Ahmad et al. *Boundary Value Problems* 2014, **2014**:249
http://www.boundaryvalueproblems.com/content/2014/1/249

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

Some existence theorems for fractional integro-differential equations and inclusions with initial and non-separated boundary conditions

Bashir Ahmad[1*], Ahmed Alsaedi[1], Sayyedeh Zahra Nazemi[2] and Shahram Rezapour[2]

*Correspondence: bashirahmad_qau@yahoo.com
[1]Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this paper, we study the existence of solutions for a new class of boundary value problems of nonlinear fractional integro-differential equations and inclusions of arbitrary order with initial and non-separated boundary conditions. In the case of inclusion, the existence results are obtained for convex as well as non-convex multifunctions. Our results rely on the standard tools of fixed point theory and are well illustrated with the aid of examples.

1 Introduction

The subject of fractional calculus has recently been investigated in an extensive manner. The publication of several books, special issues, and a huge number of articles in journals of international repute, exploring numerous aspects of this branch of mathematics, clearly indicates the popularity of the topic. One of the key factors accounting for the utility of the subject is that fractional-order operators are nonlocal in nature in contrast to the integer-order operators and can describe the hereditary properties of many underlying phenomena and processes. Owing to this characteristic, the principles of fractional calculus have played a significant role in improving the modeling techniques for several real world problems [1–4].

Many researchers have focused their attention on fractional differential equations and inclusions, and a variety of interesting and important results concerning existence and uniqueness of solutions, stability properties of solutions, analytic and numerical methods of solutions of these equations have been obtained and the surge for investigating more and more results is still under way. For details and examples, we refer the reader to a series of papers [5–29] and the references therein. Anti-periodic boundary value problems occur in the mathematical modeling of a variety of physical processes and some works have been published in this area, for instance; see [30–34] and the references therein.

In this paper, for $\alpha \in (n-1,n]$, $n \geq 5$, $n \in \mathbb{N}$, $t \in I = [0,T]$, $T > 0$, we investigate the fractional integro-differential equation

$$\begin{aligned} {}^cD^{\alpha}x(t) = f\bigl(&t, x(t), x'(t), x''(t), x'''(t), \phi x(t), \psi x(t), {}^cD^{\mu_1}x(t), {}^cD^{\mu_2}x(t), \ldots, {}^cD^{\mu_m}x(t), \\ &{}^cD^{\nu_1}x(t), {}^cD^{\nu_2}x(t), \ldots, {}^cD^{\nu_{m'}}x(t), {}^cD^{\xi_1}x(t), {}^cD^{\xi_2}x(t), \ldots, {}^cD^{\xi_{m''}}x(t)\bigr), \end{aligned} \tag{1.1}$$

and related fractional integro-differential inclusion

$$
\begin{aligned}
{}^cD^\alpha x(t) \in F\big(&t, x(t), x'(t), x''(t), x'''(t), \phi x(t), \psi x(t), {}^cD^{\mu_1}x(t), {}^cD^{\mu_2}x(t), \dots, {}^cD^{\mu_m}x(t), \\
&{}^cD^{\nu_1}x(t), {}^cD^{\nu_2}x(t), \dots, {}^cD^{\nu_{m'}}x(t), {}^cD^{\xi_1}x(t), {}^cD^{\xi_2}x(t), \dots, {}^cD^{\xi_{m''}}x(t)\big),
\end{aligned}
\tag{1.2}
$$

supplemented with initial boundary conditions

$$
\begin{aligned}
&x^{(4)}(0) = \cdots = x^{(n-1)}(0) = 0, \qquad ax(0) + bx(T) = 0, \\
&{}^cD^p x(0) = -{}^cD^p x(T), \qquad {}^cD^q x(0) = -{}^cD^q x(T), \\
&{}^cD^\gamma x(0) = -{}^cD^\gamma x(T), \quad 0 < p < 1, 1 < q < 2, 2 < \gamma < 3, a + b \neq 0, a, b \in \mathbb{R},
\end{aligned}
\tag{1.3}
$$

where ${}^cD$ denotes the Caputo fractional derivative, $f : [0,T] \times \mathbb{R}^{6+m+m'+m''} \to \mathbb{R}$ is a continuous function, $F : [0,1] \times \mathbb{R}^{6+m+m'+m''} \to P(\mathbb{R})$ is a multifunction, $P(\mathbb{R})$ is the family of all non-empty subsets of $\mathbb{R}$, and the maps ϕ and ψ are defined by

$$
\phi x(t) = \int_0^t \gamma(t,s) h_1\big(t, s, x(s), x'(s), x''(s), x'''(s), {}^cD^{\delta_1}x(s), {}^cD^{\beta_1}x(s), {}^cD^{\theta_1}x(s)\big)\, ds
$$

and

$$
\psi x(t) = \int_0^t \lambda(t,s) h_2\big(t, s, x(s), x'(s), x''(s), x'''(s), {}^cD^{\delta_2}x(s), {}^cD^{\beta_2}x(s), {}^cD^{\theta_2}x(s)\big)\, ds,
$$

where $\gamma, \lambda : [0,T] \times [0,T] \to \mathbb{R}$ and $h_1, h_2 : [0,T] \times [0,T] \times \mathbb{R}^7 \to \mathbb{R}$ are continuous maps, $0 < \mu_i < 1$ $(1 \le i \le m)$, $1 < \nu_j < 2$ $(1 \le j \le m')$, $2 < \xi_k < 3$ $(1 \le k \le m'')$, $0 < \delta_i < 1$, $1 < \beta_i < 2$, and $2 < \theta_i < 3$ for $i = 1, 2$.

The paper is organized as follows. In Section 2, we recall some preliminary facts that we used in the sequel. Section 3 deals with the existence result for single-valued initial boundary value problem, while the results for multivalued problem are presented in Section 4. We present some examples illustrating the main results in Section 5.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a normed space, $P(X)$ the set of all non-empty subsets of X, $P_{cl}(X)$ the set of all non-empty closed subsets of X, $P_b(X)$ the set of all non-empty bounded subsets of X, $P_{cp}(X)$ the set of all non-empty compact subsets of X and $P_{cp,c}(X)$ the set of all non-empty compact and convex subsets of X [35]. A multivalued map $G : X \to P(X)$ is said to be convex (closed) valued whenever $G(x)$ is convex (closed) for all $x \in X$ [35]. The multifunction G is called bounded on bounded sets whenever $G(B) = \bigcup_{x \in B} G(x)$ is bounded subset of X for all $B \in P_b(X)$, that is, $\sup_{x \in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$ for all $B \in P_b(X)$ [35]. Also, the multifunction $G : X \to P(X)$ is called upper semi-continuous whenever for each $x_0 \in X$ the set $G(x_0)$ is a non-empty closed subset of X, and for every open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$ [36, 37]. The multifunction $G : X \to P(X)$ is called compact whenever $G(B)$ is relatively compact for all $B \in P_b(X)$ and also is called completely continuous whenever G is upper semi-continuous and compact [38, 39]. It is well known that a compact multifunction G with non-empty compact valued is upper semi-continuous if and only if G has a closed graph, that is, $x_n \to x_*$, $y_n \in G(x_n)$ for all n, and $y_n \to y_*$ imply $y_* \in G(x_*)$ [37]. We say that $x_0 \in X$ is a fixed point of a multifunction G whenever $x_0 \in G(x_0)$ [40]. Let $T > 0$ and $G : [0, T] \to P_{cl}(\mathbb{R})$ a multifunction. We say that G is measurable whenever the function $t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$ is measurable for all $y \in \mathbb{R}$ [38, 39].

One can find basic notions of fractional calculus in [1] and [2]. We recall two necessary ones here.

The Riemann-Liouville fractional integral of order $q > 0$ with the lower limit zero for a function $f : [0, \infty) \to \mathbb{R}$ is defined by $I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}}\, ds$ for $t > 0$ provided the integral exists.

The Caputo fractional derivative of order $q > 0$ for a function $f \in C^n([0, \infty), \mathbb{R})$ can be written as

$$^cD^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}}\, ds = I^{n-q} f^{(n)}(t),$$

where $n - 1 < q \leq n$ and $t > 0$.

To define the solution for problems (1.1)-(1.3) and (1.2)-(1.3), we establish the following lemma.

Lemma 2.1 *Let $y \in L^1([0, T], \mathbb{R})$. Then the integral solution of the linear equation*

$$^cD^\alpha x(t) = y(t) \tag{2.1}$$

subject to the initial boundary conditions (1.3) *is given by*

$$x(t) = \int_0^T G(t, s) y(s)\, ds, \tag{2.2}$$

where

$$G(t, s) = \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + G_1(t, s), & \text{if } s \leq t, \\ G_1(t, s), & \text{if } t \leq s, \end{cases}$$

and

$$\begin{aligned} G_1(t, s) = {} & -\frac{b(T-s)^{\alpha-1}}{(a+b)\Gamma(\alpha)} + \frac{[bT - (a+b)t]\Gamma(2-p)(T-s)^{\alpha-p-1}}{(a+b)\Gamma(\alpha-p)T^{1-p}} \\ & - \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)(T-s)^{\alpha-q-1}}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \\ & - \frac{\Gamma(4-\gamma)(T-s)^{\alpha-\gamma-1}}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\ & \times \bigl(b\bigl(-6(q-p) + (2-p)(3-p)q\bigr)T^3 \\ & + (a+b)\bigl(6(p-q)T^2 t + (2-p)(3-p)\bigl(-3Tt^2 + (3-q)t^3\bigr)\bigr)\bigr). \end{aligned}$$

Proof It is well known that the solution of (2.1) can be written as

$$\begin{aligned} x(t) &= I^\alpha y(t) - b_0 - b_1 t - b_2 t^2 - b_3 t^3 - b_4 t^4 - \cdots - b_{n-1} t^{n-1} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds - b_0 - b_1 t - b_2 t^2 - b_3 t^3 - b_4 t^4 - \cdots - b_{n-1} t^{n-1}, \end{aligned} \tag{2.3}$$

where $b_0, b_1, b_2, b_3, b_4, \ldots, b_{n-1} \in \mathbb{R}$ are arbitrary constants. Using the initial conditions $x^{(4)}(0) = \cdots = x^{(n-1)}(0) = 0$, we find that $b_4 = \cdots = b_{n-1} = 0$. Since ${}^cD^p c = 0$ for all constant c, ${}^cD^p t = \frac{t^{1-p}}{\Gamma(2-p)}$, ${}^cD^p t^2 = \frac{2t^{2-p}}{\Gamma(3-p)}$, ${}^cD^p t^3 = \frac{6t^{3-p}}{\Gamma(4-p)}$, ${}^cD^q t = 0$, ${}^cD^q t^2 = \frac{2t^{2-q}}{\Gamma(3-q)}$, ${}^cD^q t^3 = \frac{6t^{3-q}}{\Gamma(4-q)}$, ${}^cD^\gamma t = 0$, ${}^cD^\gamma t^2 = 0$, ${}^cD^\gamma t^3 = \frac{6t^{3-\gamma}}{\Gamma(4-\gamma)}$, ${}^cD^p I^\alpha y(t) = I^{\alpha-p} y(t)$, ${}^cD^q I^\alpha y(t) = I^{\alpha-q} y(t)$, and ${}^cD^\gamma I^\alpha y(t) = I^{\alpha-\gamma} y(t)$, therefore

$$\begin{aligned}
{}^cD^p x(t) &= \frac{1}{\Gamma(\alpha-p)} \int_0^t (t-s)^{\alpha-p-1} y(s)\,ds - b_1 \frac{t^{1-p}}{\Gamma(2-p)} - b_2 \frac{2t^{2-p}}{\Gamma(3-p)} - b_3 \frac{6t^{3-p}}{\Gamma(4-p)}, \\
{}^cD^q x(t) &= \frac{1}{\Gamma(\alpha-q)} \int_0^t (t-s)^{\alpha-q-1} y(s)\,ds - b_2 \frac{2t^{2-q}}{\Gamma(3-q)} - b_3 \frac{6t^{3-q}}{\Gamma(4-q)}, \\
{}^cD^\gamma x(t) &= \frac{1}{\Gamma(\alpha-\gamma)} \int_0^t (t-s)^{\alpha-\gamma-1} y(s)\,ds - b_3 \frac{6t^{3-\gamma}}{\Gamma(4-\gamma)}.
\end{aligned}$$

Now using the conditions $ax(0) + bx(T) = 0$, ${}^cD^p x(0) = -{}^cD^p x(T)$, ${}^cD^q x(0) = -{}^cD^q x(T)$, and ${}^cD^\gamma x(0) = -{}^cD^\gamma x(T)$, we obtain

$$\begin{aligned}
b_0 &= \frac{b}{(a+b)} \Bigg[\frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} y(s)\,ds - \frac{\Gamma(2-p) T^p}{\Gamma(\alpha-p)} \int_0^T (T-s)^{\alpha-p-1} y(s)\,ds \\
&\quad + \frac{p\Gamma(3-q) T^q}{2(2-p)\Gamma(\alpha-q)} \int_0^T (T-s)^{\alpha-q-1} y(s)\,ds \\
&\quad + \frac{[-6(q-p) + (2-p)(3-p)q]\Gamma(4-\gamma) T^\gamma}{6(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)} \int_0^T (T-s)^{\alpha-\gamma-1} y(s)\,ds \Bigg], \\
b_1 &= \frac{\Gamma(2-p)}{\Gamma(\alpha-p) T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} y(s)\,ds \\
&\quad - \frac{\Gamma(3-q)}{(2-p)\Gamma(\alpha-q) T^{1-q}} \int_0^T (T-s)^{\alpha-q-1} y(s)\,ds \\
&\quad + \frac{(q-p)\Gamma(4-\gamma)}{(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma) T^{1-\gamma}} \int_0^T (T-s)^{\alpha-\gamma-1} y(s)\,ds, \\
b_2 &= \frac{\Gamma(3-q)}{2\Gamma(\alpha-q) T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} y(s)\,ds \\
&\quad - \frac{\Gamma(4-\gamma)}{2(3-q)\Gamma(\alpha-\gamma) T^{2-\gamma}} \int_0^T (T-s)^{\alpha-\gamma-1} y(s)\,ds, \\
b_3 &= \frac{\Gamma(4-\gamma)}{6\Gamma(\alpha-\gamma) T^{3-\gamma}} \int_0^T (T-s)^{\alpha-\gamma-1} y(s)\,ds.
\end{aligned}$$

Substituting the values of $b_0, b_1, b_2, b_3, b_4, \ldots, b_{n-1}$ in (2.3), we get the solution (2.2). □

3 Existence results for problem (1.1)-(1.3)

Consider the space $X = \{u : u \in C^3(I)\}$ endowed with the norm

$$\|u\| = \sup_{t\in I} |u(t)| + \sup_{t\in I} |u'(t)| + \sup_{t\in I} |u''(t)| + \sup_{t\in I} |u'''(t)|.$$

Obviously $(X, \|\cdot\|)$ is a Banach space.

We need the following result [40] in the sequel.

Theorem 3.1 *Let E be a Banach space, $S : E \to E$ a completely continuous operator and*

$$V = \{x \in E : x = \mu Sx, 0 \le \mu \le 1\}$$

a bounded set. Then S has a fixed point in E.

Let us define the operator $T : X \to X$ by

$$\begin{aligned}(Tx)(t) ={}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\widetilde{f}\big(s,x(s)\big)\,ds - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\widetilde{f}\big(s,x(s)\big)\,ds \\ &+ \frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}}\int_0^T (T-s)^{\alpha-p-1}\widetilde{f}\big(s,x(s)\big)\,ds \\ &- \frac{[bpT^2-(a+b)(2Tt-(2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}}\int_0^T (T-s)^{\alpha-q-1}\widetilde{f}\big(s,x(s)\big)\,ds \\ &- \bigg(\frac{[b(-6(q-p)+(2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\ &+ \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}}\bigg) \\ &\times \int_0^T (T-s)^{\alpha-\gamma-1}\widetilde{f}\big(s,x(s)\big)\,ds,\end{aligned}$$

where

$$\begin{aligned}\widetilde{f}\big(s,x(s)\big) ={}& f\big(s,x(s),x'(s),x''(s),x'''(s),\phi x(s),\psi x(s),{}^cD^{\mu_1}x(s),{}^cD^{\mu_2}x(s),\dots,{}^cD^{\mu_m}x(s), \\ &{}^cD^{\nu_1}x(s),{}^cD^{\nu_2}x(s),\dots,{}^cD^{\nu_{m'}}x(s),{}^cD^{\xi_1}x(s),{}^cD^{\xi_2}x(s),\dots,{}^cD^{\xi_{m''}}x(s)\big).\end{aligned}$$

For the sake of convenience, we set

$$\begin{aligned}M_1 ={}& \bigg[\frac{|a|+2|b|}{|a+b|\Gamma(\alpha+1)} + \frac{(|a|+2|b|)\Gamma(2-p)}{|a+b|\Gamma(\alpha-p+1)} + \frac{(|b|p+|a+b|(4-p))\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q+1)} \\ &+ \bigg(\frac{[|b|(6(q-p)+(2-p)(3-p)q)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \\ &+ \frac{|a+b|(6(q-p)+(2-p)(3-p)(6-q))]\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\bigg)\bigg]T^{\alpha} \\ &+ \bigg[\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-p)}{\Gamma(\alpha-p+1)} + \frac{(3-p)\Gamma(3-q)}{(2-p)\Gamma(\alpha-q+1)} \\ &+ \frac{[2(q-p)+(2-p)(3-p)(5-q)]\Gamma(4-\gamma)}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\bigg]T^{\alpha-1} \\ &+ \bigg[\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(3-q)}{\Gamma(\alpha-q+1)} + \frac{(4-q)\Gamma(4-\gamma)}{(3-q)\Gamma(\alpha-\gamma+1)}\bigg]T^{\alpha-2} \\ &+ \bigg[\frac{1}{\Gamma(\alpha-2)} + \frac{\Gamma(4-\gamma)}{\Gamma(\alpha-\gamma+1)}\bigg]T^{\alpha-3}.\end{aligned}$$

Theorem 3.2 *The operator $T : X \to X$ is completely continuous.*

Proof First, we show that the operator $T : X \to X$ is continuous. Let $\{x_n\}$ be a sequence in X with $x_n \to x_0$ and $0 < \mu_1, \dots, \mu_m < 1$. Then we have

$$\begin{aligned}
&\sup_{t\in I}\left|{}^cD^{\mu_i}x_n(t) - {}^cD^{\mu_i}x_0(t)\right| \\
&\quad = \sup_{t\in I}\left|\frac{1}{\Gamma(1-\mu_i)}\int_0^t (t-s)^{-\mu_i}x_n'(s)\,ds - \frac{1}{\Gamma(1-\mu_i)}\int_0^t (t-s)^{-\mu_i}x_0'(s)\,ds\right| \\
&\quad = \sup_{t\in I}\left|\frac{1}{\Gamma(1-\mu_i)}\int_0^t (t-s)^{-\mu_i}\left[x_n'(s) - x_0'(s)\right]ds\right| \\
&\quad \le \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}\sup_{t\in I}\left|x_n'(t) - x_0'(t)\right| \le \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}\|x_n - x_0\|.
\end{aligned}$$

Since $\|x_n - x\| \to 0$, $\lim_{n\to\infty} {}^cD^{\mu_i}x_n(t) = {}^cD^{\mu_i}x_0(t)$ uniformly on I. Similarly, $\lim_{n\to\infty} {}^cD^{\nu_j}x_n(t) = {}^cD^{\nu_j}x_0(t)$ uniformly on I for $1 \le j \le m'$, $\lim_{n\to\infty} {}^cD^{\xi_k}x_n(t) = {}^cD^{\xi_k}x_0(t)$ uniformly on I for $1 \le k \le m''$. Also, we get $\lim_{n\to\infty} {}^cD^{\delta_i}x_n(t) = {}^cD^{\delta_i}x_0(t)$, $\lim_{n\to\infty} {}^cD^{\beta_i}x_n(t) = {}^cD^{\beta_i}x_0(t)$, and $\lim_{n\to\infty} {}^cD^{\theta_i}x_n(t) = {}^cD^{\theta_i}x_0(t)$ uniformly on I for $i = 1, 2$. Since

$$\begin{aligned}
\|Tx_n - Tx_0\| &= \sup_{t\in I}\left|Tx_n(t) - Tx_0(t)\right| + \sup_{t\in I}\left|(Tx_n)'(t) - (Tx_0)'(t)\right| \\
&\quad + \sup_{t\in I}\left|(Tx_n)''(t) - (Tx_0)''(t)\right| + \sup_{t\in I}\left|(Tx_n)'''(t) - (Tx_0)'''(t)\right|,
\end{aligned}$$

using the continuity of f, h_1, h_2, we get $\|Tx_n - Tx\| \to 0$. Thus, T is continuous on X. Now, let $\Omega \subseteq X$ be a bounded subset. Then there exists a positive constant $L > 0$ such that $|\widetilde{f}(t, x(t))| \le L$ for all $t \in I$ and $x \in \Omega$. We show that $T\Omega$ is a bounded set. We have

$$\begin{aligned}
\left|(Tx)(t)\right| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\widetilde{f}(s,x(s))\right|ds + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|\widetilde{f}(s,x(s))\right|ds \\
&\quad + \frac{|bT-(a+b)t|\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)T^{1-p}}\int_0^T (T-s)^{\alpha-p-1}\left|\widetilde{f}(s,x(s))\right|ds \\
&\quad + \frac{|bpT^2-(a+b)(2Tt-(2-p)t^2)|\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)T^{2-q}}\int_0^T (T-s)^{\alpha-q-1}\left|\widetilde{f}(s,x(s))\right|ds \\
&\quad + \left(\frac{|b(-6(q-p)+(2-p)(3-p)q)T^3}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}}\right. \\
&\quad \left. + \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))|\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}}\right) \\
&\quad \times \int_0^T (T-s)^{\alpha-\gamma-1}\left|\widetilde{f}(s,x(s))\right|ds \\
&\le \frac{(|a|+2|b|)LT^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{(|a|+2|b|)\Gamma(2-p)LT^\alpha}{|a+b|\Gamma(\alpha-p+1)} \\
&\quad + \frac{(|b|p+|a+b|(4-p))\Gamma(3-q)LT^\alpha}{2|a+b|(2-p)\Gamma(\alpha-q+1)} \\
&\quad + \left(\frac{[|b|(6(q-p)+(2-p)(3-p)q)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\right. \\
&\quad \left. + \frac{|a+b|(6(q-p)+(2-p)(3-p)(6-q))]\Gamma(4-\gamma)LT^\alpha}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\right),
\end{aligned}$$

$$\begin{aligned}
\left|(Tx)'(t)\right| &\le \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\
&\quad + \frac{\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}}\int_0^T (T-s)^{\alpha-p-1}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\
&\quad + \frac{|T-(2-p)t|\Gamma(3-q)}{(2-p)\Gamma(\alpha-q)T^{2-q}}\int_0^T (T-s)^{\alpha-q-1}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\
&\quad + \frac{|2(q-p)T^2+(2-p)(3-p)(-2Tt+(3-q)t^2)|\Gamma(4-\gamma)}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
&\quad \times \int_0^T (T-s)^{\alpha-\gamma-1}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\
&\le \frac{T^{\alpha-1}L}{\Gamma(\alpha)} + \frac{\Gamma(2-p)LT^{\alpha-1}}{\Gamma(\alpha-p+1)} + \frac{(3-p)\Gamma(3-q)LT^{\alpha-1}}{(2-p)\Gamma(\alpha-q+1)} \\
&\quad + \frac{[2(q-p)+(2-p)(3-p)(5-q)]\Gamma(4-\gamma)LT^{\alpha-1}}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}, \\
\left|(Tx)''(t)\right| &\le \frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\
&\quad + \frac{\Gamma(3-q)}{\Gamma(\alpha-q)T^{2-q}}\int_0^T (T-s)^{\alpha-q-1}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\
&\quad + \frac{|-T+(3-q)t|\Gamma(4-\gamma)}{(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}}\int_0^T (T-s)^{\alpha-\gamma-1}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\
&\le \frac{LT^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-q)LT^{\alpha-2}}{\Gamma(\alpha-q+1)} + \frac{(4-q)\Gamma(4-\gamma)LT^{\alpha-2}}{(3-q)\Gamma(\alpha-\gamma+1)},
\end{aligned}$$

and

$$\begin{aligned}
\left|(Tx)'''(t)\right| &\le \frac{1}{\Gamma(\alpha-3)}\int_0^t (t-s)^{\alpha-4}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\
&\quad + \frac{\Gamma(4-\gamma)}{\Gamma(\alpha-\gamma)T^{3-\gamma}}\int_0^T (T-s)^{\alpha-\gamma-1}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\
&\le \frac{LT^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{\Gamma(4-\gamma)LT^{\alpha-3}}{\Gamma(\alpha-\gamma+1)}
\end{aligned}$$

for all $x \in \Omega$. Hence, we get

$$\begin{aligned}
\|Tx\| &= \sup_{t\in I}\left|(Tx)(t)\right| + \sup_{t\in I}\left|(Tx)'(t)\right| + \sup_{t\in I}\left|(Tx)''(t)\right| + \sup_{t\in I}\left|(Tx)'''(t)\right| \\
&\le \bigg[\frac{|a|+2|b|}{|a+b|\Gamma(\alpha+1)} + \frac{(|a|+2|b|)\Gamma(2-p)}{|a+b|\Gamma(\alpha-p+1)} + \frac{(|b|p+|a+b|(4-p))\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q+1)} \\
&\quad + \bigg(\frac{[|b|(6(q-p)+(2-p)(3-p)q)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \\
&\quad + \frac{|a+b|(6(q-p)+(2-p)(3-p)(6-q))]\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\bigg)\bigg]LT^{\alpha} \\
&\quad + \bigg[\frac{1}{\Gamma(\alpha)} + \frac{\Gamma(2-p)}{\Gamma(\alpha-p+1)} + \frac{(3-p)\Gamma(3-q)}{(2-p)\Gamma(\alpha-q+1)}
\end{aligned}$$

$$+\frac{[2(q-p)+(2-p)(3-p)(5-q)]\Gamma(4-\gamma)}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\Bigg]LT^{\alpha-1}$$
$$+\left[\frac{1}{\Gamma(\alpha-1)}+\frac{\Gamma(3-q)}{\Gamma(\alpha-q+1)}+\frac{(4-q)\Gamma(4-\gamma)}{(3-q)\Gamma(\alpha-\gamma+1)}\right]LT^{\alpha-2}$$
$$+\left[\frac{1}{\Gamma(\alpha-2)}+\frac{\Gamma(4-\gamma)}{\Gamma(\alpha-\gamma+1)}\right]LT^{\alpha-3}=M_1L.$$

This implies that the operator T maps bounded sets of X into bounded sets. Now, we prove that the sets $\{Tx : x\in\Omega\}$, $\{(Tx)' : x\in\Omega\}$, $\{(Tx)'' : x\in\Omega\}$, $\{(Tx)''' : x\in\Omega\}$ are equicontinuous on I. For $0\le t_1<t_2\le T$, we have

$$\begin{aligned}
&\bigl|(Tx)(t_2)-(Tx)(t_1)\bigr| \\
&\quad=\Biggl|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}\widetilde{f}\bigl(s,x(s)\bigr)\,ds-\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}\widetilde{f}\bigl(s,x(s)\bigr)\,ds \\
&\qquad-\frac{(t_2-t_1)\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}}\int_0^T(T-s)^{\alpha-p-1}\widetilde{f}\bigl(s,x(s)\bigr)\,ds \\
&\qquad+\frac{[2T(t_2-t_1)-(2-p)(t_2^2-t_1^2)]\Gamma(3-q)}{2(2-p)\Gamma(\alpha-q)T^{2-q}}\int_0^T(T-s)^{\alpha-q-1}\widetilde{f}\bigl(s,x(s)\bigr)\,ds \\
&\qquad-\frac{[6(q-p)T^2(t_2-t_1)+(2-p)(3-p)(-3T(t_2^2-t_1^2)+(3-q)(t_2^3-t_1^3))]\Gamma(4-\gamma)}{6(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
&\qquad\times\int_0^T(T-s)^{\alpha-\gamma-1}\widetilde{f}\bigl(s,x(s)\bigr)\,ds\Biggr| \\
&\quad\le\frac{L}{\Gamma(\alpha)}\int_0^{t_1}\bigl[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\bigr]\,ds+\frac{L}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds \\
&\qquad+\frac{(t_2-t_1)\Gamma(2-p)LT^{\alpha-1}}{\Gamma(\alpha-p+1)}+\frac{[2T(t_2-t_1)+(2-p)(t_2^2-t_1^2)]\Gamma(3-q)LT^{\alpha-2}}{2(2-p)\Gamma(\alpha-q+1)} \\
&\qquad+\frac{[6(q-p)T^2(t_2-t_1)+(2-p)(3-p)(3T(t_2^2-t_1^2)+(3-q)(t_2^3-t_1^3))]\Gamma(4-\gamma)LT^{\alpha-3}}{6(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \\
&\quad\le\frac{L}{\Gamma(\alpha+1)}\bigl(t_2^\alpha-t_1^\alpha\bigr) \\
&\qquad+\frac{(t_2-t_1)\Gamma(2-p)LT^{\alpha-1}}{\Gamma(\alpha-p+1)}+\frac{[2T(t_2-t_1)+(2-p)(t_2^2-t_1^2)]\Gamma(3-q)LT^{\alpha-2}}{2(2-p)\Gamma(\alpha-q+1)} \\
&\qquad+\frac{[6(q-p)T^2(t_2-t_1)+(2-p)(3-p)(3T(t_2^2-t_1^2)+(3-q)(t_2^3-t_1^3))]\Gamma(4-\gamma)LT^{\alpha-3}}{6(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}.
\end{aligned}$$

In a similar manner, one can find that

$$\begin{aligned}
\bigl|(Tx)'(t_2)-(Tx)'(t_1)\bigr|&\le\frac{L}{\Gamma(\alpha)}\bigl(t_2^{\alpha-1}-t_1^{\alpha-1}\bigr)+\frac{(t_2-t_1)\Gamma(3-q)LT^{\alpha-2}}{\Gamma(\alpha-q+1)} \\
&\quad+\frac{[2T(t_2-t_1)+(3-q)(t_2^2-t_1^2)]\Gamma(4-\gamma)LT^{\alpha-3}}{2(3-q)\Gamma(\alpha-\gamma+1)}, \\
\bigl|(Tx)''(t_2)-(Tx)''(t_1)\bigr|&\le\frac{L}{\Gamma(\alpha-1)}\bigl(t_2^{\alpha-2}-t_1^{\alpha-2}\bigr)+\frac{(t_2-t_1)\Gamma(4-\gamma)LT^{\alpha-3}}{\Gamma(\alpha-\gamma+1)},
\end{aligned}$$

and

$$\left|(Tx)'''(t_2)-(Tx)'''(t_1)\right| \le \frac{L}{\Gamma(\alpha-2)}\left(t_2^{\alpha-3}-t_1^{\alpha-3}\right).$$

Clearly the right-hand sides of the above inequalities tend to zero as $t_2 \to t_1$. So T is completely continuous. This completes the proof. □

Theorem 3.3 *Assume that there exist positive constants $d_0 > 0$, $d_i \ge 0$ $(1 \le i \le 6)$, $\zeta_i \ge 0$ $(1 \le i \le m)$, $\eta_j \ge 0$ $(1 \le j \le m')$, $\tau_k \ge 0$ $(1 \le k \le m'')$, $l_{i1}, l_{i2} \ge 0$ $(1 \le i \le 7)$, $c_{01}, c_{02} > 0$ such that*

$$\begin{aligned}
&\left|f(t,x_1,x_2,x_3,x_4,x_5,x_6,y_1,y_2,\ldots,y_m,z_1,z_2,\ldots,z_{m'},w_1,w_2,\ldots,w_{m''})\right| \\
&\quad\le d_0+d_1|x_1|+d_2|x_2|+d_3|x_3|+d_4|x_4|+d_5|x_5|+d_6|x_6| \\
&\qquad+\sum_{i=1}^{m}\zeta_i|y_i|+\sum_{j=1}^{m'}\eta_j|z_j|+\sum_{k=1}^{m''}\tau_k|w_k|
\end{aligned}$$

for all $t \in I$ and $x_1,\ldots,x_6,y_1,\ldots,y_m,z_1,\ldots,z_{m'},\tau_1,\ldots,\tau_{m''} \in \mathbb{R}$ and

$$\left|h_j(t,s,u_1,u_2,u_3,u_4,u_5,u_6,u_7)\right| \le c_{0j}+\sum_{i=1}^{7} l_{ij}|u_i|$$

for $j = 1, 2$, all $t, s \in I$, and all $u_1,\ldots,u_7 \in \mathbb{R}$. In addition, assume that

$$\begin{aligned}
M_1' = M_1\Bigg[&d_1+d_2+d_3+d_4+d_5\gamma_0\bigg(l_{11}+l_{21}+l_{31}+l_{41} \\
&+l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)}+l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}+l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg) \\
&+d_6\lambda_0\bigg(l_{12}+l_{22}+l_{32}+l_{42}+l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)}+l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}+l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\bigg) \\
&+\sum_{i=1}^{m}\zeta_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}+\sum_{j=1}^{m'}\eta_j\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}+\sum_{k=1}^{m''}\tau_k\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg] < 1,
\end{aligned}$$

where $\gamma_0 = \sup_{t\in I}\int_0^t |\gamma(t,s)|\,ds$ and $\lambda_0 = \sup_{t\in I}\int_0^t |\lambda(t,s)|\,ds$. Then problem (1.1)-(1.3) *has at least one solution.*

Proof In view of Theorem 3.2, the operator $T : X \to X$ is completely continuous. Next we show that the set $V = \{x \in X : x = \mu Tx, 0 \le \mu \le 1\}$ is bounded. Let $x \in V$ and $t \in I$. Then we have

$$\begin{aligned}
x(t) = {}&\frac{1}{\Gamma(\alpha)}\int_0^t \mu(t-s)^{\alpha-1}\widetilde{f}\big(s,x(s)\big)\,ds \\
&-\frac{b}{(a+b)\Gamma(\alpha)}\int_0^T \mu(T-s)^{\alpha-1}\widetilde{f}\big(s,x(s)\big)\,ds \\
&+\frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}}\int_0^T \mu(T-s)^{\alpha-p-1}\widetilde{f}\big(s,x(s)\big)\,ds
\end{aligned}$$

$$
\begin{aligned}
&- \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T \mu(T-s)^{\alpha-q-1}\widetilde{f}(s,x(s))\,ds \\
&- \biggl(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
&+ \frac{(a+b)(6(q-p)T^2 t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \biggr) \\
&\times \int_0^T \mu(T-s)^{\alpha-\gamma-1}\widetilde{f}(s,x(s))\,ds, \\
x'(t) = {}& \frac{1}{\Gamma(\alpha-1)} \int_0^t \mu(t-s)^{\alpha-2}\widetilde{f}(s,x(s))\,ds \\
&- \frac{\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}} \int_0^T \mu(T-s)^{\alpha-p-1}\widetilde{f}(s,x(s))\,ds \\
&+ \frac{[T-(2-p)t]\Gamma(3-q)}{(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T \mu(T-s)^{\alpha-q-1}\widetilde{f}(s,x(s))\,ds \\
&- \frac{[2(q-p)T^2 + (2-p)(3-p)(-2Tt + (3-q)t^2)]\Gamma(4-\gamma)}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
&\times \int_0^T \mu(T-s)^{\alpha-\gamma-1}\widetilde{f}(s,x(s))\,ds, \\
x''(t) = {}& \frac{1}{\Gamma(\alpha-2)} \int_0^t \mu(t-s)^{\alpha-3}\widetilde{f}(s,x(s))\,ds \\
&- \frac{\Gamma(3-q)}{\Gamma(\alpha-q)T^{2-q}} \int_0^T \mu(T-s)^{\alpha-q-1}\widetilde{f}(s,x(s))\,ds \\
&- \frac{[-T + (3-q)t]\Gamma(4-\gamma)}{(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \int_0^T \mu(T-s)^{\alpha-\gamma-1}\widetilde{f}(s,x(s))\,ds,
\end{aligned}
$$

and $x'''(t) = \frac{1}{\Gamma(\alpha-3)} \int_0^t \mu(t-s)^{\alpha-4}\widetilde{f}(s,x(s))\,ds - \frac{\Gamma(4-\gamma)}{\Gamma(\alpha-\gamma)T^{3-\gamma}} \int_0^T \mu(T-s)^{\alpha-\gamma-1}\widetilde{f}(s,x(s))\,ds$. Thus, we get

$$
\begin{aligned}
\bigl|x(t)\bigr| &= \mu\bigl|Tx(t)\bigr| \\
&\le \Biggl[d_0 + d_1\|x\| + d_2\|x\| + d_3\|x\| + d_4\|x\| \\
&\quad + d_5\gamma_0 \biggl(c_{01} + l_{11}\|x\| + l_{21}\|x\| + l_{31}\|x\| + l_{41}\|x\| \\
&\quad + l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)}\|x\| + l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}\|x\| + l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\|x\| \biggr) \\
&\quad + d_6\lambda_0 \biggl(c_{02} + l_{12}\|x\| + l_{22}\|x\| + l_{32}\|x\| + l_{42}\|x\| \\
&\quad + l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)}\|x\| + l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}\|x\| + l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\|x\| \biggr) \\
&\quad + \sum_{i=1}^{m} \zeta_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}\|x\| + \sum_{j=1}^{m'} \eta_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}\|x\| + \sum_{k=1}^{m''} \tau_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\|x\| \Biggr] \\
&\quad \times \biggl[\frac{(|a| + 2|b|)T^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{(|a| + 2|b|)\Gamma(2-p)T^\alpha}{|a+b|\Gamma(\alpha-p+1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{(|b|p+|a+b|(4-p))\Gamma(3-q)T^{\alpha}}{2|a+b|(2-p)\Gamma(\alpha-q+1)} \\
&+\left(\frac{[|b|(6(q-p)+(2-p)(3-p)q)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\right. \\
&\left.+\frac{|a+b|(6(q-p)+(2-p)(3-p)(6-q))]\Gamma(4-\gamma)T^{\alpha}}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\left|x'(t)\right| &= \mu\left|(Tx)'(t)\right| \\
&\leq \Bigg[d_0+d_1\|x\|+d_2\|x\|+d_3\|x\|+d_4\|x\| \\
&\quad+d_5\gamma_0\bigg(c_{01}+l_{11}\|x\|+l_{21}\|x\|+l_{31}\|x\|+l_{41}\|x\| \\
&\quad+l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)}\|x\|+l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}\|x\|+l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\|x\|\bigg) \\
&\quad+d_6\lambda_0\bigg(c_{02}+l_{12}\|x\|+l_{22}\|x\|+l_{32}\|x\|+l_{42}\|x\| \\
&\quad+l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)}\|x\|+l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}\|x\|+l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\|x\|\bigg) \\
&\quad+\sum_{i=1}^{m}\zeta_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}\|x\|+\sum_{j=1}^{m'}\eta_j\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}\|x\|+\sum_{k=1}^{m''}\tau_k\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\|x\|\Bigg] \\
&\quad\times\Bigg[\frac{T^{\alpha-1}}{\Gamma(\alpha)}+\frac{\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)}+\frac{(3-p)\Gamma(3-q)T^{\alpha-1}}{(2-p)\Gamma(\alpha-q+1)} \\
&\quad+\frac{[2(q-p)+(2-p)(3-p)(5-q)]\Gamma(4-\gamma)T^{\alpha-1}}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
\left|x''(t)\right| &= \mu\left|(Tx)''(t)\right| \\
&\leq \Bigg[d_0+d_1\|x\|+d_2\|x\|+d_3\|x\|+d_4\|x\| \\
&\quad+d_5\gamma_0\bigg(c_{01}+l_{11}\|x\|+l_{21}\|x\|+l_{31}\|x\|+l_{41}\|x\| \\
&\quad+l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)}\|x\|+l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}\|x\|+l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\|x\|\bigg) \\
&\quad+d_6\lambda_0\bigg(c_{02}+l_{12}\|x\|+l_{22}\|x\|+l_{32}\|x\|+l_{42}\|x\| \\
&\quad+l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)}\|x\|+l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}\|x\|+l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\|x\|\bigg) \\
&\quad+\sum_{i=1}^{m}\zeta_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}\|x\|+\sum_{j=1}^{m'}\eta_j\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}\|x\|+\sum_{k=1}^{m''}\tau_k\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\|x\|\Bigg] \\
&\quad\times\Bigg[\frac{T^{\alpha-2}}{\Gamma(\alpha-1)}+\frac{\Gamma(3-q)T^{\alpha-2}}{\Gamma(\alpha-q+1)}+\frac{(4-q)\Gamma(4-\gamma)T^{\alpha-2}}{(3-q)\Gamma(\alpha-\gamma+1)}\Bigg],
\end{aligned}
$$

and

$$\begin{aligned}
|x'''(t)| &= \mu|(Tx)'''(t)| \\
&\leq \Bigg[d_0 + d_1\|x\| + d_2\|x\| + d_3\|x\| + d_4\|x\| \\
&\quad + d_5\gamma_0\bigg(c_{01} + l_{11}\|x\| + l_{21}\|x\| + l_{31}\|x\| + l_{41}\|x\| \\
&\quad + l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)}\|x\| + l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}\|x\| + l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\|x\|\bigg) \\
&\quad + d_6\lambda_0\bigg(c_{02} + l_{12}\|x\| + l_{22}\|x\| + l_{32}\|x\| + l_{42}\|x\| \\
&\quad + l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)}\|x\| + l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}\|x\| + l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\|x\|\bigg) \\
&\quad + \sum_{i=1}^{m}\zeta_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}\|x\| + \sum_{j=1}^{m'}\eta_j\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}\|x\| + \sum_{k=1}^{m''}\tau_k\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\|x\|\Bigg] \\
&\quad \times \left[\frac{T^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{\Gamma(4-\gamma)T^{\alpha-3}}{\Gamma(\alpha-\gamma+1)}\right].
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x\| \leq M_1\Bigg[& d_1 + d_2 + d_3 + d_4 + d_5\gamma_0\bigg(l_{11} + l_{21} + l_{31} + l_{41} \\
& + l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg) \\
& + d_6\lambda_0\bigg(l_{12} + l_{22} + l_{32} + l_{42} + l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\bigg) \\
& + \sum_{i=1}^{m}\zeta_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'}\eta_j\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''}\tau_k\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg]\|x\| \\
& + M_1(d_0 + d_5\gamma_0 c_{01} + d_6\lambda_0 c_{02})
\end{aligned}$$

and so $\|x\| \leq \frac{M_1(d_0+d_5\gamma_0 c_{01}+d_6\lambda_0 c_{02})}{1-M_1'}$. Thus, the set V is bounded. Hence it follows by Theorem 3.1 that the operator T has at least one fixed point, which in turn implies that problem (1.1)-(1.3) has a solution. □

Theorem 3.4 *Assume that* $f : I \times \mathbb{R}^{6+m+m'+m''} \to \mathbb{R}$ *and* $h_1, h_2 : I \times I \times \mathbb{R}^7 \to \mathbb{R}$ *are continuous functions and there exist constants* $n_i \geq 0$ $(1 \leq i \leq 6)$, $k_i \geq 0$ $(1 \leq i \leq m)$, $k_j' \geq 0$ $(1 \leq j \leq m')$, $k_k'' \geq 0$ $(1 \leq k \leq m'')$, $e_{i1}, e_{i2} \geq 0$ $(1 \leq i \leq 7)$ *such that*

$$\begin{aligned}
&\big|f(t, x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_{m'}, w_1, w_2, \dots, w_{m''}) \\
&\qquad - f\big(t, x_1', x_2', x_3', x_4', x_5', x_6', y_1', y_2', \dots, y_m', z_1', z_2', \dots, z_{m'}', w_1', w_2', \dots, w_{m''}'\big)\big| \\
&\quad \leq n_1|x_1 - x_1'| + n_2|x_2 - x_2'| + n_3|x_3 - x_3'|
\end{aligned}$$

$$+ n_4|x_4 - x_4'| + n_5|x_5 - x_5'| + n_6|x_6 - x_6'| + \sum_{i=1}^{m} k_i|y_i - y_i'| + \sum_{j=1}^{m'} k_j'|z_j - z_j'| + \sum_{k=1}^{m''} k_k''|w_k - w_k'|$$

for all $x_1,\dots,x_6,x_1',\dots,x_6',y_1,\dots,y_m,y_1',\dots,y_m',z_1,\dots,z_{m'},z_1',\dots,z_{m'}',w_1,\dots,w_{m''},w_1',\dots,w_{m''}' \in \mathbb{R}$, *and* $t \in I$, *and also*

$$\left|h_j(t,s,u_1,u_2,u_3,u_4,u_5,u_6,u_7) - h_j\big(t,s,u_1',u_2',u_3',u_4',u_5',u_6',u_7'\big)\right| \le \sum_{i=1}^{7} e_{ij}|u_i - u_i'|$$

for $j = 1,2$, $t,s \in I$, *and* $u_1,\dots,u_7,u_1',\dots,u_7' \in \mathbb{R}$. *In addition, suppose that*

$$\begin{aligned}\Delta = M_1\Bigg[& n_1 + n_2 + n_3 + n_4 + n_5\gamma_0\Bigg(e_{11} + e_{21} + e_{31} + e_{41} \\ & + e_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\Bigg) \\ & + n_6\lambda_0\Bigg(e_{12} + e_{22} + e_{32} + e_{42} + e_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\Bigg) \\ & + \sum_{i=1}^{m} k_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k_j'\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k_k''\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg] < 1.\end{aligned}$$

Then problem (1.1)-(1.3) *has a unique solution.*

Proof Set $N = \sup_{0\le t\le T}|f(t,0,0,\dots,0)| < \infty$, $\kappa_j = \sup_{0\le t,s\le T}|h_j(t,s,0,0,\dots,0)| < \infty$ for $j = 1,2$, and choose $r \ge \frac{(N+n_5\gamma_0\kappa_1+n_6\lambda_0\kappa_2)M_1}{1-\Delta}$. We show that $T(B_r) \subseteq B_r$, where $B_r = \{x \in X : \|x\| \le r\}$. Let $x \in B_r$. Then we have

$$\begin{aligned}\left|(Tx)(t)\right| \le{} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|\widetilde{f}\big(s,x(s)\big)\right|ds + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\ & + \frac{|bT-(a+b)t|\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)T^{1-p}}\int_0^T (T-s)^{\alpha-p-1}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\ & + \frac{|bpT^2-(a+b)(2Tt-(2-p)t^2)|\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)T^{2-q}}\int_0^T (T-s)^{\alpha-q-1}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\ & + \Bigg(\frac{|b(-6(q-p)+(2-p)(3-p)q)T^3}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\ & + \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))|\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}}\Bigg) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1}\left|\widetilde{f}\big(s,x(s)\big)\right|ds \\ \le{} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[\left|\widetilde{f}\big(s,x(s)\big) - f(s,0,0,\dots,0)\right| + \left|f(s,0,0,\dots,0)\right|\Big]ds \\ & + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\Big[\left|\widetilde{f}\big(s,x(s)\big) - f(s,0,0,\dots,0)\right|\end{aligned}$$

$$
\begin{aligned}
&+ \big|f(s,0,0,\ldots,0)\big|\big]\,ds \\
&+ \frac{|bT-(a+b)t|\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)T^{1-p}}\int_0^T (T-s)^{\alpha-p-1}\big[\big|\widetilde{f}\big(s,x(s)\big)-f(s,0,0,\ldots,0)\big| \\
&+ \big|f(s,0,0,\ldots,0)\big|\big]\,ds \\
&+ \frac{|bpT^2-(a+b)(2Tt-(2-p)t^2)|\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)T^{2-q}} \\
&\times \int_0^T (T-s)^{\alpha-q-1}\big[\big|\widetilde{f}\big(s,x(s)\big)-f(s,0,0,\ldots,0)\big| + \big|f(s,0,0,\ldots,0)\big|\big]\,ds \\
&+ \bigg(\frac{|b(-6(q-p)+(2-p)(3-p)q)T^3}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
&+ \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))|\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}}\bigg) \\
&\times \int_0^T (T-s)^{\alpha-\gamma-1}\big[\big|\widetilde{f}\big(s,x(s)\big)-f(s,0,0,\ldots,0)\big| + \big|f(s,0,0,\ldots,0)\big|\big]\,ds \\
\le{}& \Bigg[\Bigg(n_1+n_2+n_3+n_4+n_5\gamma_0\bigg(e_{11}+e_{21}+e_{31}+e_{41} \\
&+ e_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg) \\
&+ n_6\lambda_0\bigg(e_{12}+e_{22}+e_{32}+e_{42}+e_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\bigg) \\
&+ \sum_{i=1}^{m} k_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg)r \\
&+ N + n_5\gamma_0\kappa_1 + n_6\lambda_0\kappa_2\Bigg] \\
&\times \bigg[\frac{(|a|+2|b|)T^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{(|a|+2|b|)\Gamma(2-p)T^\alpha}{|a+b|\Gamma(\alpha-p+1)} \\
&+ \frac{(|b|p+|a+b|(4-p))\Gamma(3-q)T^\alpha}{2|a+b|(2-p)\Gamma(\alpha-q+1)} \\
&+ \bigg(\frac{[|b|(6(q-p)+(2-p)(3-p)q)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)} \\
&+ \frac{|a+b|(6(q-p)+(2-p)(3-p)(6-q))]\Gamma(4-\gamma)T^\alpha}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\bigg)\bigg].
\end{aligned}
$$

In a similar way, we can obtain

$$
\begin{aligned}
&\big|(Tx)'(t)\big| \\
&\quad\le \Bigg[\Bigg(n_1+n_2+n_3+n_4+n_5\gamma_0\bigg(e_{11}+e_{21}+e_{31}+e_{41} \\
&\quad+ e_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+ n_6\lambda_0\left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\right) \\
&+ \sum_{i=1}^{m} k_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg)r + N + n_5\gamma_0\kappa_1 + n_6\lambda_0\kappa_2\Bigg] \\
&\times \left[\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)} + \frac{(3-p)\Gamma(3-q)T^{\alpha-1}}{(2-p)\Gamma(\alpha-q+1)}\right. \\
&\left. + \frac{[2(q-p)+(2-p)(3-p)(5-q)]\Gamma(4-\gamma)T^{\alpha-1}}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\right],
\end{aligned}
$$

$$
\begin{aligned}
&\left|(Tx)''(t)\right| \\
&\quad\le \Bigg[\Bigg(n_1 + n_2 + n_3 + n_4 + n_5\gamma_0\Bigg(e_{11} + e_{21} + e_{31} + e_{41} \\
&\quad + e_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\Bigg) \\
&\quad + n_6\lambda_0\left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\right) \\
&\quad + \sum_{i=1}^{m} k_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg)r + N + n_5\gamma_0\kappa_1 + n_6\lambda_0\kappa_2\Bigg] \\
&\quad \times \left[\frac{T^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-q)T^{\alpha-2}}{\Gamma(\alpha-q+1)} + \frac{(4-q)\Gamma(4-\gamma)T^{\alpha-2}}{(3-q)\Gamma(\alpha-\gamma+1)}\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|(Tx)'''(t)\right| \\
&\quad\le \Bigg[\Bigg(n_1 + n_2 + n_3 + n_4 + n_5\gamma_0\Bigg(e_{11} + e_{21} + e_{31} + e_{41} \\
&\quad + e_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\Bigg) \\
&\quad + n_6\lambda_0\left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\right) \\
&\quad + \sum_{i=1}^{m} k_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg)r \\
&\quad + N + n_5\gamma_0\kappa_1 + n_6\lambda_0\kappa_2\Bigg]\left[\frac{T^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{\Gamma(4-\gamma)T^{\alpha-3}}{\Gamma(\alpha-\gamma+1)}\right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|Tx\| \le \Bigg[\Bigg(& n_1 + n_2 + n_3 + n_4 + n_5\gamma_0\Bigg(e_{11} + e_{21} + e_{31} + e_{41} \\
& + e_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\Bigg)
\end{aligned}
$$

$$+ n_6\lambda_0\left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\right.$$
$$\left. + \sum_{i=1}^{m} k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k_j' \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k_k'' \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\right) r$$
$$\left. + N + n_5\gamma_0\kappa_1 + n_6\lambda_0\kappa_2\right] M_1 \le r$$

and so $\|Tx\| \le r$. Also, we have

$$\begin{aligned}
&\bigl|(Tu)(t) - (Tv)(t)\bigr| \\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl[\bigl|\widetilde{f}\bigl(s,u(s)\bigr) - \widetilde{f}\bigl(s,v(s)\bigr)\bigr|\bigr]\,ds \\
&\qquad + \frac{|b|}{|a+b|\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\bigl[\bigl|\widetilde{f}\bigl(s,u(s)\bigr) - \widetilde{f}\bigl(s,v(s)\bigr)\bigr|\bigr]\,ds \\
&\qquad + \frac{|bT-(a+b)t|\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)T^{1-p}}\int_0^T (T-s)^{\alpha-p-1}\bigl[\bigl|\widetilde{f}\bigl(s,u(s)\bigr) - \widetilde{f}\bigl(s,v(s)\bigr)\bigr|\bigr]\,ds \\
&\qquad + \frac{|bpT^2-(a+b)(2Tt-(2-p)t^2)|\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)T^{2-q}} \\
&\qquad \times \int_0^T (T-s)^{\alpha-q-1}\bigl[\bigl|\widetilde{f}\bigl(s,u(s)\bigr) - \widetilde{f}\bigl(s,v(s)\bigr)\bigr|\bigr]\,ds \\
&\qquad + \left(\frac{|b(-6(q-p)+(2-p)(3-p)q)T^3}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}}\right. \\
&\qquad \left. + \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))|\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}}\right) \\
&\qquad \times \int_0^T (T-s)^{\alpha-\gamma-1}\bigl[\bigl|\widetilde{f}\bigl(s,u(s)\bigr) - \widetilde{f}\bigl(s,v(s)\bigr)\bigr|\bigr]\,ds \\
&\quad\le \left[n_1 + n_2 + n_3 + n_4 + n_5\gamma_0\left(e_{11} + e_{21} + e_{31} + e_{41}\right.\right. \\
&\qquad \left. + e_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\right) \\
&\qquad + n_6\lambda_0\left(e_{12} + e_{22} + e_{32} + e_{42} + e_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\right) \\
&\qquad \left. + \sum_{i=1}^{m} k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k_j' \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k_k'' \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\right] \\
&\qquad \times \left[\frac{(|a|+2|b|)T^\alpha}{|a+b|\Gamma(\alpha+1)} + \frac{(|a|+2|b|)\Gamma(2-p)T^\alpha}{|a+b|\Gamma(\alpha-p+1)} + \frac{(|b|p+|a+b|(4-p))\Gamma(3-q)T^\alpha}{2|a+b|(2-p)\Gamma(\alpha-q+1)}\right. \\
&\qquad + \left(\frac{[|b|(6(q-p)+(2-p)(3-p)q)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\right. \\
&\qquad \left.\left. + \frac{|a+b|(6(q-p)+(2-p)(3-p)(6-q))]\Gamma(4-\gamma)T^\alpha}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\right)\right]\|u-v\|
\end{aligned}$$

for all $u, v \in X$ and $t \in I$. Similarly, one can obtain

$$\begin{aligned}
&\big|(Tu)'(t) - (Tv)'(t)\big| \\
&\quad\le \Bigg[n_1 + n_2 + n_3 + n_4 + n_5\gamma_0\bigg(e_{11} + e_{21} + e_{31} + e_{41} \\
&\qquad + e_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg) \\
&\qquad + n_6\lambda_0\bigg(e_{12} + e_{22} + e_{32} + e_{42} + e_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\bigg) \\
&\qquad + \sum_{i=1}^{m} k_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg] \\
&\qquad \times \bigg[\frac{T^{\alpha-1}}{\Gamma(\alpha)} + \frac{\Gamma(2-p)T^{\alpha-1}}{\Gamma(\alpha-p+1)} + \frac{(3-p)\Gamma(3-q)T^{\alpha-1}}{(2-p)\Gamma(\alpha-q+1)} \\
&\qquad + \frac{[2(q-p) + (2-p)(3-p)(5-q)]\Gamma(4-\gamma)T^{\alpha-1}}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma+1)}\bigg]\|u - v\|, \\
&\big|(Tu)''(t) - (Tv)''(t)\big| \\
&\quad\le \Bigg[n_1 + n_2 + n_3 + n_4 + n_5\gamma_0\bigg(e_{11} + e_{21} + e_{31} + e_{41} \\
&\qquad + e_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg) \\
&\qquad + n_6\lambda_0\bigg(e_{12} + e_{22} + e_{32} + e_{42} + e_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\bigg) \\
&\qquad + \sum_{i=1}^{m} k_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg] \\
&\qquad \times \bigg[\frac{T^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(3-q)T^{\alpha-2}}{\Gamma(\alpha-q+1)} + \frac{(4-q)\Gamma(4-\gamma)T^{\alpha-2}}{(3-q)\Gamma(\alpha-\gamma+1)}\bigg]\|u - v\|,
\end{aligned}$$

and

$$\begin{aligned}
&\big|(Tu)'''(t) - (Tv)'''(t)\big| \\
&\quad\le \Bigg[n_1 + n_2 + n_3 + n_4 + n_5\gamma_0\bigg(e_{11} + e_{21} + e_{31} + e_{41} \\
&\qquad + e_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg) \\
&\qquad + n_6\lambda_0\bigg(e_{12} + e_{22} + e_{32} + e_{42} + e_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\bigg) \\
&\qquad + \sum_{i=1}^{m} k_i\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg] \\
&\qquad \times \bigg[\frac{T^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{\Gamma(4-\gamma)T^{\alpha-3}}{\Gamma(\alpha-\gamma+1)}\bigg]
\end{aligned}$$

for all $u, v \in X$ and $t \in I$. This implies that

$$\begin{aligned}
&\|Tu - Tv\| \\
&\le M_1 \Bigg[n_1 + n_2 + n_3 + n_4 + n_5\gamma_0 \bigg(e_{11} + e_{21} + e_{31} + e_{41} \\
&\quad + e_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + e_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + e_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \bigg) \\
&\quad + n_6\lambda_0 \bigg(e_{12} + e_{22} + e_{32} + e_{42} + e_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + e_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + e_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \bigg) \\
&\quad + \sum_{i=1}^{m} k_i \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} k'_j \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} k''_k \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \Bigg] \|u - v\| = \Delta \|u - v\|.
\end{aligned}$$

Since $\Delta < 1$, the operator T is a contraction. In consequence, by the Banach contraction principle, T has a unique fixed point which corresponds to the unique solution of problem (1.1)-(1.3). □

4 Existence results for problem (1.2)-(1.3)

This section is concerned with the existence of solutions for problem (1.2)-(1.3). As before, let the space $X = \{u : u \in C^3(I)\}$ be endowed with the norm

$$\|u\| = \sup_{t\in I} |u(t)| + \sup_{t\in I} |u'(t)| + \sup_{t\in I} |u''(t)| + \sup_{t\in I} |u'''(t)|.$$

A multivalued map $F : I \times \mathbb{R}^{6+m+m'+m''} \to P(\mathbb{R})$ is said to be Carathéodory whenever the map

$$t \longmapsto F(t, x_1, x_2, \ldots, x_{6+m+m'+m''})$$

is measurable for all $x_i \in \mathbb{R}$ for $1 \le i \le 6 + m + m' + m''$ and the map

$$(x_1, x_2, \ldots, x_{6+m+m'+m''}) \longmapsto F(t, x_1, x_2, \ldots, x_{6+m+m'+m''})$$

is upper semi-continuous for almost all $t \in I$. Also, a Carathéodory function F is said to be L^1-Carathéodory whenever for each $l > 0$ there exists $\psi_l \in L^1(I, \mathbb{R}^+)$ such that

$$\big\|F(t, x_1, x_2, \ldots, x_{6+m+m'+m''})\big\|_p = \sup\big\{|v| : v \in F(t, x_1, x_2, \ldots, x_{6+m+m'+m''})\big\} \le \psi_l(t)$$

for all $|x_i| \le l$ ($1 \le i \le 6 + m + m' + m''$) and almost all $t \in I$ [35, 37].

Lemma 4.1 ([41]) *Let E be a Banach space, $G : I \times E \to P_{cp,c}(E)$ an L^1-Carathéodory multifunction, and θ a linear continuous mapping from $L^1(I, E)$ to $C(I, E)$. Then the operator*

$$\theta \circ S_G : C(I, E) \to P_{cp,c}\big(C(I, E)\big)$$

defined by $\theta \circ S_G(x) = \theta(S_{G,x})$ is a closed graph operator, where

$$S_{G,x} = \big\{ w \in L^1(I, E) : w(t) \in G\big(t, x(t)\big) \text{ for almost all } t \in I \big\}.$$

Let E be a Banach space. The multivalued map $G : I \times E \to P_{cp}(E)$ is said to be lower semi-continuous (l.s.c) type whenever $S_G : C(I,E) \to P(L^1(I,E))$ is lower semi-continuous and has non-empty closed and decomposable values [42].

Now we state some well-known results which are needed in the sequel.

Lemma 4.2 ([42]) *Let Y be a separable metric space and $N : Y \to P(L^1(I,\mathbb{R}))$ a lower semi-continuous multivalued map with closed decomposable values. Then N has a continuous selection, that is, there exists a continuous mapping $g : Y \to L^1(I,\mathbb{R})$ such that $g(y) \in N(y)$ for all $y \in Y$.*

Theorem 4.3 (Nonlinear alternative of Leray-Schauder type [43]) *Let E be a Banach space, C a closed and convex subset of E, and U an open subset of C such that $0 \in U$. If $F : \overline{U} \to P_{cp,c}(C)$ is an upper semi-continuous compact map, then either F has a fixed point in $\overline{U}$ or there is a $x \in \partial U$ and $\lambda \in (0,1)$ such that $x \in \lambda F(x)$.*

Theorem 4.4 (Covitz and Nadler [44]) *Let (M,d) be a complete metric space. If $F : M \to P_{cl}(M)$ is a contraction, then F has a fixed point.*

For $t \in I$ and $x \in X$, let the multifunction $\widetilde{F}$ be defined by

$$\begin{aligned}\widetilde{F}\big(t,x(t)\big) = F\big(&t,x(t),x'(t),x''(t),x'''(t),\phi x(t),\psi x(t),{}^cD^{\mu_1}x(t),{}^cD^{\mu_2}x(t),\dots,{}^cD^{\mu_m}x(t),\\ &{}^cD^{\nu_1}x(t),{}^cD^{\nu_2}x(t),\dots,{}^cD^{\nu_{m'}}x(t),{}^cD^{\xi_1}x(t),{}^cD^{\xi_2}x(t),\dots,{}^cD^{\xi_{m''}}x(t)\big)\end{aligned}$$

and the set of selections of F by $S_{F,x} = \{v \in L^1(I,\mathbb{R}) : v(t) \in \widetilde{F}(t,x(t))$ for almost all $t \in I\}$. For the sake of brevity, we set

$$\begin{aligned}M_2 = {}&\bigg[\frac{|a|+2|b|}{|a+b|\Gamma(\alpha)} + \frac{(|a|+2|b|)\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)} + \frac{(|b|p+|a+b|(4-p))\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)}\\ &+ \bigg(\frac{[|b|(6(q-p)+(2-p)(3-p)q)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)}\\ &+ \frac{|a+b|(6(q-p)+(2-p)(3-p)(6-q))]\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)}\bigg)\bigg]T^{\alpha-1}\\ &+ \bigg[\frac{1}{\Gamma(\alpha-1)} + \frac{\Gamma(2-p)}{\Gamma(\alpha-p)} + \frac{(3-p)\Gamma(3-q)}{(2-p)\Gamma(\alpha-q)}\\ &+ \frac{[2(q-p)+(2-p)(3-p)(5-q)]\Gamma(4-\gamma)}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)}\bigg]T^{\alpha-2}\\ &+ \bigg[\frac{1}{\Gamma(\alpha-2)} + \frac{\Gamma(3-q)}{\Gamma(\alpha-q)} + \frac{(4-q)\Gamma(4-\gamma)}{(3-q)\Gamma(\alpha-\gamma)}\bigg]T^{\alpha-3}\\ &+ \bigg[\frac{1}{\Gamma(\alpha-3)} + \frac{\Gamma(4-\gamma)}{\Gamma(\alpha-\gamma)}\bigg]T^{\alpha-4}.\end{aligned}$$

Now, we are in a position to give our first existence result for problem (1.2)-(1.3).

Theorem 4.5 *Suppose that $F : I \times \mathbb{R}^{6+m+m'+m''} \to P_{cp,c}(\mathbb{R})$ is a Carathéodory multivalued map and there exist continuous nondecreasing functions $\varphi_i : [0,\infty) \to (0,\infty)$ for $1 \le i \le 6$, $\psi_i, \psi'_j, \psi''_k : [0,\infty) \to (0,\infty)$ for $1 \le i \le m$, $1 \le j \le m'$, and $1 \le k \le m''$ and nonnegative* *functions* $q_i \in L^1(I)$ *for* $1 \le i \le 6$, $\rho_i, \rho'_j, \rho''_k \in L^1(I)$ *for* $1 \le i \le m$, $1 \le j \le m'$ *and* $1 \le k \le m''$ *such that*

$$\begin{aligned}
&\bigl\|F(t,x_1,x_2,x_3,x_4,x_5,x_6,y_1,y_2,\ldots,y_m,z_1,z_2,\ldots,z_{m'},w_1,w_2,\ldots,w_{m''})\bigr\|_p \\
&\quad= \sup\bigl\{|y| : y \in F(t,x_1,x_2,x_3,x_4,x_5,x_6,y_1,y_2,\ldots,y_m,z_1,z_2,\ldots,z_{m'},w_1,w_2,\ldots,w_{m''})\bigr\} \\
&\quad\le q_1(t)\varphi_1\bigl(|x_1|\bigr) + q_2(t)\varphi_2\bigl(|x_2|\bigr) + q_3(t)\varphi_3\bigl(|x_3|\bigr) + q_4(t)\varphi_4\bigl(|x_4|\bigr) \\
&\qquad + q_5(t)\varphi_5\bigl(|x_5|\bigr) + q_6(t)\varphi_6\bigl(|x_6|\bigr) \\
&\qquad + \sum_{i=1}^{m} \rho_i(t)\psi_i\bigl(|y_i|\bigr) + \sum_{j=1}^{m'} \rho'_j(t)\psi'_j\bigl(|z_j|\bigr) + \sum_{k=1}^{m''} \rho''_k(t)\psi''_k\bigl(|w_k|\bigr),
\end{aligned}$$

for all $t \in I$, $x_i, y_i, z_j, w_k \in \mathbb{R}$ *for* $1 \le i \le 6$, $1 \le i \le m$, $1 \le j \le m'$, *and* $1 \le k \le m''$. *Assume that there exist positive constants* $c_{01}, c_{02} > 0$ *and* $l_{i1}, l_{i2} \ge 0$ *for* $1 \le i \le 7$ *such that*

$$\bigl|h_j(t,s,u_1,u_2,u_3,u_4,u_5,u_6,u_7)\bigr| \le c_{0j} + \sum_{i=1}^{7} l_{ij}|u_i|$$

for $j = 1, 2$, *all* $t, s \in I$, *and all* $u_i \in \mathbb{R}$ *for* $1 \le i \le 7$. *If there exists a constant* $D > 0$ *such that* $\frac{D}{M_2 A(D)} > 1$, *then problem* (1.2)-(1.3) *has at least one solution, where*

$$\begin{aligned}
A(D) = {}& \|q_1\|_1\varphi_1(D) + \|q_2\|_1\varphi_2(D) + \|q_3\|_1\varphi_3(D) + \|q_4\|_1\varphi_4(D) \\
&+ \|q_5\|_1\varphi_5\biggl(c_{01}\gamma_0 + \gamma_0\biggl[l_{11} + l_{21} + l_{31} + l_{41} + l_{51}\frac{T^{1-\gamma_1}}{\Gamma(2-\gamma_1)} \\
&+ l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\biggr]D\biggr) \\
&+ \|q_6\|_1\varphi_6\biggl(c_{02}\lambda_0 + \lambda_0\biggl[l_{12} + l_{22} + l_{32} + l_{42} + l_{52}\frac{T^{1-\gamma_2}}{\Gamma(2-\gamma_2)} \\
&+ l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\biggr]D\biggr) \\
&+ \sum_{i=1}^{m}\|\rho_i\|_1\psi_i\biggl(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}D\biggr) + \sum_{j=1}^{m'}\bigl\|\rho'_j\bigr\|_1\psi'_j\biggl(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}D\biggr) \\
&+ \sum_{k=1}^{m''}\bigl\|\rho''_k\bigr\|_1\psi''_k\biggl(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}D\biggr).
\end{aligned}$$

Proof Let $x \in X$. Observe that the first property of the multifunction F and Theorem 1.3.5 in [45] imply that $S_{F,x}$ is non-empty. Define an operator $\Omega : X \to P(X)$ by

$$\Omega(x) = \bigl\{g \in X : \text{ there exists } f \in S_{F,x} \text{ such that } g(t) = \gamma_f(t) \text{ for all } t \in I\bigr\},$$

where

$$\begin{aligned}
\gamma_f(t) = {}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}f(s)\,ds \\
&+ \frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}}\int_0^T (T-s)^{\alpha-p-1}f(s)\,ds
\end{aligned}$$

$$- \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f(s)\,ds$$
$$- \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right.$$
$$\left. + \frac{(a+b)(6(q-p)T^2 t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right)$$
$$\times \int_0^T (T-s)^{\alpha-\gamma-1} f(s)\,ds.$$

We show that the operator Ω satisfies the hypothesis of the nonlinear alternative of the Leray-Schauder type result (Theorem 4.3). First, we show that $\Omega(x)$ is convex for all $x \in X$. Let $g_1, g_2 \in \Omega(x)$ and $w \in [0,1]$. Choose $f_1, f_2 \in S_{F,x}$ such that

$$g_i(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_i(s)\,ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f_i(s)\,ds$$
$$+ \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f_i(s)\,ds$$
$$- \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f_i(s)\,ds$$
$$- \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right.$$
$$\left. + \frac{(a+b)(6(q-p)T^2 t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right)$$
$$\times \int_0^T (T-s)^{\alpha-\gamma-1} f_i(s)\,ds$$

for all $t \in I$. Then we have

$$\big[wg_1 + (1-w)g_2\big](t)$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big[wf_1(s) + (1-w)f_2(s)\big]\,ds$$
$$- \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \big[wf_1(s) + (1-w)f_2(s)\big]\,ds$$
$$+ \frac{[bT - (a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} \big[wf_1(s) + (1-w)f_2(s)\big]\,ds$$
$$- \frac{[bpT^2 - (a+b)(2Tt - (2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}}$$
$$\times \int_0^T (T-s)^{\alpha-q-1} \big[wf_1(s) + (1-w)f_2(s)\big]\,ds$$
$$- \left(\frac{[b(-6(q-p) + (2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right.$$
$$\left. + \frac{(a+b)(6(q-p)T^2 t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right)$$
$$\times \int_0^T (T-s)^{\alpha-\gamma-1} \big[wf_1(s) + (1-w)f_2(s)\big]\,ds$$

for all $t \in I$. Since F has convex values, it is easy to check that $S_{F,x}$ is convex and so $wg_1 + (1-w)g_2 \in \Omega(x)$. Now, we show that Ω maps bounded sets into bounded sets in X. Let $B_r = \{x \in X : \|x\| \leq r\}$, $x \in B_r$, and $g \in \Omega(x)$. Choose $f \in S_{F,x}$ such that

$$\begin{aligned}
\bigl|g(t)\bigr| \leq{}& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\bigl|f(s)\bigr|\,ds + \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1}\bigl|f(s)\bigr|\,ds \\
&+ \frac{|bT-(a+b)t|\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1}\bigl|f(s)\bigr|\,ds \\
&+ \frac{|bpT^2-(a+b)(2Tt-(2-p)t^2)|\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1}\bigl|f(s)\bigr|\,ds \\
&+ \biggl(\frac{|b(-6(q-p)+(2-p)(3-p)q)T^3}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
&+ \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))|\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \biggr) \\
&\times \int_0^T (T-s)^{\alpha-\gamma-1}\bigl|f(s)\bigr|\,ds \\
\leq{}& \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \biggl[q_1(s)\varphi_1\bigl(\bigl|x(s)\bigr|\bigr) + q_2(s)\varphi_2\bigl(\bigl|x'(s)\bigr|\bigr) \\
&+ q_3(s)\varphi_3\bigl(\bigl|x''(s)\bigr|\bigr) + q_4(s)\varphi_4\bigl(\bigl|x'''(s)\bigr|\bigr) \\
&+ q_5(s)\varphi_5\bigl(\bigl|\phi x(s)\bigr|\bigr) + q_6(s)\varphi_6\bigl(\bigl|\psi x(s)\bigr|\bigr) + \sum_{i=1}^{m} \rho_i(s)\psi_i\bigl(\bigl|{}^cD^{\mu_i}x(s)\bigr|\bigr) \\
&+ \sum_{j=1}^{m'} \rho'_j(s)\psi'_j\bigl(\bigl|{}^cD^{\nu_j}x(s)\bigr|\bigr) + \sum_{k=1}^{m''} \rho''_k(s)\psi''_k\bigl(\bigl|{}^cD^{\xi_k}x(s)\bigr|\bigr) \biggr]\,ds \\
&+ \frac{|b|}{|a+b|\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} \biggl[q_1(s)\varphi_1\bigl(\bigl|x(s)\bigr|\bigr) + q_2(s)\varphi_2\bigl(\bigl|x'(s)\bigr|\bigr) \\
&+ q_3(s)\varphi_3\bigl(\bigl|x''(s)\bigr|\bigr) + q_4(s)\varphi_4\bigl(\bigl|x'''(s)\bigr|\bigr) + q_5(s)\varphi_5\bigl(\bigl|\phi x(s)\bigr|\bigr) \\
&+ q_6(s)\varphi_6\bigl(\bigl|\psi x(s)\bigr|\bigr) + \sum_{i=1}^{m} \rho_i(s)\psi_i\bigl(\bigl|{}^cD^{\mu_i}x(s)\bigr|\bigr) \\
&+ \sum_{j=1}^{m'} \rho'_j(s)\psi'_j\bigl(\bigl|{}^cD^{\nu_j}x(s)\bigr|\bigr) + \sum_{k=1}^{m''} \rho''_k(s)\psi''_k\bigl(\bigl|{}^cD^{\xi_k}x(s)\bigr|\bigr) \biggr]\,ds \\
&+ \frac{|bT-(a+b)t|\Gamma(2-p)}{|a+b|\Gamma(\alpha-p)T^{1-p}} \\
&\times \int_0^T (T-s)^{\alpha-p-1} \biggl[q_1(s)\varphi_1\bigl(\bigl|x(s)\bigr|\bigr) + q_2(s)\varphi_2\bigl(\bigl|x'(s)\bigr|\bigr) \\
&+ q_3(s)\varphi_3\bigl(\bigl|x''(s)\bigr|\bigr) + q_4(s)\varphi_4\bigl(\bigl|x'''(s)\bigr|\bigr) \\
&+ q_5(s)\varphi_5\bigl(\bigl|\phi x(s)\bigr|\bigr) + q_6(s)\varphi_6\bigl(\bigl|\psi x(s)\bigr|\bigr) + \sum_{i=1}^{m} \rho_i(s)\psi_i\bigl(\bigl|{}^cD^{\mu_i}x(s)\bigr|\bigr) \\
&+ \sum_{j=1}^{m'} \rho'_j(s)\psi'_j\bigl(\bigl|{}^cD^{\nu_j}x(s)\bigr|\bigr) + \sum_{k=1}^{m''} \rho''_k(s)\psi''_k\bigl(\bigl|{}^cD^{\xi_k}x(s)\bigr|\bigr) \biggr]\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{|bpT^2 - (a+b)(2Tt - (2-p)t^2)|\Gamma(3-q)}{2|a+b|(2-p)\Gamma(\alpha-q)T^{2-q}} \\
&\times \int_0^T (T-s)^{\alpha-q-1}\Bigg[q_1(s)\varphi_1(|x(s)|) + q_2(s)\varphi_2(|x'(s)|) \\
&+ q_3(s)\varphi_3(|x''(s)|) + q_4(s)\varphi_4(|x'''(s)|) \\
&+ q_5(s)\varphi_5(|\phi x(s)|) + q_6(s)\varphi_6(|\psi x(s)|) + \sum_{i=1}^{m}\rho_i(s)\psi_i(|{}^cD^{\mu_i}x(s)|) \\
&+ \sum_{j=1}^{m'}\rho_j'(s)\psi_j'(|{}^cD^{\nu_j}x(s)|) + \sum_{k=1}^{m''}\rho_k''(s)\psi_k''(|{}^cD^{\xi_k}x(s)|)\Bigg] ds \\
&+ \bigg(\frac{|b(-6(q-p)+(2-p)(3-p)q)T^3}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
&+ \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))|\Gamma(4-\gamma)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \bigg) \\
&\times \int_0^T (T-s)^{\alpha-\gamma-1}\Bigg[q_1(s)\varphi_1(|x(s)|) + q_2(s)\varphi_2(|x'(s)|) \\
&+ q_3(s)\varphi_3(|x''(s)|) + q_4(s)\varphi_4(|x'''(s)|) \\
&+ q_5(s)\varphi_5(|\phi x(s)|) + q_6(s)\varphi_6(|\psi x(s)|) + \sum_{i=1}^{m}\rho_i(s)\psi_i(|{}^cD^{\mu_i}x(s)|) \\
&+ \sum_{j=1}^{m'}\rho_j'(s)\psi_j'(|{}^cD^{\nu_j}x(s)|) + \sum_{k=1}^{m''}\rho_k''(s)\psi_k''(|{}^cD^{\xi_k}x(s)|)\Bigg] ds \\
\le{}& \Bigg[\|q_1\|_1\varphi_1(r) + \|q_2\|_1\varphi_2(r) + \|q_3\|_1\varphi_3(r) + \|q_4\|_1\varphi_4(r) \\
&+ \|q_5\|_1\varphi_5\bigg(c_{01}\gamma_0 + \gamma_0\bigg[l_{11} + l_{21} + l_{31} + l_{41} + l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \\
&+ l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg] r\bigg) \\
&+ \|q_6\|_1\varphi_6\bigg(c_{02}\lambda_0 + \lambda_0\bigg[l_{12} + l_{22} + l_{32} + l_{42} + l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \\
&+ l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\bigg] r\bigg) \\
&+ \sum_{i=1}^{m}\|\rho_i\|_1\psi_i\bigg(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}r\bigg) + \sum_{j=1}^{m'}\|\rho_j'\|_1\psi_j'\bigg(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}r\bigg) \\
&+ \sum_{k=1}^{m''}\|\rho_k''\|_1\psi_k''\bigg(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}r\bigg)\Bigg] \\
&\times \bigg[\frac{(|a|+2|b|)T^{\alpha-1}}{|a+b|\Gamma(\alpha)} + \frac{(|a|+2|b|)\Gamma(2-p)T^{\alpha-1}}{|a+b|\Gamma(\alpha-p)} \\
&+ \frac{(|b|p + |a+b|(4-p))\Gamma(3-q)T^{\alpha-1}}{2|a+b|(2-p)\Gamma(\alpha-q)}
\end{aligned}
$$

$$+\left(\frac{[|b|(6(q-p)+(2-p)(3-p)q)}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)}\right.$$
$$\left.\left.+\frac{|a+b|(6(q-p)+(2-p)(3-p)(6-q))]\Gamma(4-\gamma)T^{\alpha-1}}{6|a+b|(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)}\right)\right]$$

for all $t \in I$. In a similar manner, we obtain

$$\begin{aligned}
\left|g'(t)\right| \le \Bigg[& \|q_1\|_1\varphi_1(r) + \|q_2\|_1\varphi_2(r) + \|q_3\|_1\varphi_3(r) + \|q_4\|_1\varphi_4(r) \\
& + \|q_5\|_1\varphi_5\bigg(c_{01}\gamma_0 + \gamma_0\bigg[l_{11} + l_{21} + l_{31} + l_{41} + l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \\
& + l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg]r\bigg) \\
& + \|q_6\|_1\varphi_6\bigg(c_{02}\lambda_0 + \lambda_0\bigg[l_{12} + l_{22} + l_{32} + l_{42} + l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \\
& + l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\bigg]r\bigg) \\
& + \sum_{i=1}^{m}\|\rho_i\|_1\psi_i\bigg(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}r\bigg) + \sum_{j=1}^{m'}\big\|\rho_j'\big\|_1\psi_j'\bigg(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}r\bigg) \\
& + \sum_{k=1}^{m''}\big\|\rho_k''\big\|_1\psi_k''\bigg(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}r\bigg)\Bigg] \\
& \times\bigg[\frac{T^{\alpha-2}}{\Gamma(\alpha-1)} + \frac{\Gamma(2-p)T^{\alpha-2}}{\Gamma(\alpha-p)} + \frac{(3-p)\Gamma(3-q)T^{\alpha-2}}{(2-p)\Gamma(\alpha-q)} \\
& + \frac{[2(q-p)+(2-p)(3-p)(5-q)]\Gamma(4-\gamma)T^{\alpha-2}}{2(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)}\bigg], \\
\left|g''(t)\right| \le \Bigg[& \|q_1\|_1\varphi_1(r) + \|q_2\|_1\varphi_2(r) + \|q_3\|_1\varphi_3(r) + \|q_4\|_1\varphi_4(r) \\
& + \|q_5\|_1\varphi_5\bigg(c_{01}\gamma_0 + \gamma_0\bigg[l_{11} + l_{21} + l_{31} + l_{41} + l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \\
& + l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg]r\bigg) \\
& + \|q_6\|_1\varphi_6\bigg(c_{02}\lambda_0 + \lambda_0\bigg[l_{12} + l_{22} + l_{32} + l_{42} + l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \\
& + l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\bigg]r\bigg) \\
& + \sum_{i=1}^{m}\|\rho_i\|_1\psi_i\bigg(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}r\bigg) + \sum_{j=1}^{m'}\big\|\rho_j'\big\|_1\psi_j'\bigg(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}r\bigg) \\
& + \sum_{k=1}^{m''}\big\|\rho_k''\big\|_1\psi_k''\bigg(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}r\bigg)\Bigg] \\
& \times\bigg[\frac{T^{\alpha-3}}{\Gamma(\alpha-2)} + \frac{\Gamma(3-q)T^{\alpha-3}}{\Gamma(\alpha-q)} + \frac{(4-q)\Gamma(4-\gamma)T^{\alpha-3}}{(3-q)\Gamma(\alpha-\gamma)}\bigg],
\end{aligned}$$

and

$$\begin{aligned}
\left|g'''(t)\right| \le{}& \Bigg[\|q_1\|_1\varphi_1(r) + \|q_2\|_1\varphi_2(r) + \|q_3\|_1\varphi_3(r) + \|q_4\|_1\varphi_4(r) \\
&+ \|q_5\|_1\varphi_5\Bigg(c_{01}\gamma_0 + \gamma_0\Bigg[l_{11} + l_{21} + l_{31} + l_{41} + l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \\
&+ l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\Bigg]r\Bigg) \\
&+ \|q_6\|_1\varphi_6\Bigg(c_{02}\lambda_0 + \lambda_0\Bigg[l_{12} + l_{22} + l_{32} + l_{42} + l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \\
&+ l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\Bigg]r\Bigg) \\
&+ \sum_{i=1}^{m}\|\rho_i\|_1\psi_i\left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}r\right) + \sum_{j=1}^{m'}\left\|\rho_j'\right\|_1\psi_j'\left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}r\right) \\
&+ \sum_{k=1}^{m''}\left\|\rho_k''\right\|_1\psi_k''\left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}r\right)\Bigg]\left[\frac{T^{\alpha-4}}{\Gamma(\alpha-3)} + \frac{\Gamma(4-\gamma)T^{\alpha-4}}{\Gamma(\alpha-\gamma)}\right]
\end{aligned}$$

for all $t \in I$. Thus, we get

$$\begin{aligned}
\|g\| \le M_2\Bigg[&\|q_1\|_1\varphi_1(r) + \|q_2\|_1\varphi_2(r) + \|q_3\|_1\varphi_3(r) + \|q_4\|_1\varphi_4(r) \\
&+ \|q_5\|_1\varphi_5\Bigg(c_{01}\gamma_0 + \gamma_0\Bigg[l_{11} + l_{21} + l_{31} + l_{41} + l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \\
&+ l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\Bigg]r\Bigg) \\
&+ \|q_6\|_1\varphi_6\Bigg(c_{02}\lambda_0 + \lambda_0\Bigg[l_{12} + l_{22} + l_{32} + l_{42} + l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \\
&+ l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\Bigg]r\Bigg) \\
&+ \sum_{i=1}^{m}\|\rho_i\|_1\psi_i\left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}r\right) + \sum_{j=1}^{m'}\left\|\rho_j'\right\|_1\psi_j'\left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}r\right) \\
&+ \sum_{k=1}^{m''}\left\|\rho_k''\right\|_1\psi_k''\left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}r\right)\Bigg].
\end{aligned}$$

This implies that Ω maps bounded sets into bounded sets in X. Now, we show that Ω maps bounded sets of X into equicontinuous sets. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, $x \in B_r$, and $g \in \Omega(x)$. Then we have

$$\begin{aligned}
&\left|g(t_2) - g(t_1)\right| \\
&\quad\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right]\left|f(s)\right|ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|f(s)\right|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(t_2-t_1)\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}}\int_0^T (T-s)^{\alpha-p-1}\bigl|f(s)\bigr|\,ds\\
&+\frac{[2T(t_2-t_1)+(2-p)(t_2^2-t_1^2)]\Gamma(3-q)}{2(2-p)\Gamma(\alpha-q)T^{2-q}}\int_0^T (T-s)^{\alpha-q-1}\bigl|f(s)\bigr|\,ds\\
&+\frac{[6(q-p)T^2(t_2-t_1)+(2-p)(3-p)(3T(t_2^2-t_1^2)+(3-q)(t_2^3-t_1^3))]\Gamma(4-\gamma)}{6(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}}\\
&\times\int_0^T (T-s)^{\alpha-\gamma-1}\bigl|f(s)\bigr|\,ds\\
\le{}&\Biggl[\varphi_1(r)+\varphi_2(r)+\varphi_3(r)+\varphi_4(r)\\
&+\varphi_5\biggl(c_{01}\gamma_0+\gamma_0\biggl[l_{11}+l_{21}+l_{31}+l_{41}+l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)}\\
&+l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}+l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\biggr]r\biggr)\\
&+\varphi_6\biggl(c_{02}\lambda_0+\lambda_0\biggl[l_{12}+l_{22}+l_{32}+l_{42}+l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)}\\
&+l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}+l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\biggr]r\biggr)\\
&+\sum_{i=1}^{m}\psi_i\biggl(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}r\biggr)+\sum_{j=1}^{m'}\psi_j'\biggl(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}r\biggr)+\sum_{k=1}^{m''}\psi_k''\biggl(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}r\biggr)\Biggr]\\
&\times\Biggl[\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\bigl[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\bigr]\\
&\times\Biggl[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\Biggr]\,ds\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\Biggl[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\Biggr]\,ds\\
&+\frac{(t_2-t_1)\Gamma(2-p)}{\Gamma(\alpha-p)T^{1-p}}\int_0^T (T-s)^{\alpha-p-1}\\
&\times\Biggl[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\Biggr]\,ds\\
&+\frac{[2T(t_2-t_1)+(2-p)(t_2^2-t_1^2)]\Gamma(3-q)}{2(2-p)\Gamma(\alpha-q)T^{2-q}}\\
&\times\int_0^T (T-s)^{\alpha-q-1}\Biggl[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\Biggr]\,ds\\
&+\frac{[6(q-p)T^2(t_2-t_1)+(2-p)(3-p)(3T(t_2^2-t_1^2)+(3-q)(t_2^3-t_1^3))]\Gamma(4-\gamma)}{6(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}}\\
&\times\int_0^T (T-s)^{\alpha-\gamma-1}\Biggl[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\Biggr]\,ds\Biggr].
\end{aligned}
$$

Proceeding as before, one can obtain

$$\begin{aligned}
&\left|g'(t_2)-g'(t_1)\right| \\
&\quad\le\Bigg[\varphi_1(r)+\varphi_2(r)+\varphi_3(r)+\varphi_4(r) \\
&\qquad+\varphi_5\biggl(c_{01}\gamma_0+\gamma_0\biggl[l_{11}+l_{21}+l_{31}+l_{41}+l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \\
&\qquad+l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}+l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\biggr]r\biggr) \\
&\qquad+\varphi_6\biggl(c_{02}\lambda_0+\lambda_0\biggl[l_{12}+l_{22}+l_{32}+l_{42}+l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \\
&\qquad+l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}+l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\biggr]r\biggr) \\
&\qquad+\sum_{i=1}^{m}\psi_i\biggl(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}r\biggr)+\sum_{j=1}^{m'}\psi_j'\biggl(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}r\biggr)+\sum_{k=1}^{m''}\psi_k''\biggl(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}r\biggr)\Bigg] \\
&\qquad\times\Bigg[\frac{1}{\Gamma(\alpha-1)}\int_0^{t_1}\bigl[(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}\bigr] \\
&\qquad\times\Biggl[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\Biggr]ds \\
&\qquad+\frac{1}{\Gamma(\alpha-1)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}\Biggl[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\Biggr]ds \\
&\qquad+\frac{(t_2-t_1)\Gamma(3-q)}{\Gamma(\alpha-q)T^{2-q}}\int_0^{T}(T-s)^{\alpha-q-1} \\
&\qquad\times\Biggl[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\Biggr]ds \\
&\qquad+\frac{[2T(t_2-t_1)+(3-q)(t_2^2-t_1^2)]\Gamma(4-\gamma)}{2(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
&\qquad\times\int_0^{T}(T-s)^{\alpha-\gamma-1}\Biggl[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\Biggr]ds\Bigg],
\end{aligned}$$

$$\begin{aligned}
&\left|g''(t_2)-g''(t_1)\right| \\
&\quad\le\Bigg[\varphi_1(r)+\varphi_2(r)+\varphi_3(r)+\varphi_4(r)+\varphi_5\biggl(c_{01}\gamma_0+\gamma_0\biggl[l_{11}+l_{21}+l_{31}+l_{41}+l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \\
&\qquad+l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}+l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\biggr]r\biggr) \\
&\qquad+\varphi_6\biggl(c_{02}\lambda_0+\lambda_0\biggl[l_{12}+l_{22}+l_{32}+l_{42}+l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} \\
&\qquad+l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}+l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\biggr]r\biggr)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{m}\psi_i\left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}r\right)+\sum_{j=1}^{m'}\psi_j'\left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}r\right)+\sum_{k=1}^{m''}\psi_k''\left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}r\right)\Bigg]\\
&\times\left[\frac{1}{\Gamma(\alpha-2)}\int_0^{t_1}\big[(t_2-s)^{\alpha-3}-(t_1-s)^{\alpha-3}\big]\right.\\
&\times\left[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\right]ds\\
&+\frac{1}{\Gamma(\alpha-2)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-3}\left[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\right]ds\\
&+\frac{(t_2-t_1)\Gamma(4-\gamma)}{\Gamma(\alpha-\gamma)T^{3-\gamma}}\int_0^{T}(T-s)^{\alpha-\gamma-1}\\
&\times\left.\left[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\right]ds\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\big|g'''(t_2)-g'''(t_1)\big|\\
&\quad\le\Bigg[\varphi_1(r)+\varphi_2(r)+\varphi_3(r)+\varphi_4(r)+\varphi_5\Bigg(c_{01}\gamma_0+\gamma_0\Bigg[l_{11}+l_{21}+l_{31}+l_{41}+l_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)}\\
&\qquad+l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}+l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\Bigg]r\Bigg)\\
&\qquad+\varphi_6\Bigg(c_{02}\lambda_0+\lambda_0\Bigg[l_{12}+l_{22}+l_{32}+l_{42}+l_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)}\\
&\qquad+l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}+l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\Bigg]r\Bigg)\\
&\qquad+\sum_{i=1}^{m}\psi_i\left(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}r\right)+\sum_{j=1}^{m'}\psi_j'\left(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}r\right)+\sum_{k=1}^{m''}\psi_k''\left(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}r\right)\Bigg]\\
&\qquad\times\left[\frac{1}{\Gamma(\alpha-3)}\int_0^{t_1}\big[(t_2-s)^{\alpha-4}-(t_1-s)^{\alpha-4}\big]\right.\\
&\qquad\times\left[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\right]ds\\
&\qquad+\left.\frac{1}{\Gamma(\alpha-3)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-4}\left[\sum_{l=1}^{6}q_l(s)+\sum_{i=1}^{m}\rho_i(s)+\sum_{j=1}^{m'}\rho_j'(s)+\sum_{k=1}^{m''}\rho_k''(s)\right]ds\right].
\end{aligned}
$$

It is easy to see that the right-hand side of the above inequalities tends to zero as $t_2-t_1\to 0$ (independent on $x\in B_r$). Thus, by using the Arzela-Ascoli theorem, we see that $\Omega: X\to P(X)$ is a compact multivalued map. Next, we show that Ω has a closed graph. Let $x_n\to x_*$, $g_n\in\Omega(x_n)$ for all n and $g_n\to g_*$. We show that $g_*\in\Omega(x_*)$. Since $g_n\in\Omega(x_n)$ for all n, there exists $f_n \in S_{F,x_n}$ such that

$$\begin{aligned} g_n(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_n(s)\,ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f_n(s)\,ds \\ &\quad + \frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f_n(s)\,ds \\ &\quad - \frac{[bpT^2-(a+b)(2Tt-(2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f_n(s)\,ds \\ &\quad - \biggl(\frac{[b(-6(q-p)+(2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\ &\quad + \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \biggr) \\ &\quad \times \int_0^T (T-s)^{\alpha-\gamma-1} f_n(s)\,ds \end{aligned}$$

for all $t \in I$. Thus, we have to show that there exists $f_* \in S_{F,x_*}$ such that

$$\begin{aligned} g_*(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_*(s)\,ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f_*(s)\,ds \\ &\quad + \frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f_*(s)\,ds \\ &\quad - \frac{[bpT^2-(a+b)(2Tt-(2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f_*(s)\,ds \\ &\quad - \biggl(\frac{[b(-6(q-p)+(2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\ &\quad + \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \biggr) \\ &\quad \times \int_0^T (T-s)^{\alpha-\gamma-1} f_*(s)\,ds \end{aligned}$$

for all $t \in I$. Consider the linear continuous operator $\theta : L^1(I,\mathbb{R}) \to X$ defined by $f \longmapsto \theta(f)(t)$, where

$$\begin{aligned} \theta(f)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s)\,ds \\ &\quad + \frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f(s)\,ds \\ &\quad - \frac{[bpT^2-(a+b)(2Tt-(2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f(s)\,ds \\ &\quad - \biggl(\frac{[b(-6(q-p)+(2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\ &\quad + \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \biggr) \\ &\quad \times \int_0^T (T-s)^{\alpha-\gamma-1} f(s)\,ds \end{aligned}$$

for all $t \in I$. Since θ is a linear continuous map, therefore, by Lemma 4.1, it follows that $\theta \circ S_F$ is a closed graph operator. Note that $g_n \in \theta \circ S_F(x_n)$ for all n. Since $x_n \to x_*$ and $g_n \to g_*$, there exists $f_* \in S_F(x_*)$ such that

$$\begin{aligned}
g_*(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_*(s)\,ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f_*(s)\,ds \\
& + \frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f_*(s)\,ds \\
& - \frac{[bpT^2-(a+b)(2Tt-(2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f_*(s)\,ds \\
& - \biggl(\frac{[b(-6(q-p)+(2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
& + \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \biggr) \\
& \times \int_0^T (T-s)^{\alpha-\gamma-1} f_*(s)\,ds
\end{aligned}$$

for all $t \in I$. Let $\lambda \in (0,1)$ and $x \in \lambda\Omega(x)$, then there exists $f \in S_{F,x}$ such that

$$\begin{aligned}
x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t \lambda(t-s)^{\alpha-1} f(s)\,ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T \lambda(T-s)^{\alpha-1} f(s)\,ds \\
& + \frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T \lambda(T-s)^{\alpha-p-1} f(s)\,ds \\
& - \frac{[bpT^2-(a+b)(2Tt-(2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T \lambda(T-s)^{\alpha-q-1} f(s)\,ds \\
& - \biggl(\frac{[b(-6(q-p)+(2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
& + \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \biggr) \\
& \times \int_0^T \lambda(T-s)^{\alpha-\gamma-1} f(s)\,ds
\end{aligned}$$

for all $t \in I$. Hence

$$\begin{aligned}
\|x\| = {} & \sup_{t\in I}\bigl|x(t)\bigr| + \sup_{t\in I}\bigl|x'(t)\bigr| + \sup_{t\in I}\bigl|x''(t)\bigr| + \sup_{t\in I}\bigl|x'''(t)\bigr| \\
\leq {} & M_2 \biggl[\|q_1\|_1 \varphi_1\bigl(\|x\|\bigr) + \|q_2\|_1 \varphi_2\bigl(\|x\|\bigr) + \|q_3\|_1 \varphi_3\bigl(\|x\|\bigr) + \|q_4\|_1 \varphi_4\bigl(\|x\|\bigr) \\
& + \|q_5\|_1 \varphi_5 \biggl(c_{01}\gamma_0 + \gamma_0 \biggl[l_{11} + l_{21} + l_{31} + l_{41} + l_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} \\
& + l_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \biggr] \|x\| \biggr) \\
& + \|q_6\|_1 \varphi_6 \biggl(c_{02}\lambda_0 + \lambda_0 \biggl[l_{12} + l_{22} + l_{32} + l_{42} + l_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)}
\end{aligned}$$

$$
\begin{aligned}
&+ l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\biggr]\|x\|\biggr) \\
&+ \sum_{i=1}^{m}\|\rho_i\|_1\psi_i\biggl(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}\|x\|\biggr) + \sum_{j=1}^{m'}\|\rho'_j\|_1\psi'_j\biggl(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}\|x\|\biggr) \\
&+ \sum_{k=1}^{m''}\|\rho''_k\|_1\psi''_k\biggl(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\|x\|\biggr)\biggr] \\
&= M_2A\bigl(\|x\|\bigr)
\end{aligned}
$$

and so $\frac{\|x\|}{M_2A(\|x\|)} \leq 1$. Letting $U = \{x \in X : \|x\| < D\}$, the operator $\Omega : \overline{U} \to P_{cp,c}(X)$ is upper semi-continuous and compact. In view of the choice of U, there is no $x \in \partial U$ such that $x \in \lambda\Omega(x)$ for some $\lambda \in (0,1)$ and so Ω has a fixed point $x \in \overline{U}$ by virtue of Theorem 4.3. Obviously, each fixed point of Ω is a solution of problem (1.2)-(1.3). This completes the proof. □

Our next result deals with the case that F is not necessary convex valued.

Theorem 4.6 *Suppose that $F : I \times \mathbb{R}^{6+m+m'+m''} \to P_{cp}(\mathbb{R})$ is a multifunction such that the map*

$$
(t, x_1, x_2, \ldots, x_{6+m+m'+m''}) \longmapsto F(t, x_1, x_2, \ldots, x_{6+m+m'+m''})
$$

is $L(I) \otimes B(\mathbb{R}) \otimes \cdots \otimes B(\mathbb{R}) \otimes B(\mathbb{R})$ measurable and the map

$$
(x_1, x_2, \ldots, x_{6+m+m'+m''}) \longmapsto F(t, x_1, x_2, \ldots, x_{6+m+m'+m''})
$$

is lower semi-continuous for almost all $t \in I$. Assume that there exist continuous non-decreasing functions $\varphi_i : [0,\infty) \to (0,\infty)$ for $1 \leq i \leq 6$, $\psi_i, \psi'_j, \psi''_k : [0,\infty) \to (0,\infty)$ for $1 \leq i \leq m$, $1 \leq j \leq m'$, and $1 \leq k \leq m''$ and nonnegative functions $q_i \in L^1(I)$ for $1 \leq i \leq 6$, $\rho_i, \rho'_j, \rho''_k \in L^1(I)$ for $1 \leq i \leq m$, $1 \leq j \leq m'$, and $1 \leq k \leq m''$ such that

$$
\begin{aligned}
&\bigl\|F(t, x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_{m'}, w_1, w_2, \ldots, w_{m''})\bigr\|_p \\
&\quad= \sup\bigl\{|y| : y \in F(t, x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, \ldots, y_m, z_1, z_2, \ldots, z_{m'}, w_1, w_2, \ldots, w_{m''})\bigr\} \\
&\quad\leq q_1(t)\varphi_1\bigl(|x_1|\bigr) + q_2(t)\varphi_2\bigl(|x_2|\bigr) + q_3(t)\varphi_3\bigl(|x_3|\bigr) \\
&\qquad+ q_4(t)\varphi_4\bigl(|x_4|\bigr) + q_5(t)\varphi_5\bigl(|x_5|\bigr) + q_6(t)\varphi_6\bigl(|x_6|\bigr) \\
&\qquad+ \sum_{i=1}^{m}\rho_i(t)\psi_i\bigl(|y_i|\bigr) + \sum_{j=1}^{m'}\rho'_j(t)\psi'_j\bigl(|z_j|\bigr) + \sum_{k=1}^{m''}\rho''_k(t)\psi''_k\bigl(|w_k|\bigr),
\end{aligned}
$$

for all $t \in I$, $x_i, y_i, z_j, w_k \in \mathbb{R}$ for $1 \leq i \leq 6$, $1 \leq i \leq m$, $1 \leq j \leq m'$, and $1 \leq k \leq m''$. Furthermore, there exist positive constants $c_{01}, c_{02} > 0$, and $l_{i1}, l_{i2} \geq 0$ for $1 \leq i \leq 7$ such that

$$
\bigl|h_j(t, s, u_1, u_2, u_3, u_4, u_5, u_6, u_7)\bigr| \leq c_{0j} + \sum_{i=1}^{7} l_{ij}|u_i|
$$

for $j = 1, 2$, $t, s \in I$ *and all* $u_i \in \mathbb{R}$ *for* $1 \le i \le 7$. *If there exists a constant* $D > 0$ *such that* $\frac{D}{M_2 A(D)} > 1$, *then problem* (1.2)-(1.3) *has at least one solution, where*

$$\begin{aligned}
A(D) = {} & \|q_1\|_1\varphi_1(D) + \|q_2\|_1\varphi_2(D) + \|q_3\|_1\varphi_3(D) + \|q_4\|_1\varphi_4(D) \\
& + \|q_5\|_1\varphi_5\biggl(c_{01}\gamma_0 + \gamma_0\biggl[l_{11} + l_{21} + l_{31} + l_{41} + l_{51}\frac{T^{1-\gamma_1}}{\Gamma(2-\gamma_1)} \\
& + l_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + l_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\biggr]D\biggr) \\
& + \|q_6\|_1\varphi_6\biggl(c_{02}\lambda_0 + \lambda_0\biggl[l_{12} + l_{22} + l_{32} + l_{42} + l_{52}\frac{T^{1-\gamma_2}}{\Gamma(2-\gamma_2)} \\
& + l_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + l_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\biggr]D\biggr) \\
& + \sum_{i=1}^{m}\|\rho_i\|_1\psi_i\biggl(\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)}D\biggr) + \sum_{j=1}^{m'}\|\rho'_j\|_1\psi'_j\biggl(\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)}D\biggr) \\
& + \sum_{k=1}^{m''}\|\rho''_k\|_1\psi''_k\biggl(\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}D\biggr).
\end{aligned}$$

Proof In view of the given assumptions and Lemma 4.1 in [46], it follows that F is of lower semi-continuous type. Thus, by Lemma 4.2, there exists a continuous function $f : X \to L^1(I, \mathbb{R})$ such that $f(x) \in S_F(x)$ for all $x \in X$. Now, consider the equation

$$^cD^\alpha x(t) = f(x)(t) \tag{4.1}$$

supplemented with boundary conditions (1.3). Note that each solution of problem (4.1)-(1.3) with the given conditions is a solution of problem (1.2)-(1.3). Define the operator $\overline{\Omega} : X \to X$ by

$$\begin{aligned}
\overline{\Omega}x(t) = {} & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(x)(s)\,ds - \frac{b}{(a+b)\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}f(x)(s)\,ds \\
& + \frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}}\int_0^T (T-s)^{\alpha-p-1}f(x)(s)\,ds \\
& - \frac{[bpT^2-(a+b)(2Tt-(2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}}\int_0^T (T-s)^{\alpha-q-1}f(x)(s)\,ds \\
& - \biggl(\frac{[b(-6(q-p)+(2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\
& + \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}}\biggr) \\
& \times \int_0^T (T-s)^{\alpha-\gamma-1}f(x)(s)\,ds
\end{aligned}$$

for all $t \in I$. Following the procedure employed in the last result, one can show that $\overline{\Omega}$ is continuous and completely continuous and satisfies all conditions of the nonlinear alternative of Leary-Schauder type for single-valued maps. Consequently, there exists a solution for problem (4.1)-(1.3). This completes the proof. □

Finally, we establish the existence of a solution for the case that the right-hand side of (1.2) is non-convex valued.

Theorem 4.7 *Assume that $F : I \times \mathbb{R}^{6+m+m'+m''} \to P_{cp}(\mathbb{R})$ is a multifunction such that the map*

$$t \longmapsto F(t, x_1, x_2, \dots, x_{6+m+m'+m''})$$

is measurable for all $x_1, \dots, x_{6+m+m'+m''} \in \mathbb{R}$, the map $t \longmapsto d_H(0, F(t, 0, 0, \dots, 0))$ is integrably bounded for almost all $t \in I$ and there are nonnegative functions $k_1, \dots, k_6 \in L^1(I)$, $m_1, \dots, m_m, m'_1, \dots, m'_{m'}, m''_1, \dots, m''_{m''} \in L^1(I)$ such that

$$\begin{aligned} &d_H\bigl(F(t, x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, \dots, y_m, z_1, z_2, \dots, z_{m'}, w_1, w_2, \dots, w_{m''}), \\ &\quad F\bigl(t, x'_1, x'_2, x'_3, x'_4, x'_5, x'_6, y'_1, y'_2, \dots, y'_m, z'_1, z'_2, \dots, z'_{m'}, w'_1, w'_2, \dots, w'_{m''}\bigr)\bigr) \\ &\quad \le k_1(t)\bigl|x_1 - x'_1\bigr| + k_2(t)\bigl|x_2 - x'_2\bigr| + k_3(t)\bigl|x_3 - x'_3\bigr| \\ &\qquad + k_4(t)\bigl|x_4 - x'_4\bigr| + k_5(t)\bigl|x_5 - x'_5\bigr| + k_6(t)\bigl|x_6 - x'_6\bigr| \\ &\qquad + \sum_{i=1}^{m} m_i(t)\bigl|y_i - y'_i\bigr| + \sum_{j=1}^{m'} m'_j(t)\bigl|z_j - z'_j\bigr| + \sum_{k=1}^{m''} m''_k(t)\bigl|w_k - w'_k\bigr| \end{aligned}$$

for almost all $t \in I$ and all $x_1, \dots, x_6, x'_1, \dots, x'_6, y_1, \dots, y_m, y'_1, \dots, y'_m, z_1, \dots, z_{m'}, z'_1, \dots, z'_{m'}$, $w_1, \dots, w_{m''}$, $w'_1, \dots, w'_{m''} \in \mathbb{R}$. Also, suppose that there exist $\eta_{11}, \dots, \eta_{71}, \eta_{12}, \dots, \eta_{72} \ge 0$ such that

$$\bigl|h_j(t, s, u_1, u_2, u_3, u_4, u_5, u_6, u_7) - h_j\bigl(t, s, u'_1, u'_2, u'_3, u'_4, u'_5, u'_6, u'_7\bigr)\bigr| \le \sum_{i=1}^{7} \eta_{ij}\bigl|u_i - u'_i\bigr|$$

for all $t, s \in I$, $u_1, \dots, u_7, u'_1, \dots, u'_7 \in \mathbb{R}$, and $j = 1, 2$. If $\Theta < 1$, then problem (1.2)-(1.3) *has at least one solution, where*

$$\begin{aligned} \Theta = M_2\Biggl[&\|k_1\|_1 + \|k_2\|_1 + \|k_3\|_1 + \|k_4\|_1 \\ &+ \|k_5\|_1 \gamma_0 \biggl(\eta_{11} + \eta_{21} + \eta_{31} + \eta_{41} + \eta_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + \eta_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + \eta_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \biggr) \\ &+ \|k_6\|_1 \lambda_0 \biggl(\eta_{12} + \eta_{22} + \eta_{32} + \eta_{42} + \eta_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + \eta_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + \eta_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \biggr) \\ &+ \sum_{i=1}^{m} \|m_i\|_1 \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} \bigl\|m'_j\bigr\|_1 \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} \bigl\|m''_k\bigr\|_1 \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \Biggr]. \end{aligned}$$

Proof With the given assumptions and Theorem III-6 (the measurable selection theorem) in [47], one can infer that F admits a measurable selection $f : I \to \mathbb{R}$. Since F is integrably bounded, $f \in L^1(I, \mathbb{R})$, so $S_{F,x} \neq \emptyset$ for all $x \in X$. Now, we show that the operator Ω satisfies the assumptions of Theorem 4.4. First, we show that $\Omega(x) \in P_{cl}(X)$ for all $x \in X$. Let $u_n \in$

$\Omega(x)$ for all $n \geq 0$ and $u_n \to u$ for some $u \in X$. For each n, choose $v_n \in S_{F,x}$ such that

$$\begin{aligned} u_n(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s)\,ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v_n(s)\,ds \\ & + \frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} v_n(s)\,ds \\ & - \frac{[bpT^2-(a+b)(2Tt-(2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} v_n(s)\,ds \\ & - \biggl(\frac{[b(-6(q-p)+(2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\ & + \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \biggr) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} v_n(s)\,ds \end{aligned}$$

for all $t \in I$. Since F has compact values, there is a subsequence of $\{v_n\}$ that converges to v in $L^1(I,\mathbb{R})$. Thus, $v \in S_{F,x}$ and

$$\begin{aligned} u_n(t) \to u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)\,ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} v(s)\,ds \\ & + \frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} v(s)\,ds \\ & - \frac{[bpT^2-(a+b)(2Tt-(2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} v(s)\,ds \\ & - \biggl(\frac{[b(-6(q-p)+(2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \\ & + \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \biggr) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} v(s)\,ds \end{aligned}$$

for all $t \in I$. This implies that $u \in \Omega(x)$. Next, we show that there exists $\Theta < 1$ such that

$$d_H\bigl(\Omega(z), \Omega(\tilde{z})\bigr) \leq \Theta \|z - \tilde{z}\|$$

for all $z, \tilde{z} \in X$. Let $z, \tilde{z} \in X$ and $g_1 \in \Omega(z)$. Choose $f_1 \in S_{F,z}$ such that

$$\begin{aligned} g_1(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s)\,ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f_1(s)\,ds \\ & + \frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f_1(s)\,ds \\ & - \frac{[bpT^2-(a+b)(2Tt-(2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f_1(s)\,ds \\ & - \biggl(\frac{[b(-6(q-p)+(2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \end{aligned}$$

$$+ \frac{(a+b)(6(q-p)T^2t + (2-p)(3-p)(-3Tt^2 + (3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}}\bigg) \times \int_0^T (T-s)^{\alpha-\gamma-1} f_1(s)\,ds$$

for all $t \in I$. On the other hand we have

$$\begin{aligned} &d_H\big(\widetilde{F}\big(t,z(t)\big),\widetilde{F}\big(t,\tilde{z}(t)\big)\big) \\ &\leq \Bigg[k_1(t)+k_2(t)+k_3(t)+k_4(t) \\ &\quad + k_5(t)\gamma_0\bigg(\eta_{11}+\eta_{21}+\eta_{31}+\eta_{41}+\eta_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)}+\eta_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}+\eta_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg) \\ &\quad + k_6(t)\lambda_0\bigg(\eta_{12}+\eta_{22}+\eta_{32}+\eta_{42}+\eta_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)}+\eta_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}+\eta_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\bigg) \\ &\quad + \sum_{i=1}^{m} m_i(t)\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} m_j'(t)\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} m_k''(t)\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg]\|z-\tilde{z}\| \end{aligned}$$

for almost all $t \in I$. Hence, there exists $w_t \in \widetilde{F}(t,\tilde{z}(t))$ such that

$$\begin{aligned} &\big|f_1(t)-w_t\big| \\ &\leq \Bigg[k_1(t)+k_2(t)+k_3(t)+k_4(t) \\ &\quad + k_5(t)\gamma_0\bigg(\eta_{11}+\eta_{21}+\eta_{31}+\eta_{41}+\eta_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)}+\eta_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}+\eta_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg) \\ &\quad + k_6(t)\lambda_0\bigg(\eta_{12}+\eta_{22}+\eta_{32}+\eta_{42}+\eta_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)}+\eta_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}+\eta_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\bigg) \\ &\quad + \sum_{i=1}^{m} m_i(t)\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} m_j'(t)\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} m_k''(t)\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg]\|z-\tilde{z}\| := M_t \end{aligned}$$

for almost all $t \in I$. Define $V : I \to P(\mathbb{R})$ by $V(t) = \{u \in \mathbb{R} : |f_1(t) - u| \leq M_t\}$ for all $t \in I$. By Theorem III-41 in [47], it follows that V is measurable. Since the multivalued operator $t \longmapsto V(t) \cap \widetilde{F}(t,\tilde{z}(t))$ is measurable (Proposition III-4 in [47]), there exists a function $f_2 \in S_{F,\tilde{z}}$ such that

$$\begin{aligned} &\big|f_1(t)-f_2(t)\big| \\ &\leq \Bigg[k_1(t)+k_2(t)+k_3(t)+k_4(t) \\ &\quad + k_5(t)\gamma_0\bigg(\eta_{11}+\eta_{21}+\eta_{31}+\eta_{41}+\eta_{51}\frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)}+\eta_{61}\frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)}+\eta_{71}\frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)}\bigg) \\ &\quad + k_6(t)\lambda_0\bigg(\eta_{12}+\eta_{22}+\eta_{32}+\eta_{42}+\eta_{52}\frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)}+\eta_{62}\frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)}+\eta_{72}\frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)}\bigg) \\ &\quad + \sum_{i=1}^{m} m_i(t)\frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} m_j'(t)\frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} m_k''(t)\frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)}\Bigg]\|z-\tilde{z}\| \end{aligned}$$

for almost all $t \in I$. Define

$$\begin{aligned} g_2(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s)\,ds - \frac{b}{(a+b)\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f_2(s)\,ds \\ & + \frac{[bT-(a+b)t]\Gamma(2-p)}{(a+b)\Gamma(\alpha-p)T^{1-p}} \int_0^T (T-s)^{\alpha-p-1} f_2(s)\,ds \\ & - \frac{[bpT^2-(a+b)(2Tt-(2-p)t^2)]\Gamma(3-q)}{2(a+b)(2-p)\Gamma(\alpha-q)T^{2-q}} \int_0^T (T-s)^{\alpha-q-1} f_2(s)\,ds \\ & - \left(\frac{[b(-6(q-p)+(2-p)(3-p)q)T^3}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right. \\ & \left. + \frac{(a+b)(6(q-p)T^2t+(2-p)(3-p)(-3Tt^2+(3-q)t^3))]\Gamma(4-\gamma)}{6(a+b)(2-p)(3-p)(3-q)\Gamma(\alpha-\gamma)T^{3-\gamma}} \right) \\ & \times \int_0^T (T-s)^{\alpha-\gamma-1} f_2(s)\,ds \end{aligned}$$

for all $t \in I$. Then we have

$$\begin{aligned} & \|g_1-g_2\| \\ & \quad = \sup_{t\in I}\left|g_1(t)-g_2(t)\right| + \sup_{t\in I}\left|g_1'(t)-g_2'(t)\right| + \sup_{t\in I}\left|g_1''(t)-g_2''(t)\right| + \sup_{t\in I}\left|g_1'''(t)-g_2'''(t)\right| \\ & \quad \le M_2 \Bigg[\|k_1\|_1 + \|k_2\|_1 + \|k_3\|_1 + \|k_4\|_1 \\ & \qquad + \|k_5\|_1 \gamma_0 \left(\eta_{11} + \eta_{21} + \eta_{31} + \eta_{41} + \eta_{51} \frac{T^{1-\delta_1}}{\Gamma(2-\delta_1)} + \eta_{61} \frac{T^{2-\beta_1}}{\Gamma(3-\beta_1)} + \eta_{71} \frac{T^{3-\theta_1}}{\Gamma(4-\theta_1)} \right) \\ & \qquad + \|k_6\|_1 \lambda_0 \left(\eta_{12} + \eta_{22} + \eta_{32} + \eta_{42} + \eta_{52} \frac{T^{1-\delta_2}}{\Gamma(2-\delta_2)} + \eta_{62} \frac{T^{2-\beta_2}}{\Gamma(3-\beta_2)} + \eta_{72} \frac{T^{3-\theta_2}}{\Gamma(4-\theta_2)} \right) \\ & \qquad + \sum_{i=1}^{m} \|m_i\|_1 \frac{T^{1-\mu_i}}{\Gamma(2-\mu_i)} + \sum_{j=1}^{m'} \left\|m_j'\right\|_1 \frac{T^{2-\nu_j}}{\Gamma(3-\nu_j)} + \sum_{k=1}^{m''} \left\|m_k''\right\|_1 \frac{T^{3-\xi_k}}{\Gamma(4-\xi_k)} \Bigg] \|z-\tilde{z}\| \\ & \quad = \Theta\|z-\tilde{z}\|. \end{aligned}$$

Further, interchanging the roles of z and $\tilde{z}$, we get

$$d_H\big(\Omega(z),\Omega(\tilde{z})\big) \le \Theta\|z-\tilde{z}\|.$$

Since $\Theta < 1$, Ω is a contraction and so by Theorem 4.4, Ω has a fixed point which corresponds to a solution of problem (1.2)-(1.3). □

5 Examples

This section is devoted to the illustration of Theorems 4.5 and 4.7.

Example 5.1 Consider the fractional differential inclusion

$$\begin{aligned} {}^cD^{\frac{9}{2}}x(t) \in F\big(& t,x(t),x'(t),x''(t),x'''(t),\varphi x(t), \\ & {}^cD^{\frac{2}{5}}x(t), {}^cD^{\frac{4}{3}}x(t), {}^cD^{\frac{7}{3}}x(t), {}^cD^{\frac{13}{6}}x(t)\big), \quad t \in [0,1] \end{aligned} \tag{5.1}$$

supplemented with the boundary conditions $x^{(4)}(0) = 0$, $\frac{1}{5}x(0) + \frac{1}{2}x(1) = 0$, ${}^cD^{\frac{1}{3}}x(0) = -{}^cD^{\frac{1}{3}}x(1)$, ${}^cD^{\frac{7}{5}}x(0) = -{}^cD^{\frac{7}{5}}x(1)$, and ${}^cD^{\frac{9}{4}}x(0) = -{}^cD^{\frac{9}{4}}x(1)$ where

$$\begin{aligned}\varphi x(t) = \int_0^t \frac{e^{-(s-t)/2}}{900}\bigg[&\frac{e^{-\pi t}}{16(1+t^2)} + \frac{2x(s)}{517(1+t)(2+\sin x(s))} + \frac{5e^{-st}x'(s)}{327(s^2+7)}\\ &+ \frac{\sqrt[3]{\pi}x'(s)x''(s)}{984(1+|x'(s)|)} + \frac{3e^{-\cos^2 x(s)}x'''(s)}{895(s^2+1)} + \frac{\sin x(s)^cD^{\frac{1}{4}}x(s)}{1{,}875\sqrt{1+|{}^cD^{\frac{1}{4}}x(s)|+|{}^cD^{\frac{5}{3}}x(s)|}}\\ &+ \frac{e^{-\pi}\cos^2 x(s)^cD^{\frac{5}{3}}x(s)}{3{,}641(s^2+2s+1)} + \frac{{}^cD^{\frac{11}{5}}x(s)}{6{,}795(1+|x'(s)|)}\bigg]\,ds.\end{aligned}$$

Here $\alpha = \frac{9}{2}$, $\mu_1 = \frac{2}{5}$, $\nu_1 = \frac{4}{3}$, $\xi_1 = \frac{7}{3}$, $\xi_2 = \frac{13}{6}$, $p = \frac{1}{3}$, $q = \frac{7}{5}$, $\gamma = \frac{9}{4}$, $a = \frac{1}{5}$, $b = \frac{1}{2}$, $\delta_1 = \frac{1}{4}$, $\beta_1 = \frac{5}{3}$, $\theta_1 = \frac{11}{5}$, $c_{01} = \frac{1}{16}$, $l_{11} = \frac{2}{517}$, $l_{21} = \frac{5}{2{,}289}$, $l_{31} = \frac{\sqrt[3]{\pi}}{984}$, $l_{41} = \frac{3}{895}$, $l_{51} = \frac{1}{1{,}875}$, $l_{61} = \frac{e^{-\pi}}{3{,}641}$, $l_{71} = \frac{1}{6{,}795}$, $\gamma_0 = \frac{\sqrt{e}-1}{450}$. Define the multifunction $F : [0,1] \times \mathbb{R}^9 \to P(\mathbb{R})$ by

$$\begin{aligned}&F(t,x_1,x_2,x_3,x_4,x_5,y_1,z_1,w_1,w_2)\\ &\quad= \bigg\{u \in \mathbb{R} : -\frac{|x_1|^3}{4(3+|x_1|^3)} - \frac{e^{-\pi t}}{5}\sin^2 x_2 - \frac{|x_5|}{13(4+\sin^2 x_1)^2} - \frac{9|y_1|}{10(4+|y_1|)}\\ &\qquad - \frac{t^3}{9}\cos^2 z_1 - \frac{e^{-\pi t^2}}{81(t^4+3)}|w_1|\\ &\qquad - \frac{t^{\frac{1}{3}}}{20(1+|w_2|)} \le u \le \frac{1}{3}e^{-|x_1|} + \frac{7e^{-\pi t^2}}{18(1+tx_2^2)} + \frac{|x_4|}{10(1+|x_4|)}\\ &\qquad + \frac{t^{\frac{3}{2}}|x_5+\sin x_5|}{119} + \frac{e^{-t}}{t^4+2}\sin^4 y_1 + \frac{e^{-\frac{3}{2}t}}{5\sqrt{1+|z_1|^{\frac{5}{2}}}}\\ &\qquad + \frac{t}{135(t^2+1)}\left|w_1 + \frac{x_3}{1+|x_3|}\right| + \frac{31|w_2|^3}{140(1+|w_2|^3)} + \frac{3}{2}\bigg\}\end{aligned}$$

and note that

$$\begin{aligned}\big\|F(t,x_1,x_2,x_3,x_4,x_5,y_1,z_1,w_1,w_2)\big\|_p &= \sup\big\{|v| : v \in F(t,x_1,x_2,x_3,x_4,x_5,y_1,z_1,w_1,w_2)\big\}\\ &\le \frac{4{,}591}{1{,}260} + \frac{1}{119}\big(|x_5|+1\big) + \frac{1}{135}\big(|w_1|+1\big)\end{aligned}$$

for all $t \in [0,1]$ and $x_1,x_2,x_3,x_4,x_5,y_1,z_1,w_1,w_2 \in \mathbb{R}$. It is clear that F has convex and compact values and is of Carathéodory type. Let $q_1(t) = 1$, $\varphi_1(|x_1|) = \frac{4}{5}$, $q_2(t) = 1$, $\varphi_2(|x_2|) = \frac{1}{2}$, $q_3(t) = 1$, $\varphi_3(|x_3|) = \frac{1}{5}$, $q_4(t) = 1$, $\varphi_4(|x_4|) = \frac{1}{2}$, $q_5(t) = 1$, $\varphi_5(|x_5|) = \frac{1}{119}(|x_5|+1)$, $\rho_1(t) = 1$, $\psi_1(|y_1|) = \frac{8}{9}$, $\rho_1'(t) = 1$, $\psi_1'(|z_1|) = \frac{1}{60}$, $\rho_1''(t) = 1$, $\psi_1''(|w_1|) = \frac{1}{135}(|w_1|+1)$, $\rho_2''(t) = 1$, and $\psi_2''(|w_2|) = \frac{31}{42}$ for all $t \in [0,1]$ and $x_1,x_2,x_3,x_4,x_5,y_1,z_1,w_1,w_2 \in \mathbb{R}$. Hence,

$$\begin{aligned}&\big\|F(t,x_1,x_2,x_3,x_4,x_5,y_1,z_1,w_1,w_2)\big\|_p\\ &\quad= \sup\big\{|v| : v \in F(t,x_1,x_2,x_3,x_4,x_5,y_1,z_1,w_1,w_2)\big\}\\ &\quad\le q_1(t)\varphi_1\big(|x_1|\big) + q_2(t)\varphi_2\big(|x_2|\big) + q_3(t)\varphi_3\big(|x_3|\big) + q_4(t)\varphi_4\big(|x_4|\big)\\ &\qquad + q_5(t)\varphi_5\big(|x_5|\big) + \rho_1(t)\psi_1\big(|y_1|\big) + \rho_1'(t)\psi_1'\big(|z_1|\big) + \rho_1''(t)\psi_1''\big(|w_1|\big) + \rho_2''(t)\psi_2''\big(|w_2|\big)\end{aligned}$$

for all $t \in [0,1]$ and $x_1, x_2, x_3, x_4, x_5, y_1, z_1, w_1, w_2 \in \mathbb{R}$. With the given values, it is found that $M_2 \cong 8.609986787$. Letting $D > 33.90317546$, all the conditions of Theorem 4.5 hold and consequently problem (5.1) has at least one solution.

Example 5.2 Consider the fractional differential inclusion

$$\begin{aligned} {}^cD^{\frac{16}{3}}x(t) &\in F\big(t, x(t), x'(t), x''(t), x'''(t), \varphi x(t), \\ &\quad {}^cD^{\frac{1}{19}}x(t), {}^cD^{\frac{2}{15}}x(t), {}^cD^{\frac{20}{17}}x(t), {}^cD^{\frac{11}{5}}x(t)\big), \quad t \in [0,1] \end{aligned} \tag{5.2}$$

with the boundary value problems $x^{(4)}(0) = x^{(5)}(0) = 0$, $x(0) - 3x(1) = 0$, ${}^cD^{\frac{3}{20}}x(0) = -{}^cD^{\frac{3}{20}}x(1)$, ${}^cD^{\frac{15}{14}}x(0) = -{}^cD^{\frac{15}{14}}x(1)$, and ${}^cD^{\frac{19}{9}}x(0) = -{}^cD^{\frac{19}{9}}x(1)$, where

$$\begin{aligned} \varphi x(t) &= \int_0^t \frac{(s-t)^2 e^{-(s-t)^3}}{1{,}350}\left[\frac{t^3\cos^2 t}{e^{\pi t}(1+t^2)} + \frac{|x(s) + x'(s) + {}^cD^{\frac{1}{12}}x(s)|}{9{,}416\pi(1 + |x(s) + x'(s) + {}^cD^{\frac{1}{12}}x(s)|)}\right. \\ &\quad + \frac{t^3\sin^2 t\cos s}{759(36\sqrt{\pi} + e^{3s})}\arctan\left(\frac{3}{2} + \frac{|x''(s) + {}^cD^{\frac{7}{4}}x(s)|}{1 + |x''(s) + {}^cD^{\frac{7}{4}}x(s)|}\right) \\ &\quad \left. + \frac{e^{st}}{8{,}190(1 + e^{st})}\cos x'''(s) + \frac{s^2 e^{-\pi s^3}|{}^cD^{\frac{41}{20}}x(s)|}{(1{,}200 + \arcsin(\frac{1}{3})e^{3t^2})(1 + |{}^cD^{\frac{41}{20}}x(s)|)}\right] ds. \end{aligned}$$

Here $\alpha = \frac{16}{3}$, $\mu_1 = \frac{1}{19}$, $\mu_2 = \frac{2}{15}$, $\nu_1 = \frac{20}{17}$, $\xi_1 = \frac{11}{5}$, $p = \frac{3}{20}$, $q = \frac{15}{14}$, $\gamma = \frac{19}{9}$, $a = 1$, $b = -3$, $\delta_1 = \frac{1}{12}$, $\beta_1 = \frac{7}{4}$, $\theta_1 = \frac{41}{20}$, $n_{11} = n_{21} = n_{51} = \frac{1}{9{,}416\pi}$, $n_{31} = n_{61} = \frac{1}{759(36\sqrt{\pi}+1)}$, $n_{41} = \frac{1}{8{,}190}$, $n_{71} = \frac{1}{1{,}200+\arcsin(\frac{1}{3})}$, $\gamma_0 = \frac{e-1}{4{,}050}$. We define the multifunction $F : [0,1] \times \mathbb{R}^9 \to P(\mathbb{R})$ as

$$\begin{aligned} &F(t, x_1, x_2, x_3, x_4, x_5, y_1, y_2, z_1, w_1) \\ &\quad = \left[-\frac{e^{-\pi t}}{1+t^2} - \arctan\left(1 + \frac{e^t|x_1 + x_2 + x_3 + x_4|}{9{,}600(\frac{1}{3} + e^t)(1 + |x_1 + x_2 + x_3 + x_4|)}\right), \cos^2 t\right] \\ &\quad \cup \left[7 + \frac{\sin\pi t}{\sqrt{2 + t^3}}, 20\big(t^3 + 1\big) + \cos\left(t^3 + \frac{e^t|y_1 + y_2|}{8{,}719\pi(1 + |y_1 + y_2|)}\right)\right. \\ &\quad \left. + \frac{|z_1 + w_1|}{5{,}170(9 + t)^4(1 + |z_1 + w_1|)}\right] \\ &\quad \cup \left[\frac{3}{2}, \frac{|x_5|}{7{,}491(t + 25)^5(1 + |x_5|)} + t^2 + \frac{5}{2}\right] \end{aligned}$$

for all $t \in [0,1]$ and $x_1, x_2, x_3, x_4, x_5, y_1, y_1, z_1, w_1 \in \mathbb{R}$. It is clear that F has compact values and

$$\begin{aligned} &d_H\big(F(t, x_1, x_2, x_3, x_4, x_5, y_1, y_2, z_1, w_1), F\big(t, x_1', x_2', x_3', x_4', x_5', y_1', y_2', z_1', w_1'\big)\big) \\ &\quad \le \frac{e^t}{9{,}600(\frac{1}{3} + e^t)}\big(|x_1 - x_1'| + |x_2 - x_2'| + |x_3 - x_3'| + |x_4 - x_4'|\big) \\ &\quad + \frac{1}{7{,}491(t + 25)^5}|x_5 - x_5'| + \frac{e^t}{8{,}719\pi}\big(|y_1 - y_1'| + |y_2 - y_2'|\big) \\ &\quad + \frac{1}{5{,}170(9 + t)^4}\big(|z_1 - z_1'| + |w_1 - w_1'|\big) \end{aligned}$$

for all $t \in [0,1]$ and $x_1, x_2, x_3, x_4, x_5, x'_1, x'_2, x'_3, x'_4, x'_5, y_1, y_2, y'_1, y'_2, z_1, z'_1, w_1, w'_1 \in \mathbb{R}$. Fix $k_1(t) = k_2(t) = k_3(t) = k_4(t) = \frac{e^t}{9{,}600(\frac{1}{3}+e^t)}$, $k_5(t) = \frac{1}{7{,}491(t+25)^5}$, $m_1(t) = m_2(t) = \frac{e^t}{8{,}719\pi}$, $m'_1(t) = m''_1(t) = \frac{1}{5{,}170(9+t)^4}$. As in the previous example, it is found that $M_2 \cong 4.046590862$ and

$$\begin{aligned}\Theta = M_2\Bigg[& \|k_1\|_1 + \|k_2\|_1 + \|k_3\|_1 + \|k_4\|_1 \\ & + \|k_5\|_1\gamma_0\left(n_{11} + n_{21} + n_{31} + n_{41} + n_{51}\frac{1}{\Gamma(2-\delta_1)} + n_{61}\frac{1}{\Gamma(3-\beta_1)} + n_{71}\frac{1}{\Gamma(3-\theta_1)}\right) \\ & + \|m_1\|_1\frac{1}{\Gamma(2-\mu_1)} + \|m_2\|_1\frac{1}{\Gamma(2-\mu_2)} + \|m'_1\|_1\frac{1}{\Gamma(3-\nu_1)} + \|m''_1\|_1\frac{1}{\Gamma(4-\xi_1)}\Bigg] \\ \cong {} & 1.922566928 \times 10^{-3} < 1.\end{aligned}$$

As all the conditions of Theorem 4.7 are satisfied, the inclusion problem (5.2) has at least one solution.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [2]Department of Mathematics, Azarbaijan Shahid Madani University, Azarshahr, Tabriz, Iran.

Acknowledgements
This paper was funded by Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia. The authors, therefore, acknowledge technical and financial support of KAU.

Received: 19 September 2014 Accepted: 13 November 2014 Published online: 09 December 2014

References
1. Podlubny, I: Fractional Differential Equations. Academic Press, San Diego (1999)
2. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)
3. Sabatier, J, Agrawal, OP, Machado, JAT (eds.): Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht (2007)
4. Baleanu, D, Diethelm, K, Scalas, E, Trujillo, JJ: Fractional Calculus Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos. World Scientific, Boston (2012)
5. Agarwal, RP, Benchohra, M, Hamani, S: A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl. Math. **109**, 973-1033 (2010)
6. Bhalekar, S, Daftardar-Gejji, V, Baleanu, D, Magin, R: Fractional Bloch equation with delay. Comput. Math. Appl. **61**, 1355-1365 (2011)
7. Ahmad, B, Nieto, JJ: Sequential fractional differential equations with three-point boundary conditions. Comput. Math. Appl. **64**, 3046-3052 (2012)
8. Bai, ZB, Sun, W: Existence and multiplicity of positive solutions for singular fractional boundary value problems. Comput. Math. Appl. **63**, 1369-1381 (2012)
9. Sakthivel, R, Mahmudov, NI, Nieto, JJ: Controllability for a class of fractional-order neutral evolution control systems. Appl. Math. Comput. **218**, 10334-10340 (2012)
10. Agarwal, RP, O'Regan, D, Stanek, S: Positive solutions for mixed problems of singular fractional differential equations. Math. Nachr. **285**, 27-41 (2012)
11. Ahmad, B, Ntouyas, SK: Fractional differential inclusions with fractional separated boundary conditions. Fract. Calc. Appl. Anal. **15**(3), 362-382 (2012)
12. Liu, X, Liu, Z: Existence results for fractional differential inclusions with multivalued term depending on lower-order derivative. Abstr. Appl. Anal. **2012**, Article ID 423796 (2012)
13. Wang, JR, Zhou, Y, Medved, M: Qualitative analysis for nonlinear fractional differential equations via topological degree method. Topol. Methods Nonlinear Anal. **40**, 245-271 (2012)
14. Cabada, A, Wang, G: Positive solutions of nonlinear fractional differential equations with integral boundary value conditions. J. Math. Anal. Appl. **389**, 403-411 (2012)
15. Ahmad, B, Ntouyas, SK, Alsaedi, A: A study of nonlinear fractional differential equations of arbitrary order with Riemann-Liouville type multistrip boundary conditions. Math. Probl. Eng. **2013**, Article ID 320415 (2013)

16. Cernea, A: On a multi point boundary value problem for fractional order differential inclusion. Arab J. Math. Sci. **19**(1), 73-83 (2013)
17. Zhang, L, Ahmad, B, Wang, G, Agarwal, RP: Nonlinear fractional integro-differential equations on unbounded domains in a Banach space. J. Comput. Appl. Math. **249**, 51-56 (2013)
18. Baleanu, D, Rezapour, S, Mohammadi, H: Some existence results on nonlinear fractional differential equations. Philos. Trans. R. Soc. A, Math. Phys. Eng. Sci. **371**(1990), 20120144 (2013)
19. Baleanu, D, Agarwal, RP, Mohammadi, H, Rezapour, S: Some existence results for a nonlinear fractional differential equation on partially ordered Banach spaces. Bound. Value Probl. **2013**, Article ID 112 (2013)
20. Zhou, WX, Chu, YD, Baleanu, D: Uniqueness and existence of positive solutions for a multi-point boundary value problem of singular fractional differential equations. Adv. Differ. Equ. **2013**, Article ID 114 (2013)
21. Ahmad, B, Ntouyas, SK: Existence results for higher order fractional differential inclusions with multi-strip fractional integral boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2013**, 20 (2013)
22. Baleanu, D, Nazemi, SZ, Rezapour, S: Existence and uniqueness of solutions for multi-term nonlinear fractional integro-differential equations. Adv. Differ. Equ. **2013**, Article ID 368 (2013)
23. Nieto, JJ, Ouahab, A, Prakash, P: Extremal solutions and relaxation problems for fractional differential inclusions. Abstr. Appl. Anal. **2013**, Article ID 292643 (2013)
24. Zhai, C, Hao, M: Mixed monotone operator methods for the existence and uniqueness of positive solutions to Riemann-Liouville fractional differential equation boundary value problems. Bound. Value Probl. **2013**, Article ID 85 (2013)
25. Graef, JR, Kong, L, Wang, M: Existence and uniqueness of solutions for a fractional boundary value problem on a graph. Fract. Calc. Appl. Anal. **17**, 499-510 (2014)
26. Baleanu, D, Nazemi, SZ, Rezapour, S: Attractivity for a k-dimensional system of fractional functional differential equations and global attractivity for a k-dimensional system of nonlinear fractional differential equations. J. Inequal. Appl. **2014**, Article ID 31 (2014)
27. Baleanu, D, Nazemi, SZ, Rezapour, S: The existence of solution for a k-dimensional system of multiterm fractional integrodifferential equations with antiperiodic boundary value problems. Abstr. Appl. Anal. **2014**, Article ID 896871 (2014)
28. Punzo, F, Terrone, G: On the Cauchy problem for a general fractional porous medium equation with variable density. Nonlinear Anal. **98**, 27-47 (2014)
29. Liu, X, Liu, Z, Fu, X: Relaxation in nonconvex optimal control problems described by fractional differential equations. J. Math. Anal. Appl. **409**, 446-458 (2014)
30. Ahmad, B: Existence of solutions for fractional differential equations of order $q \in (2,3]$ with anti-periodic boundary conditions. J. Appl. Math. Comput. **34**, 385-391 (2010)
31. Wang, F, Liu, Z: Anti-periodic fractional boundary value problems for nonlinear differential equations of fractional order. Adv. Differ. Equ. **2012**, Article ID 116 (2012)
32. Alsaedi, A, Ahmad, B, Assolami, A: On antiperiodic boundary value problems for higher-order fractional differential equations. Abstr. Appl. Anal. **2012**, Article ID 325984 (2012)
33. Wang, X, Guo, X, Tang, G: Anti-periodic fractional boundary value problems for nonlinear differential equations of fractional order. J. Appl. Math. Comput. **41**, 367-375 (2013)
34. Ahmad, B, Nieto, JJ, Alsaedi, A, Mohamad, N: On a new class of antiperiodic fractional boundary value problems. Abstr. Appl. Anal. **2013**, Article ID 606454 (2013)
35. Deimling, K: Multivalued Differential Equations. de Gruyter, Berlin (1992)
36. Hu, S, Papageorgiou, NS: Handbook of Multivalued Analysis. Kluwer Academic, Dordrecht (1997)
37. Kisielewicz, M: Differential Inclusions and Optimal Control. Kluwer Academic, Dordrecht (1991)
38. Aubin, JP, Cellina, A: Differential Inclusions. Springer, Berlin (1984)
39. Aubin, JP, Frankowska, H: Set-Valued Analysis. Birkhäuser, Boston (1990)
40. Smart, DR: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)
41. Lasota, A, Opial, Z: An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations. Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys. **13**, 781-786 (1965)
42. Bressan, A, Colombo, G: Extensions and selections of maps with decomposable values. Stud. Math. **90**, 69-86 (1988)
43. Granas, A, Dugundji, J: Fixed Point Theory. Springer, Berlin (2005)
44. Covitz, H, Nadler, SB: Multi-valued contraction mappings in generalized metric spaces. Isr. J. Math. **8**, 5-11 (1970)
45. Kamenskii, M, Obukhovskii, V, Zecca, P: Condensing Multi-Valued Maps and Semilinear Differential Inclusions in Banach Spaces. de Gruyter, Berlin (2001)
46. Tolstonogov, AA: Bogolyubov's theorem under constraints generated by a controlled second-order evolution system. Izv. Ross. Akad. Nauk, Ser. Mat. **67**(5), 177-206 (2003)
47. Castaing, C, Valadier, M: Convex Analysis and Measurable Multifunctions. Springer, Berlin (1977)

doi:10.1186/s13661-014-0249-5
Cite this article as: Ahmad et al.: Some existence theorems for fractional integro-differential equations and inclusions with initial and non-separated boundary conditions. *Boundary Value Problems* 2014 **2014**:249.