,t_1)\big] \\
&\qquad+ \int_\tau^{t_1} k'(t_1-s)\big[u(x_1,s)-u(x_1,t_1)\big]\,\mathrm{d}s.
\end{aligned} \tag{2.2}$$

Since $k \in K$ and $u_t \in C(\overline{Q}_{\tau,T})$, we have

$$\begin{aligned}
&\lim_{s\to t_1} k(t_1-s)\big[u(x_1,s)-u(x_1,t_1)\big] \\
&\quad= -\lim_{s\to t_1} (t_1-s)^\gamma k(t_1-s)\frac{u(x_1,s)-u(x_1,t_1)}{s-t_1}(t_1-s)^{1-\gamma} = 0.
\end{aligned}$$

Together with $k(t) > 0$, $k'(t) < 0$ for $t \in (0,T]$ and $u(x_1,t_1)$ is the minimum value of u on $\overline{Q}_{\tau,T}$, we obtain (2.1) from (2.2). This completes the proof of Lemma 2.1. □

Lemma 2.2 *Let $\tau \in [0,T]$ and $u \in C^{2,1}(\overline{Q}_{\tau,T})$ satisfy $L_{\tau,t}[u](x,t) \ge 0$ in $Q_{\tau,T}$. Then we have*

$$\min_{\overline{Q}_{\tau,T}} u(x,t) = \min_{\partial_p Q_{\tau,T}} u(x,t), \tag{2.3}$$

where $\partial_p Q_{\tau,T} := (\overline{\Omega} \times \{t=\tau\}) \cup \Sigma_{\tau,T}$.

Proof We first prove (2.3) when $L_{\tau,t}[u] > 0$ in $Q_{\tau,T}$. Assume that (2.3) does not hold. Then there exist $x_2 \in \Omega$ and $t_2 \in (\tau, T]$ such that $u(x_2,t_2) = \min_{(x,t)\in\overline{Q}_{\tau,T}} u(x,t)$. Therefore $u_t(x_2,t_2) \leq 0$ and $A[u](x_2,t_2) \geq 0$. Additionally, by Lemma 2.1 we have $\int_\tau^{t_2} k(t_2 - s)u_t(x_2,s)\,\mathrm{d}s \leq 0$. Thus $L_{\tau,t}[u](x_2,t_2) \leq 0$, which contradicts with $L_{\tau,t}[u](x,t) > 0$ in $Q_{\tau,T}$.

Next we consider the general case of $L_{\tau,t}[u] \geq 0$ in $Q_{\tau,T}$. Let $v = u - \varepsilon e^{-t}$ with some $\varepsilon > 0$. Then we have

$$L_{\tau,t}[v] = L_{\tau,t}[u] + \varepsilon e^{-t} + \varepsilon \int_\tau^t k(t-s)e^{-s}\,\mathrm{d}s > 0. \tag{2.4}$$

According to the proved conclusion, it follows that

$$\min_{\overline{Q}_{\tau,T}} u(x,t) \geq \min_{\overline{Q}_{\tau,T}} v(x,t) = \min_{\partial_p Q_{\tau,T}} v(x,t) \geq \min_{\partial_p Q_{\tau,T}} u(x,t) - \varepsilon e^{-\tau}. \tag{2.5}$$

Letting $\varepsilon \to 0$, we get the desired conclusion and the proof is complete. □

By using Lemma 2.2, we can prove the following lemma.

Lemma 2.3 *Let* (1.6) *be held and u be the solution of the problem* (1.1)-(1.2). *Then we have*

$$u_t(x,t) > 0, \quad (x,t) \in Q_{0,T}. \tag{2.6}$$

Proof By the equation of u and (1.6), we have

$$u_t(x,0) = A[u_0](x) + f(x,0) = 0, \quad x \in \Omega. \tag{2.7}$$

Letting $v = u_t$ and differentiating the equation in (1.1) with respect to t, we find that

$$\begin{cases} L_{0,t}[v](x,t) = f_t(x,t), & (x,t) \in Q_{0,T}, \\ v(x,0) = 0, & x \in \Omega, \\ v(x,t) = g_t(x,t), & (x,t) \in \Sigma_{0,T}, \end{cases} \tag{2.8}$$

where we have used

$$\left(\int_0^t k(t-s)u_t(x,s)\,\mathrm{d}s \right)_t = k(t)u_t(x,0) + \int_0^t k(s)u_{tt}(x,t-s)\,\mathrm{d}s. \tag{2.9}$$

According to Theorem A.1 in Appendix A, we have $v \in C^{2,1}(\overline{Q}_{0,T})$ under $f_t \in C(\overline{Q}_{0,T})$ and $g_t \in C^{2,1}(\overline{Q}_{0,T})$. In addition, by (1.6) we have $L_{0,t}[v](x,t) \geq 0$ in $Q_{0,T}$ and $\min_{\partial_p Q_{0,T}} v(x,t) \geq 0$. Then applying Lemma 2.2, we obtain

$$v(x,t) \geq 0, \quad (x,t) \in \overline{Q}_{0,T}.$$

Now we are ready to prove (2.6). If (2.6) does not hold, then v attains the minimum value 0 on $\overline{Q}_{0,T}$ at $(x_3,t_3) \in Q_{0,T}$, and we have $v_t(x_3,t_3) \leq 0$ and $Av(x_3,t_3) \geq 0$. Additionally, by Lemma 2.1 we obtain $\int_0^{t_3} k(t_3 - s)v_t(x_3,s)\,\mathrm{d}s \leq 0$. Therefore,

$$0 \geq v_t(x_3,t_3) - Av(x_3,t_3) + \int_0^{t_3} k(t_3-s)v_t(x_3,s)\,\mathrm{d}s = f_t(x_3,t_3). \tag{2.10}$$

This contradicts with $f_t(x,t) > 0$ in $Q_{0,T}$. The proof of Lemma 2.3 is complete. □

Now we prove Theorem 1.1.

Proof of Theorem 1.1 Let (u_1, k_1) and (u_2, k_2) be two solutions of the inverse problem (1.1)-(1.3). This implies that $\hat{u} := u_1 - u_2$ and $\hat{k} = k_1 - k_2$ satisfy

$$\begin{cases} L_{0,t}[\hat{u}](x,t) = -\int_0^t \hat{k}(t-s)(u_2)_t(x,s)\,\mathrm{d}s, & (x,t) \in Q_{0,T}, \\ \hat{u}(x,0) = 0, & x \in \Omega, \\ \hat{u}(x,t) = 0, & (x,t) \in \Sigma_{0,T}, \end{cases} \tag{2.11}$$

and

$$\hat{u}(x_0,t) = 0, \quad t \in [0,T]. \tag{2.12}$$

Here we have used k_1 as the kernel function in $L_{0,t}$ and this has no impact on the following proof. In order to prove the uniqueness of the inverse problem (1.1)-(1.3), it is sufficient to show that

$$t^\gamma \hat{k} = 0, \quad t \in [0,T]. \tag{2.13}$$

Indeed, if (2.13) holds, then we have $\hat{k}(t) = 0$ for $t \in (0,T]$, *i.e.* $\int_0^t \hat{k}(t-s)(u_2)_t(x,s)\,\mathrm{d}s = 0$. Therefore, by Theorem 1.1, we obtain $\hat{u}(x,t) = 0$ for $(x,t) \in \overline{Q}_{0,T}$.

We now prove (2.13) by contradiction. We assume that (2.13) does not hold and then set

$$t_0 = \inf\{t \in [0,T] \mid t^\gamma \hat{k}(t) \neq 0\}. \tag{2.14}$$

Since $t^\gamma k_1, t^\gamma k_2 \in C[0,T]$, there exists sufficiently small $\delta > 0$ such that $t^\gamma k_1(t) \neq t^\gamma k_2(t)$ for $t \in [t_0, t_0+\delta]$. Without loss of generality, we can assume that $t^\gamma k_1(t) < t^\gamma k_2(t)$ for $t \in [t_0, t_0+\delta]$. Next we prove

$$\hat{u}(x,t) = 0, \quad (x,t) \in \overline{Q}_{0,t_0}. \tag{2.15}$$

Obviously, the conclusion is correct for $t_0 = 0$. When $t_0 > 0$, we consider the direct problem (1.1) and (1.2) in Q_{0,t_0}. By $\int_0^t \hat{k}(t-s)(u_2)_t(x,s)\,\mathrm{d}s = 0$ in $[0,t_0]$ and Theorem A.1 we obtain $\|\hat{u}\|_{C^{2,1}(\overline{Q}_{0,t_0})} = 0$. Therefore, $\hat{u}(x,t) = 0$ for $(x,t) \in \overline{Q}_{0,t_0}$.

Now on the basis of (2.11), (2.14), and (2.15), we have

$$\begin{cases} L_{t_0,t}[\hat{u}](x,t) = -\int_{t_0}^t \hat{k}(t-s)(u_2)_t(x,s)\,\mathrm{d}s, & (x,t) \in Q_{t_0,t_0+\delta}, \\ \hat{u}(x,t_0) = 0, & x \in \Omega, \\ \hat{u}(x,t) = 0, & (x,t) \in \Sigma_{t_0,t_0+\delta}. \end{cases} \tag{2.16}$$

By Lemma 2.3, we have $(u_2)_t(x,t) > 0$ for $(x,t) \in Q_{t_0,t_0+\delta}$. Therefore, $L_{t_0,t}[\hat{u}](x,t) > 0$ in $Q_{t_0,t_0+\delta}$. Applying Lemma 2.2, we have $\hat{u}(x,t) \geq 0$ in $\overline{Q}_{t_0,t_0+\delta}$. Furthermore, we can obtain

$$\hat{u}(x,t) > 0, \quad (x,t) \in Q_{t_0,t_0+\delta}. \tag{2.17}$$

Otherwise, there exists $(x_4,t_4) \in Q_{t_0,t_0+\delta}$ such that $\hat{u}(x_4,t_4)$ is the minimum value of $\hat{u}$ on $\overline{Q}_{t_0,t_0+\delta}$. Then we have $\hat{u}(x_4,t_4)_t = 0$ and $Au(x_4,t_4) \geq 0$. Additionally, Lemma 2.1 gives

$\int_{t_0}^{t_4} k_1(t_4 - s)\hat{u}_t(x,s)\,\mathrm{d}s \le 0$ and we find that

$$0 \ge \hat{u}(x_4,t_4)_t - Au(x_4,t_4) + \int_{t_0}^{t_4} k_1(t_4 - s)\hat{u}_t(x_4,s)\,\mathrm{d}s = L_{t_0,t}[\hat{u}](x_4,t_4),$$

which contradicts with $L_{t_0,t}[\hat{u}](x,t) > 0$ in $Q_{t_0,t_0+\delta}$. Thus (2.17) follows. However, by (2.12), $\hat{u}(x_0,t) = 0$ for $x_0 \in \Omega$ and $t \in (t_0, t_0 + \delta)$. This is a contradiction. Thus the proof of Theorem 1.1 is complete. □

3 Conclusion

In this paper we study an inverse weakly singular memory kernel problem for an integro-differential equation related to the Basset problem. In order to determine the weakly singular term, we only use the measurement data at a fixed point $x_0 \in \Omega$, rather than the usual measurement data on the whole or part of Ω in previous studies of inverse kernel problems. The uniqueness of our inverse problem is shown by using a maximum principle related to an integro-differential operator. In addition, the existence and uniqueness of the direct Basset problem with general kernel function are also given, which extends the results in [1] and [2].

Appendix A

Here, we study the existence and uniqueness of the following direct problem:

$$\begin{cases} L_{0,t}[u](x,t) = f(x,t), & (x,t) \in Q_{0,T}, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = g(x,t), & (x,t) \in \Sigma_{0,T}. \end{cases} \tag{A.1}$$

We can prove the following.

Theorem A.1 *Let $k \in C_\gamma[0,T]$ with some $\gamma \in (0,1)$, $u_0 \in C^2(\overline{\Omega})$, $f \in C(\overline{Q}_{0,T})$ and $g \in C^{2,1}(\overline{Q}_{0,T})$. Then the direct problem* (A.1) *has a unique solution $u \in C^{2,1}(\overline{Q}_{0,T})$ such that*

$$\begin{aligned} &\|u\|_{C^{2,1}(\overline{Q}_{0,t})} \\ &\quad \le C\bigl(\|u_0\|_{C^2(\overline{\Omega})} + \|f\|_{C(\overline{Q}_{0,t})} + \|g\|_{C^{2,1}(\overline{Q}_{0,t})}\bigr)E_{1-\gamma}\bigl(\|k\|_{C_\gamma[0,T]}\Gamma(1-\gamma)t^{1-\gamma}\bigr), \end{aligned} \tag{A.2}$$

where $E_{1-\gamma}(z)$ is the Mittag-Leffler function defined by $E_{1-\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n(1-\gamma)+1)}$.

Remark A.1 When $k = \frac{1}{\Gamma(1-\alpha)}t^{-\alpha}$ with $0 < \alpha < 1$, the equation of u can be rewritten as

$$u_t(x,t) - A[u](x,t) + \partial_t^\alpha u(x,t) = f(x,t), \quad (x,t) \in Q_{0,T}. \tag{A.3}$$

This is a time fractional parabolic equation to describe the Basset problem [6].

Proof We will use a fixed point argument to prove this theorem. To do this, we set

$$\begin{aligned} V_r = \bigl\{&v \in C^{2,1}(\overline{Q}_{0,T}) \mid v(x,0) = u_0(x),\ x \in \Omega, v(x,t) = g(x,t), (x,t) \in \Sigma_{0,T}, \\ &\|v\|_{C^{2,1}(\overline{Q}_{0,T})} \le r\bigr\} \end{aligned}$$

with some $r > 0$, which will be specified below. For given $v \in V_r$, we consider

$$\begin{cases} u_t(x,t) - A[u](x,t) = G[v](x,t) + f(x,t), & (x,t) \in Q_{0,T}, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = g(x,t), & (x,t) \in \Sigma_{0,T}, \end{cases} \tag{A.4}$$

with $G[v](x,t) = -\int_0^t k(t-s)v(x,s)\,\mathrm{d}s$. The standard result for linear parabolic equation [22] shows that there exists a unique solution $u \in C^{2,1}(\overline{Q}_{0,T})$ of the problem (A.4) such that

$$\|u\|_{C^{2,1}(\overline{Q}_{0,t})} \le C\bigl(\|u_0\|_{C^2(\overline{\Omega})} + \bigl\|G[v] + f\bigr\|_{C(\overline{Q}_{0,t})} + \|g\|_{C^{2,1}(\overline{Q}_{0,t})}\bigr), \quad 0 \le t \le T. \tag{A.5}$$

Therefore, the following mapping:

$$\Phi : V_r \to C^{2,1}(\overline{Q}_{0,T}), \quad v \mapsto u \tag{A.6}$$

is well defined.

We want to choose T small enough to prove Φ is a contraction on V_r, which implies that Φ has a unique fixed point u in V_r. By $k \in C_\gamma[0,T]$, we have

$$\begin{aligned} \bigl\|G[v]\bigr\|_{C(\overline{Q}_{0,t})} &\le \int_0^t k(t-s)\|v\|_{C^{2,1}(\overline{Q}_{0,s})}\,\mathrm{d}s \\ &\le C\|k\|_{C_\gamma[0,T]} \int_0^t (t-s)^{-\gamma}\|v\|_{C^{2,1}(\overline{Q}_{0,s})}\,\mathrm{d}s. \end{aligned} \tag{A.7}$$

Substituting (A.7) into (A.5) yields

$$\|u\|_{C^{2,1}(\overline{Q}_{0,t})} \le C\bigl(\|u_0\|_{C^2(\overline{\Omega})} + \|f\|_{C(\overline{Q}_{0,T})} + \|g\|_{C^{2,1}(\overline{Q}_{0,T})}\bigr) + \frac{Ct^{1-\gamma}}{1-\gamma}\|k\|_{C_\gamma[0,T]}r. \tag{A.8}$$

We fix

$$2r = C\bigl(\|u_0\|_{C^2(\overline{\Omega})} + \|f\|_{C(\overline{Q}_{0,T})} + \|g\|_{C^{2,1}(\overline{Q}_{0,T})}\bigr). \tag{A.9}$$

Then we can choose T_1 to satisfy

$$\|u\|_{C^{2,1}(\overline{Q}_{0,T})} \le r \tag{A.10}$$

for all $T \in [0, T_1]$, and from which it follows that $\Phi(V_r) \subset V_r$. On the other hand, given $v_1, v_2 \in V_r$, $U := \Phi(v_1) - \Phi(v_2)$ satisfies

$$\begin{cases} U_t(x,t) - A[U](x,t) = G[v_1](x,t) - G[v_2](x,t), & (x,t) \in Q_{0,T}, \\ U(x,0) = 0, & x \in \Omega, \\ U(x,t) = 0, & (x,t) \in \Sigma_{0,T}. \end{cases} \tag{A.11}$$

Therefore, applying (A.5) we have

$$\|U\|_{C^{2,1}(\overline{Q}_{0,t})} \le C\bigl(\bigl\|G[v_1] - G[v_2]\bigr\|_{C(\overline{Q}_{0,t})}\bigr) \le \frac{Ct^{1-\gamma}}{1-\gamma}\|k\|_{C_\gamma[0,T]}\|v_1 - v_2\|_{C^{2,1}(\overline{Q}_{0,t})} \tag{A.12}$$

for all $t \in [0, T]$. Then there exists sufficiently small T_2 such that

$$\left\| \Phi(v_1) - \Phi(v_2) \right\|_{C^{2,1}(\overline{Q}_{0,T})} \leq 1/2 \|v_1 - v_2\|_{C^{2,1}(\overline{Q}_{0,T})} \tag{A.13}$$

for all $T \in [0, T_2]$. By (A.10) and (A.13), we find that $\Phi : V_r \to V_r$ is a contraction mapping for $T \leq \min\{T_1, T_2\}$. Thus there exists a local unique solution $u \in C^{2,1}(\overline{Q}_{0,T})$ of the problem (A.1) for sufficiently small T.

In order to obtain the global existence, it is sufficient to prove that the solution u of the problem (A.1) satisfies

$$\|u\|_{C^{2,1}(\overline{Q}_{0,T})} < \infty \tag{A.14}$$

for any T. Indeed, if (A.14) holds, then we can extend the local solution repeatedly to the whole interval $[0, T]$ by the above fixed point arguments. By the estimate of Schauder type for parabolic equation [23], we find that for any $t \in [0, T]$,

$$\begin{aligned} \|u\|_{C^{2,1}(\overline{Q}_{0,t})} &\leq C\big(\|u_0\|_{C^2(\overline{\Omega})} + \big\|G[u] + f\big\|_{C(\overline{Q}_{0,t})} + \|g\|_{C^{2,1}(\overline{Q}_{0,t})}\big) \\ &\leq C\big(\|u_0\|_{C^2(\overline{\Omega})} + \|f\|_{C(\overline{Q}_{0,t})} + \|g\|_{C^{2,1}(\overline{Q}_{0,t})}\big) \\ &\quad + C\|k\|_{C_\gamma[0,T]} \int_0^t (t-s)^{-\gamma} \|u\|_{C^{2,1}(\overline{Q}_{0,s})}\, \mathrm{d}s, \end{aligned}$$

which implies (A.2) by the weakly singular Gronwall inequalities [24]. Since $E_{1-\gamma}(z)$ is continuous on $[0, T]$, (A.14) holds for any T. This completes the proof of Theorem A.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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