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Existence of solutions for nonlinear Robin problems with the p-Laplacian and hemivariational inequality

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Abstract

In this paper, we show the existence of at least three nontrivial solutions for a nonlinear elliptic equation driven by the p-Laplacian with a nonsmooth potential (hemivariational inequality) and Robin boundary condition. Two of these solutions are of constant sign (one is positive, the other negative). We mainly use a variational approach together with a sub-sup solution method.

Keywords: p-Laplacian; nonsmooth potential; hemivariational inequality; sub-sup solution method; second deformation theorem

1 Introduction

Consider the problem

$$\begin{cases} -\Delta_p x + \alpha|x|^{p-2}x \in \partial j(z,x), & z \in Z, \\ |\nabla x|^{p-2}\frac{\partial x}{\partial n} + b(z)|x|^{p-2}x = 0, & z \in \partial Z, \end{cases} \tag{1.1}$$

where $Z \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary ∂Z, $\Delta_p x = \operatorname{div}(|\nabla x|^{p-2}\nabla x)$ $(1 < p < \infty)$ is the p-Laplacian operator, $\alpha > 0$, $b(z) \in L^\infty(\partial Z)$, $b(z) \geq 0$, and $b(z) \not\equiv 0$ on ∂Z. $j(z,x)$ is a measurable potential function on $Z \times \mathbb{R}$, which is locally Lipschitz in the $x \in \mathbb{R}$, $\partial j(z,x)$ stands for the generalized subdifferential of $x \mapsto j(z,x)$. Also $\frac{\partial x}{\partial n}$ denotes the outer normal derivative of x with respect to ∂Z. The aim of this paper is to prove the existence of two constant sign solutions and furthermore prove the existence of at least three nontrivial solutions for problem (1.1).

A multiplicity of solutions for problems driven by the p-Laplacian has been obtained by Ambrosetti *et al.* [1] and Garcia Azorero *et al.* [2]. In these works, the authors deal with a right-hand side nonlinearity of the form $-\Delta_p x = \lambda|x|^{q-2}x + |x|^{r-2}x$ with $\lambda > 0$ being a real parameter, $1 < q < p < r < p^*$ ($p^* = \frac{Np}{N-p}$ if $p < N$; $p^* = +\infty$ otherwise) and prove the existence of positive and negative solutions. The question of the existence of a p-Laplacian Robin problem $-\Delta_p x + \alpha|x|^{p-2}x = j(z,x)$ was also present in the work of Zhang *et al.* [3] for $p = 2$, the authors show that the Robin problem has at least four nontrivial solutions using a sub-sup solution method, the Fucík spectrum, the mountain pass theorem, and the degree theorem together. In the work of Zhang *et al.* [4, 5] for $p > 2$, the authors show that the oscillating equations with the p-Laplacian Robin problem has infinitely many nontrivial solutions. In Anello [6] and Ricceri [7], the main tool is an abstract variational principle of Ricceri and its use is made possible by the hypothesis that $p > N$; by the fact that Sobolev space $W^{1,p}(Z)$ is compactly embedded in $C(\overline{Z})$, the authors obtain infinitely many weak solutions for p-Laplacian Neumann problem.

In all the aforementioned works, the nonlinearity is a Carathéodory function, a.e. $j(z,x)$ is continuously differentiable in the variable x. In Barletta and Papageorgiou [8], the authors consider a nonsmooth potential with an asymmetric behavior at $+\infty$ and at $-\infty$ to get two nontrivial solutions using degree methods. Also, in Dancer and Du [9], the authors use the critical point theory and a sub-sup solution method on smooth critical point theory.

In this paper, we use a combination of nonsmooth critical point theory with sub-sup solution methods. We also use the nonsmooth version of the second deformation theorem due to Corvellec [10]. Thus, we can extend the works of [9, 11–13] to a hemivariational inequality with the Robin boundary condition.

2 Preliminaries

Now we recall the subdifferential theory for locally Lipschitz functions and the corresponding nonsmooth critical point theory. Let X be a Banach space and let X^* be its topological dual. We denote by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X,X^*). The generalized directional derivative $\varphi^0(x;h)$ of a locally Lipschitz function $\varphi : X \to \mathbb{R}$ at $x \in X$ along the direction $h \in X$ is defined as follows:

$$\varphi^0(x;h) = \limsup_{y\to x,\lambda\to 0} \frac{\varphi(y+\lambda h)-\varphi(y)}{\lambda}.$$

It is well known that $\varphi^0(x;\cdot)$ is sublinear continuous and it is the support function of a nonempty, convex, and w^*-compact set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) = \bigl\{x^* \in X^* : \langle x^*, h\rangle \le \varphi^0(x;h), \forall h \in X\bigr\}.$$

The function $\partial\varphi(x)$ is the 'generalized subdifferential' of φ. If $\varphi \in C^1(X)$, then φ is locally Lipschitz and $\partial\varphi(x) = \{\varphi'(x)\}$. Moreover, if φ is also convex, then $\partial\varphi(x)$ coincides with the subdifferential in the sense of convex analysis, $\partial_c\varphi(x)$, which is defined by

$$\partial_c\varphi(x) = \bigl\{x^* \in X^* : \langle x^*, h\rangle \le \varphi(x+h)-\varphi(x), \forall h \in X\bigr\}.$$

If $0 \in \partial\varphi(x)$, then we call $x \in X$ critical point of φ. It is easy to see that if $x \in X$ is a local minimum or a local maximum of φ, then $x \in X$ is a critical point of φ.

A locally Lipschitz function φ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$, if every sequence $\{x_n\}_{n\ge 1} \subseteq X$ satisfying $\varphi(x_n) \to c$ and $\inf\{\|x^*\| : x^* \in \partial\varphi(x_n)\} \to 0$ as $n \to \infty$ has a strongly convergent subsequence. If φ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ for all $c \in \mathbb{R}$, then we say that it satisfies the Palais-Smale condition. For the details, we refer to [14].

In the following study, denote $R(z,x) = |\nabla x|^{p-2}\frac{\partial x}{\partial n} + b(z)|x|^{p-2}x$, and we will use the following spaces:

$$W_n^{1,p}(Z) = \bigl\{x \in W^{1,p}(Z) : \exists\{x_n\} \subset C^\infty(Z), x_n \to x \text{ in } W^{1,p}(Z), R(z,x_n) = 0, \forall z \in \partial Z\bigr\},$$

$$C_n^1(\overline{Z}) = \{x \in C^1(\overline{Z}) : R(z,x) = 0, \forall z \in \partial Z\}.$$

Both are ordered Banach spaces, and we denote

$$\begin{aligned}
W^+ &= \{x \in W_n^{1,p}(Z) : x(z) \geq 0 \text{ a.e. } z \in Z\},\\
C^+ &= \{x \in C_n^1(\overline{Z}) : x(z) \geq 0, \forall z \in \overline{Z}\},\\
\operatorname{int}(C^+) &= \{x \in C^+ : x(z) > 0, \forall z \in Z\}.
\end{aligned}$$

It is well known that the principal eigenfunction $e \in \operatorname{int}(C^+)$, so $\operatorname{int}(C^+) \neq \emptyset$.

Furthermore, define u_1 as the normalized principal eigenfunction of $(-\Delta_p, W_n^{1,p}(Z))$ (see [15]). It is well known that $u_1(z) \geq 0$, a.e. $z \in Z$, from the nonlinear regularity $u_1 \in C_n^1(\overline{Z})$ (see Di Benedetto [16], [17, Chapter IX]), furthermore $u_1 \in \operatorname{int}(C^+)$ by virtue of the strong maximum principle of Vazquez [18].

We give the following minimax characterization (see [19]), suited for our purpose.

Proposition 2.1 *Let $S = W_n^{1,p}(Z) \cap \partial B_1$ and $\Gamma = \{\gamma \in C([0,1], S) : \gamma(0) = -u_1, \gamma(1) = u_1\}$, where $\partial B_1 = \{x \in L^p(Z) : \|x\|_p = 1\}$. Then the first eigenvalue λ_1 of $(-\Delta_p, W_n^{1,p}(Z))$ equals*

$$\lambda_1 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \|D\gamma(t)\|_p^p.$$

Next we recall the definitions of sub-sup solutions for problem (1.1).

(1) A function $\overline{x} \in W^{1,p}(Z)$ with $R(z,\overline{x}(z)) \geq 0$ is called a 'sup solution', if

$$\begin{aligned}
&\int_Z |D\overline{x}(z)|^{p-2}(D\overline{x}(z), Dy(z))\,dz + \alpha \int_Z |\overline{x}(z)|^{p-2}(\overline{x}(z), y(z))\,dz\\
&\quad + \int_{\partial Z} b(z)|\overline{x}(z)|^{p-2}(\overline{x}(z), y(z))\,dz \geq \int_Z u(z)y(z)\,dz
\end{aligned}$$

for all $y \in W_n^{1,p}(Z)$, $y(z) \geq 0$ a.e. on Z and for some $u \in L^q(Z)$, $u(z) \in \partial j(z, \overline{x}(z))$ a.e. on Z for some $1 < q < \frac{Np}{N-p}$ if $N > p$, $q = +\infty$ if $N \leq p$.

(2) A function $\underline{x} \in W^{1,p}(Z)$ with $R(z,\underline{x}(z)) \leq 0$ is called a 'sub-solution', if

$$\begin{aligned}
&\int_Z |D\underline{x}(z)|^{p-2}(D\underline{x}(z), Dy(z))\,dz + \alpha \int_Z |\underline{x}(z)|^{p-2}(\underline{x}(z), y(z))\,dz\\
&\quad + \int_{\partial Z} b(z)|\underline{x}(z)|^{p-2}(\underline{x}(z), y(z))\,dz \leq \int_Z u(z)y(z)\,dz
\end{aligned}$$

for all $y \in W_n^{1,p}(Z)$, $y(z) \geq 0$ a.e. on Z and for some $u \in L^q(Z)$, $u(z) \in \partial j(z, \underline{x}(z))$ a.e. on Z for some $1 < q < \frac{Np}{N-p}$ if $N > p$, $q = +\infty$ if $N \leq p$.

Finally we recall the following topological notion which is crucial in critical point theory.

Definition 2.2 [20] Let S, Q be closed subsets of a Banach space X, Q with relative boundary ∂Q. We say S and ∂Q link if

(1) $S \cap \partial Q = \emptyset$, and

(2) for any map $h \in C^0(X,X)$ such that $h|_{\partial Q} = \mathrm{id}$ we have $h(Q) \cap S \neq \emptyset$.

From the definition, we give the following general minimax principle for the critical values of a locally Lipschitz function φ.

Proposition 2.3 [20] *Suppose φ is locally Lipschitz and satisfies the (PS)-condition. Consider closed subsets $S, Q \subset X$ and Q with relative boundary ∂Q. Suppose*

(1) *S and ∂Q link,*
(2) $\inf_S \varphi > \sup_{\partial Q} \varphi$.

Let

$$\Gamma = \{h \in C^0(X,X) : h|_{\partial Q} = \mathrm{id}\}.$$

Then the number

$$\beta = \inf_{h\in\Gamma} \sup_{u\in Q} \varphi(h(u))$$

defines a critical value $\beta \geq \inf_S \varphi$ of φ.

Remark 2.4 From the above general minimax principle, a nonsmooth version of the mountain pass theorem, the saddle point theorem, and the generalized mountain pass theorem are available by choosing the link sets appropriately (see [10, 14]).

The following result is the so-called 'second deformation theorem' for a nonsmooth setting. In fact, this result is due to Corvellec [10]. We give the following sets:

$$K = \{x \in X : 0 \in \partial\varphi(x)\},$$
$$K_c = \{x \in X : 0 \in \partial\varphi(x), \varphi(x) = c\},$$
$$\varphi^c = \{x \in X : \varphi(x) < c\}.$$

We know that K, K_c, and φ^c are the critical set of φ, the critical set at level $c \in \mathbb{R}$ of φ, and the strict sublevel set of φ at c, respectively.

Proposition 2.5 *Let X be a Banach space, $\varphi : X \to \mathbb{R}$ be locally Lipschitz satisfying the Palais-Smale condition. $a, b \in \mathbb{R}$ with $a < b$. Assume also that $K \cap \varphi^{-1}((a,b)) = \emptyset$ and K_a is a finite set containing only local minimizers of φ.*

Then there exists a continuous deformation $\Phi : [0,1] \times \varphi^b \to \varphi^b$ such that

(1) *$\Phi(t,x) = x$ for all $t \in [0,1]$, $x \in K_a$,*
(2) $\Phi(1,\varphi^b) \subseteq \varphi^a \cup K_a$,
(3) *$\varphi(\Phi(t,x)) \leq \varphi(x)$ for all $t \in [0,1]$, $x \in \varphi^b$.*

Definition 2.6 [21] Let X be a topological space and A a subspace of X. A weak deformation retraction from X to A is a homology $F : X \times I \to X$ such that for all $x \in X$ and $a \in A$, we have $F(x,0) = x$, $F(a,1) = a$, and $F(x,1) \in A$.

In particular, the set $\varphi^a \cup K_a$ is a weak deformation retract of φ^b.

We now recall another notion, which will be useful in the following. Suppose W is a Banach space and $A : W \to W^*$ is a mapping, we say that A is a type $(S)_+$ if for every sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \rightharpoonup x \in W$ and $\limsup_{n\to\infty} \langle A(x_n), x_n - x\rangle \leq 0$.

Considering the nonlinear mapping $A : W_n^{1,p}(Z) \to W_n^{1,p}(Z)^*$ defined for all $x \in W_n^{1,p}(Z)$ by

$$\langle A(x), y\rangle = \int_Z \left|Dx(z)\right|^{p-2} Dx(z) \cdot Dy(z)\, dz. \tag{2.1}$$

We have the following result (see [8, Proposition 4.1]).

Proposition 2.7 *The mapping* (2.1) *is continuous and of the type* $(S)_+$.

Definition 2.8 [14] Given a functional $\varphi : W_n^{1,p}(Z) \to \mathbb{R}$, $x_0 \in W_n^{1,p}(Z)$ is called a W-local minimizer of φ if there exists $r > 0$ satisfying for all $y \in W_n^{1,p}(Z)$ with $\|y\|_{W_n^{1,p}(Z)} \le r$, we have

$$\varphi(x_0) \le \varphi(x_0 + y).$$

Definition 2.9 [14] $x_0 \in W_n^{1,p}(Z)$ is called a C-local minimizer of φ if there exists $r > 0$ satisfying for all $y \in C_n^1(\overline{Z})$ with $\|y\|_{C_n^1(\overline{Z})} \le r$, we have

$$\varphi(x_0) \le \varphi(x_0 + y).$$

As the study of problems like (1.1) is reduced to seeking the critical points of corresponding energy functional on $W_n^{1,p}(Z)$ or on $C_n^1(\overline{Z})$, in this section we introduce the notations used along the paper together with the main abstract results that we will use later on for a C-local minimizer to be a W-local minimizer. Such a result for $p > 2$ was first proved in [2]. Then it has been extended to the Neumann boundary condition and a nonsmooth potential by [8].

We denote $\psi : W_n^{1,p}(Z) \to \mathbb{R}$ for all $x \in W_n^{1,p}(Z)$

$$\psi(x) = \frac{1}{p}\|Dx\|_p^p + \frac{1}{p}\int_{\partial Z} b(z)|x|^p\, ds - \int_Z j\big(z, x(z)\big)\, dz.$$

From Clarke [22, pp.32-34], we know that ψ is locally Lipschitz. By [12], we know that if we let $x_0 \in W_n^{1,p}(Z)$ be a C-local minimizer of ψ, then $x_0 \in C_n^1(\overline{Z})$ and it is a W-local minimizer of ψ.

3 Solutions of constant sign

In this section, by using a sub-sup solution method, we get two solutions of (1.1) with constant sign, one positive and the other negative.

Our general assumptions on the nonsmooth potential $j(z,x)$ are the following:

$A(j)$
- (i) $z \mapsto j(z,x)$ is measurable for all $x \in \mathbb{R}$;
- (ii) $x \mapsto j(z,x)$ is locally Lipschitz for a.e. $z \in Z$;
- (iii) $|u| \le \gamma(z) + C|x|^{p-1}$ for a.e. $z \in Z$, all $x \in \mathbb{R}$, $u \in \partial j(z,x)$, with $\gamma \in L^\infty(Z)_+$ and $C > 0$;
- (iv) $\limsup_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \le \omega(z)$ for a.e. $z \in Z$, all $u \in \partial j(z,x)$, with $\omega \in L^\infty(Z)_+$ satisfying $\omega(z) \le \alpha$ a.e. in Z and $\omega(z) < \alpha$ in some set of positive measure;

(v) $\eta(z) + \alpha \leq \liminf_{x\to 0} \frac{u}{|x|^{p-2}x}$ for a.e. $z \in Z$, all $u \in \partial j(z,x)$, with $\eta \in L^\infty(Z)_+$ satisfying $\lambda_1 \leq \eta(z)$ a.e. in Z and $\lambda_1 < \eta(z)$ in some set of positive measure, λ_1 is the first eigenvalue of $-\Delta_p$ with Robin boundary condition;

(vi) $ux \geq 0$ for a.e. $z \in Z$, all $x \in \mathbb{R}$, $u \in \partial j(z,x)$.

Theorem 3.1 *Assume that* $A(j)$(i)-(vi) *hold. Problem* (1.1) *has at least two solutions* $x_0 \in \operatorname{int}(C_+)$ *and* $x_* \in -\operatorname{int}(C_+)$.

Example The following potential function j satisfies assumptions $A(j)$ (for the sake of simplicity we drop the z-dependence):

$$j(x) = \begin{cases} \frac{\eta+\alpha}{p}|x|^p, & |x| \leq 1, \\ \frac{\omega}{p}|x|^p + C\ln|x|^p + \frac{\eta+\alpha-\omega}{p}, & |x| > 1, \end{cases}$$

where $0 < \omega < \alpha$, $\eta > \lambda_1$, and $C > 0$. Note that, if $C = \frac{\eta+\alpha-\omega}{p}$, then $j \in C^1(\mathbb{R})$.

Note that

$$\partial j(x) = \begin{cases} [-\omega - pC, -\eta - \alpha], & x = -1, \\ (\eta+\alpha)|x|^{p-2}x, & |x| \leq 1, \\ 0, & x = 0, \\ \omega|x|^{p-2}x + pC\frac{x}{|x|^2}, & |x| > 1, \\ [\eta+\alpha, \omega + pC], & x = 1. \end{cases}$$

It is easy to see that j satisfies $A(j)$(i)-(iii), (vi). For all $u \in \partial j(x)$, we have

$$\limsup_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \leq \omega$$

and

$$\eta + \alpha \leq \liminf_{x\to 0} \frac{u}{|x|^{p-2}x}.$$

Then the potential function j satisfies assumptions $A(j)$.

Remark 3.2 In fact, problem (1.1) has the trivial solution $0 \in \partial j(z,0)$ for a.e. $z \in Z$ according to assumption $A(j)$(vi) and the upper semicontinuity of the subdifferential $\partial j(z,\cdot)$ (see Clarke [22]). What we are interesting in is whether it has nontrivial solutions.

We introduce a useful extension of the notion of maximal monotonicity (see [14, p.83]).

Definition 3.3 Let X be a reflexive Banach space and $A : X \to 2^{X^*}$ an operator. We say that A is pseudomonotone if

(1) A has nonempty, bounded and convex values;
(2) A is upper semicontinuous for every finite dimensional subspace of X into X^*;
(3) if $x_n \rightharpoonup x$ in X, $x_n^* \in A(x_n)$, and $\limsup_{n\to+\infty}\langle x_n^*, x_n - x\rangle_X \leq 0$, then for every $y \in X$, there exists $u^*(y) \in A(x)$, such that

$$\langle u^*(y), x - y\rangle_X \leq \liminf_{n\to+\infty}\langle x_n^*, x_n - y\rangle_X.$$

Definition 3.4 [14] A is said to be demicontinuous on X if $\{x_n\} \subset X$ and $x_n \to x \in X$ together imply $A(x_n) \rightharpoonup A(x)$.

It is well known that (1.1) is the Euler-Lagrange equation of the functional $\varphi : W_n^{1,p}(Z) \to \mathbb{R}$,

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p + \frac{\alpha}{p}\|x\|_p^p + \frac{1}{p}\int_{\partial Z} b(z)|x|^p\,ds - \int_Z j\bigl(z, x(z)\bigr)\,dz, \quad \forall x \in W_n^{1,p}(Z).$$

We introduce the truncation function $\nu_+ : \mathbb{R} \to \mathbb{R}_+$ by

$$\nu_+(x) = \begin{cases} x, & x > 0, \\ 0, & x \le 0; \end{cases}$$

then define the locally Lipschitz functional $\varphi_+ : W_n^{1,p}(Z) \to \mathbb{R}$ by

$$\varphi_+(x) = \frac{1}{p}\|Dx\|_p^p + \frac{\alpha}{p}\|x\|_p^p + \frac{1}{p}\int_{\partial Z} b(z)|x|^p\,ds - \int_Z j_+\bigl(z, x(z)\bigr)\,dz, \quad \forall x \in W_n^{1,p}(Z),$$

where $j_+(z,x) = j(z, \nu_+(x))$ for all $z \in \mathbb{R}$, $x \in \mathbb{R}$, which is locally Lipschitz.

We consider the nonlinear Robin problem for given $\varepsilon > 0$ and $\delta_\varepsilon(z) \in L^\infty(Z)_+$, $\delta_\varepsilon \neq 0$:

$$\begin{cases} -\Delta_p x + \alpha|x|^{p-2}x = (\omega(z) + \varepsilon)|x|^{p-2}x + \delta_\varepsilon(z), & z \in Z, \\ |\nabla x|^{p-2}\frac{\partial x}{\partial n} + b(z)|x|^{p-2}x = 0, & z \in \partial Z. \end{cases} \tag{3.1}$$

Define the mapping $I : W_n^{1,p}(Z) \to W_n^{1,p}(Z)^*$ for all $x, y \in W_n^{1,p}(Z)$ by

$$\begin{aligned} \langle I(x), y\rangle &= \int_Z \bigl|Dx(z)\bigr|^{p-2} Dx(z) \cdot Dy(z)\,dz + \alpha \int_Z \bigl|x(z)\bigr|^{p-2} x(z) \cdot y(z)\,dz \\ &\quad + \int_{\partial Z} b(z)\bigl|x(z)\bigr|^{p-2} x(z) \cdot y(z)\,ds. \end{aligned}$$

It is well known that I is strictly monotone and demicontinuous, furthermore, maximal monotone (see [23]). We denote $K_\varepsilon : L^p(Z) \to L^{p'}(Z)$ ($\frac{1}{p} + \frac{1}{p'} = 1$) and we have

$$K_\varepsilon(x)(\cdot) = \bigl(\omega(\cdot) + \varepsilon\bigr)\bigl|x(\cdot)\bigr|^{p-2} x(\cdot),$$

which is bounded and continuous. Then the mapping $I(x) - K_\varepsilon(x)$ is pseudomonotone from $W_n^{1,p}(Z)$ into $W_n^{-1,p'}(Z)$, in fact, $W_n^{1,p}(Z) \hookrightarrow L^p(Z)$ is compact embedding and $K_\varepsilon : W_n^{1,p}(Z) \to L^{p'}(Z)$ is completely continuous.

Next, we will show that (3.1) has a solution $\overline{x} \in \operatorname{int}(C_+)$.

Lemma 3.5 *Let $\omega \in L^\infty(Z)_+$ satisfy $\omega(z) \le \alpha$ a.e. in Z and $\omega(z) < \alpha$ in some set of positive measure. Then* (3.1) *has a solution $\overline{x} \in \operatorname{int}(C_+)$ for $\varepsilon > 0$ small enough.*

Proof First, we claim that there exists $\xi_0 > 0$ such that

$$J(x) = \|Dx\|_p^p + \alpha\|x\|_p^p + \int_{\partial Z} b(z)|x|^p\,ds - \int_Z \omega(z)\bigl|x(z)\bigr|^p\,dz \ge \xi_0\|x\|_p^p, \quad \forall x \in W_n^{1,p}(Z).$$

In fact, from assumption $b \geq 0$, we know that $J(x) \geq 0$, for all $x \in W_n^{1,p}(Z)$. Suppose the conclusion is false, we have $x_n \in W_n^{1,p}(Z)$, $J(x_n) < \frac{1}{n}\|x_n\|_p^p$, $x_n \neq 0$. If we set $x_n' = \frac{x_n}{\|x_n\|}$, then $J(x_n') < \frac{1}{n}$ (J is p-homogeneous). We may assume $x_n' \rightharpoonup x$ in $W_n^{1,p}(Z)$, $x_n' \to x$ in $L^p(Z)$ by passing to a subsequence if necessary. Then

$$\|x\|_p^p = \lim_{n\to\infty}\|x_n'\|_p^p, \qquad \|Dx\|_p^p \leq \liminf_{n\to\infty}\|Dx_n'\|_p^p,$$
$$\int_{\partial Z} b(z)|x|^p\,ds - \int_Z \omega(z)\big|x(z)\big|^p\,dz = \lim_{n\to\infty}\left(\int_{\partial Z} b(z)\big|x_n'\big|^p\,ds - \int_Z \omega(z)\big|x_n'(z)\big|^p\,dz\right).$$

So, by passing to the limit of J, we have

$$\|Dx\|_p^p + \alpha\|x\|_p^p + \int_{\partial Z} b(z)|x|^p\,ds - \int_Z \omega(z)\big|x(z)\big|^p\,dz \leq 0.$$

This implies

$$\|Dx\|_p^p \leq \int_Z \big(\omega(z) - \alpha\big)\big|x(z)\big|^p\,dz - \int_{\partial Z} b(z)|x|^p\,ds \leq 0.$$

Hence, we have $x(z) = C$ for a.e. $z \in Z$ where $C \in \mathbb{R}$. In fact, $C = 0$, if not, from the above inequality,

$$0 \leq |C|^p\left[\int_Z \big(\omega(z) - \alpha\big)\,dz - \int_{\partial Z} b(z)\,ds\right] < 0.$$

It produces a contradiction. On the other hand,

$$\big\|Dx_n'\big\|_p^p = J\big(x_n'\big) - \alpha\big\|x_n'\big\|_p^p - \int_{\partial Z} b(z)\big|x_n'\big|^p\,ds - \int_Z \omega(z)\big|x_n'(z)\big|^p\,dz.$$

We have $\|Dx_n'\|_p \to 0$, together with $x_n \to x$ in $L^p(Z)$, so $x_n \to 0$ in $W_n^{1,p}(Z)$, but $\|x_n\|_{W_n^{1,p}(Z)} = 1$, $n \in Z$. So the assumption is false, we have the conclusion.

For all $x \in W_n^{1,p}(Z)$, from the above discussion, we get

$$\begin{aligned}\big\langle I(x) - K_\varepsilon(x), x\big\rangle &= \|Dx\|_p^p + \alpha\|x\|_p^p + \int_{\partial Z} b(z)|x|^p\,ds - \int_Z \omega(z)\big|x(z)\big|^p\,dz - \varepsilon\|x\|_p^p \\ &\geq (\varepsilon_0 - \varepsilon)\|x\|_p^p.\end{aligned}$$

So if $\varepsilon < \varepsilon_0$ small enough, we have $I(\cdot) - K_\varepsilon(\cdot)$ is coercive. But a pseudomonotone coercive operator is surjective (see [23, Theorem 9.57]), for δ_ε, we can find $\overline{x} \in W_n^{1,p}(Z)$ such that

$$I(\overline{x}) - K_\varepsilon(\overline{x}) = \delta_\varepsilon.$$

That is,

$$\begin{cases} -\Delta_p\overline{x} + \alpha|\overline{x}|^{p-2}\overline{x} = (\omega(z) + \varepsilon)|\overline{x}|^{p-2}\overline{x} + \delta_\varepsilon(z), & z \in Z, \\ |\nabla\overline{x}|^{p-2}\frac{\partial\overline{x}}{\partial n} + b(z)|\overline{x}|^{p-2}\overline{x} = 0, & z \in \partial Z. \end{cases} \tag{3.2}$$

It follows that $\overline{x} \in W_n^{1,p}(Z)$ is a solution of (3.1).

Next we show $\overline{x} \in \operatorname{int}(C_+)$. Take $-\overline{x}_- = -\max\{-\overline{x}, 0\} \in W_n^{1,p}(Z)$ for $\delta_\varepsilon \geq 0$, then

$$\langle I(\overline{x}) - K_\varepsilon(\overline{x}), -\overline{x}_-\rangle = -\|D\overline{x}_-\|_p^p - \alpha\|\overline{x}_-\|_p^p - \int_{\partial Z} b(z)|\overline{x}_-|^p\,ds + \int_Z \omega(z)\left|\overline{x}_-(z)\right|^p dz + \varepsilon\|\overline{x}_-\|_p^p \geq 0.$$

So

$$\varepsilon_0\|\overline{x}_-\|_p^p \leq \|D\overline{x}_-\|_p^p + \alpha\|\overline{x}_-\|_p^p + \int_{\partial Z} b(z)|\overline{x}_-|^p\,ds - \int_Z \omega(z)\left|\overline{x}_-(z)\right|^p dz \leq \varepsilon\|\overline{x}_-\|_p^p.$$

But $\varepsilon < \varepsilon_0$, we have $\overline{x}_- = 0$, that is, $\overline{x} \geq 0$. Since $\delta_\varepsilon > 0$, from (3.2), we have $\overline{x} \neq 0$ and $\overline{x} \in C_n^1(\overline{Z})$ (nonlinear regularity theorem, see [24]), furthermore, $\Delta_p\overline{x} \leq 0$ on Z, so $\overline{x} \in \operatorname{int}(C_+)$. □

Now we prove that the solution $\overline{x}$ of (3.1) is a strict sup solution of (1.1) for $\varepsilon > 0$ small enough.

Lemma 3.6 *Let assumptions* $A(j)$(i)-(iv) *hold. Then the solution* $\overline{x}$ *of* (3.1) *is a strict sup solution of* (1.1) *for* $\varepsilon > 0$ *small enough.*

Proof From $A(j)$(iv), for given $\varepsilon > 0$, we can find $M_1 > 0$, such that for all $z \in Z$, $x \geq M_1$, $u \in \partial j(z,x)$, we have

$$\frac{u}{x^{p-1}} \leq \omega(z) + \varepsilon.$$

From $A(j)$(iii), we can find $\delta_\varepsilon \in L^\infty(Z)_+$, $\delta_\varepsilon \neq 0$, such that for all $z \in Z$, $x \in [0, M_1]$, $u \in \partial j(z,x)$, we have

$$u < \delta_\varepsilon(z).$$

So for all $z \in Z$, $x \geq 0$, $u \in \partial j(z,x)$, we have

$$u < \big(\omega(z) + \varepsilon\big)x^{p-1} + \delta_\varepsilon(z).$$

From Lemma 3.5, we see that (3.1) has a solution $\overline{x} \in \operatorname{int}(C_+)$, so when $\varepsilon < \varepsilon_0$ small enough, for all $z \in Z$, $x \in L^{p'}(Z)_+$, $u \in \partial j(z, \overline{x}(z))$, we have

$$u < \big(\omega(z) + \varepsilon\big)\overline{x}^{p-1} + \delta_\varepsilon(z),$$

that is,

$$u < -\Delta_p\overline{x} + \alpha|\overline{x}|^{p-2}\overline{x},$$

and from the definition of a sup solution, we know that $\overline{x}$ is a sup solution of (1.1). □

Remark 3.7 We have found a sup solution of (1.1) and $\partial j(z,0) = \{0\}$ a.e. on Z, we also find $\underline{x} \equiv 0$ is a sub-solution of (1.1). Define the set

$$W = \{x \in W_n^{1,p}(Z) : 0 \le x(z) \le \overline{x}(z), \text{a.e. on } Z\}.$$

Next, we will find a nontrivial solution of (1.1) in W.

Proof of Theorem 3.1

Step 1: Claim: We can find $x_0 \in W$ which is a local minimizer of φ_+ and of φ.

From the discussion of Lemma 3.6, for a.e. $z \in Z$, all $x \ge 0$, $u \in \partial j_+(z,x) = \partial j(z,x)$, we have

$$u < (\omega(z) + \varepsilon)x^{p-1} + \delta_\varepsilon(z).$$

Furthermore, for a.e. $z \in Z$, all $x \ge 0$, from assumptions $A(j)$(i), (ii),

$$\frac{d}{dx} j_+(z,x) < (\omega(z) + \varepsilon)x^{p-1} + \delta_\varepsilon(z)$$

then for a.e. $z \in Z$, all $x \ge 0$, we have

$$j_+(z,x) < \frac{1}{p}(\omega(z) + \varepsilon)|x|^p + \delta_\varepsilon(z)|x|.$$

So, for some $C > 0$, we have

$$\begin{aligned}
\varphi_+(x) &= \frac{1}{p}\|Dx\|_p^p + \frac{\alpha}{p}\|x\|_p^p + \frac{1}{p}\int_{\partial Z} b(z)|x|^p\, ds - \int_Z j_+(z,x(z))\, dz \\
&> \frac{1}{p}\|Dx\|_p^p + \frac{\alpha}{p}\|x\|_p^p + \frac{1}{p}\int_{\partial Z} b(z)|x|^p\, ds - \frac{1}{p}\int_Z \omega(z)|x(z)|^p\, dz - \frac{\varepsilon}{p}\|x\|_p^p - C\|x\|_p \\
&\ge \frac{1}{p}(\varepsilon_0 - \varepsilon)\|x\|_p^p - C\|x\|_p.
\end{aligned}$$

Because of $\varepsilon < \varepsilon_0$, $p > 1$, we see that φ_+ is coercive, and together with φ_+ weakly lower semicontinuous on W. Thus by the Weierstrass theorem, we can find $x_0 \in W$, satisfying

$$\varphi_+(x_0) = \inf_W \varphi_+.$$

We claim that $x_0 \ne 0$. In fact, from assumption $A(j)$(v), we see that, for given $\varepsilon > 0$, we can find some $\delta > 0$, for a.e. $z \in Z$, all $x \in [0,\delta]$, $u \in \partial j_+(z,x) = \partial j(z,x)$,

$$\frac{u}{x^{p-1}} \ge \eta(z) + \alpha - \varepsilon;$$

then for a.e. $z \in Z$ and all $x \in [0,\delta]$, we get

$$j_+(z,x) \ge \frac{1}{p}(\eta(z) + \alpha - \varepsilon)x^p.$$

Furthermore, let e_1 be the first eigenfunction of Robin problem of $-\Delta_p$ (see [15]), then for $\overline{x} \in \operatorname{int}(C^+)$, we can find $\theta > 0$, such that

$$\theta e_1(z) \leq \min\{\overline{x}(z), \delta\}, \quad \forall z \in \overline{Z}.$$

Then $\theta e_1 \in \operatorname{int}(C^+)$, and

$$\begin{aligned}\varphi_+(\theta e_1) &= \frac{\theta^p}{p}\|De_1\|_p^p + \frac{\alpha\theta^p}{p}\|e_1\|_p^p + \frac{\theta^p}{p}\int_{\partial Z} b(z)|e_1|^p\,ds - \int_Z j_+\bigl(z, \theta e_1(z)\bigr)\,dz \\ &\leq \frac{\theta^p}{p}\|De_1\|_p^p - \frac{\theta^p}{p}\int_Z \eta(z)|e_1|^p\,dz + \frac{\theta^p}{p}\int_{\partial Z} b(z)|e_1|^p\,ds + \frac{\varepsilon\theta^p}{p}\|e_1\|_p^p \\ &= \frac{\theta^p}{p}\left[\int_Z \bigl(\lambda_1 - \eta(z)\bigr)|e_1|^p\,dz + \varepsilon\|e_1\|_p^p\right].\end{aligned}$$

From assumption $A(j)$(v) and $e_1 > 0$, we have

$$\int_Z \bigl(\lambda_1 - \eta(z)\bigr)|e_1|^p\,dz < 0.$$

If we choose ε small enough, we can get $\varphi_+(\theta e_1) < 0$ for all $\theta > 0$ small enough. So, we have

$$\varphi_+(x_0) = \inf_W \varphi_+ \leq \varphi_+(\theta e_1) < 0 = \varphi_+(0),$$

then we have $x_0 \neq 0$, $x_0 \in W$.

Step 2: The local minimizer of φ_+, $x_0 \in W_n^{1,p}(Z)$ is a nontrivial solution of (1.1).

Firstly, we claim that x_0 is also a local $W_n^{1,p}(Z)$-minimizer of φ_+. In fact, the nonlinear regularity theory (see for example [24]) assures that $x_0 \in C^1(\overline{Z})$. Hence, as the boundary relation is understood in a pointwise sense and we get $x_0 \in C_n^1(\overline{Z})$, also, by $x_0 \neq 0$, $x_0 \geq 0$, and the nonlinear strong maximum principle of Vazquez, $x_0 \in \operatorname{int}(C_+)$, $\overline{x} - x_0 \in \operatorname{int}(C_+)$. So we can find $\delta > 0$ satisfying

$$\begin{aligned}&B_\delta(x_0) = \bigl\{x \in C_n^1(\overline{Z}) : \|x - x_0\|_{C_n^1(\overline{Z})} < \delta\bigr\} \subseteq \operatorname{int}(C_+), \\ &B_\delta(\overline{x} - x_0) = \bigl\{x \in C_n^1(\overline{Z}) : \bigl\|x - (\overline{x} - x_0)\bigr\|_{C_n^1(\overline{Z})} < \delta\bigr\} \subseteq \operatorname{int}(C_+).\end{aligned}$$

Then

$$x_0 + B_\delta \subseteq \operatorname{int}(C_+), \qquad \overline{x} - x_0 + B_\delta \subseteq \operatorname{int}(C_+).$$

So, x_0 is also a local minimizer of φ_+ on $C_n^1(\overline{Z})$; also from [24], x_0 is also a local $W_n^{1,p}(Z)$-minimizer of φ_+ and of φ too.

Also, from [25], there exists $\omega(z) \in \partial j_+(z, x_0(z)) = \partial j(z, x_0(z))$, $u \in L^{p'}(Z)$ satisfying

$$0 \leq \bigl\langle I(x_0), y - x_0\bigr\rangle - \int_Z u(z)(y - x_0)(z)\,dz, \quad \forall y \in W.$$

Using

$$y(z) = \begin{cases} \overline{x}(z), & z \in \{\overline{x} \leq x_0 + \varepsilon v\} = A, \\ x_0(z) + \varepsilon v(z), & z \in \{0 < x_0 + \varepsilon v < \overline{x}\} = B, \\ 0, & z \in \{x_0 + \varepsilon v \leq 0\} = C. \end{cases}$$

We have $y \in W$ for all $v \in W_n^{1,p}(Z)$, $\varepsilon > 0$, then we have

$$\begin{aligned} 0 \leq{}& \int_A |Dx_0|^{p-2}\big\langle Dx_0, D(\overline{x} - x_0)\big\rangle\, dz + \alpha \int_A |x_0|^{p-2}\langle x_0, \overline{x} - x_0\rangle\, dz - \int_A u(\overline{x} - x_0)\, dz \\ &+ \varepsilon \int_B |Dx_0|^{p-2}\langle Dx_0, Dv\rangle\, dz + \alpha \int_B |x_0|^{p-2}\langle x_0, \varepsilon v\rangle\, dz - \int_B u(\varepsilon v)\, dz \\ &+ \int_{\partial B} b(z)|x_0|^{p-2}\langle x_0, \varepsilon v\rangle\, ds \\ &- \int_C |Dx_0|^p\, dz - \alpha \int_C |x_0|^p\, dz + \int_C u x_0\, dz - \int_{\partial C} b(z)|x_0|^p\, ds \\ &+ \int_{\partial A} b(z)|x_0|^{p-2}\langle x_0, \overline{x} - x_0\rangle\, ds \\ ={}& \varepsilon \int_Z |Dx_0|^{p-2}\langle Dx_0, Dv\rangle\, dz + \varepsilon\alpha \int_Z |x_0|^{p-2}\langle x_0, v\rangle\, dz - \varepsilon \int_Z uv\, dz \\ &+ \varepsilon \int_{\partial Z} b(z)|x_0|^{p-2}\langle x_0, v\rangle\, ds \\ &- \int_C |Dx_0|^p\, dz - \alpha \int_C |x_0|^p\, dz - \varepsilon \int_C |Dx_0|^{p-2}\langle Dx_0, Dv\rangle\, dz - \int_{\partial C} b(z)|x_0|^p\, ds \\ &- \int_A |D\overline{x}|^{p-2}\big\langle D\overline{x}, D(x_0 + \varepsilon v - \overline{x})\big\rangle\, dz - \alpha \int_A |\overline{x}|^{p-2}\langle \overline{x}, x_0 + \varepsilon v - \overline{x}\rangle\, dz \\ &+ \int_A \overline{u}(x_0 + \varepsilon v - \overline{x})\, dz \\ &- \int_{\partial A} b(z)|\overline{x}|^{p-2}\langle \overline{x}, x_0 + \varepsilon v - \overline{x}\rangle\, ds + \int_C u(x_0 + \varepsilon v)\, dz + \int_A (\overline{u} - u)(\overline{x} - x_0 - \varepsilon v)\, dz \\ &+ \int_A \big\langle |D\overline{x}|^{p-2}D\overline{x} - |Dx_0|^{p-2}Dx_0, D(x_0 - \overline{x})\big\rangle\, dz + \alpha \int_A \big\langle |\overline{x}|^{p-2}\overline{x} - |x_0|^{p-2}x_0, x_0 - \overline{x}\big\rangle\, dz \\ &+ \int_{\partial A} b(z)\big\langle |\overline{x}|^{p-2}\overline{x} - |x_0|^{p-2}x_0, x_0 - \overline{x}\big\rangle\, ds + \varepsilon \int_C \big\langle |D\overline{x}|^{p-2}D\overline{x} - |Dx_0|^{p-2}Dx_0, Dv\big\rangle\, dz \\ &- \varepsilon \int_{\partial A} b(z)|x_0|^{p-2}\langle x_0, v\rangle\, ds - \varepsilon \int_{\partial C} b(z)|x_0|^{p-2}\langle x_0, v\rangle\, ds + \varepsilon \int_{\partial A} b(z)|\overline{x}|^{p-2}\langle \overline{x}, v\rangle\, ds \\ &- \varepsilon\alpha \int_A |x_0|^{p-2}\langle x_0, v\rangle\, dx - \varepsilon\alpha \int_C |x_0|^{p-2}\langle x_0, v\rangle\, dx + \varepsilon\alpha \int_A |\overline{x}|^{p-2}\langle \overline{x}, v\rangle\, dx. \end{aligned}$$

From the monotonicity of I, we have

$$\begin{aligned} &\int_A \big\langle |D\overline{x}|^{p-2}D\overline{x} - |Dx_0|^{p-2}Dx_0, D(x_0 - \overline{x})\big\rangle\, dz + \alpha \int_A \big\langle |\overline{x}|^{p-2}\overline{x} - |x_0|^{p-2}x_0, x_0 - \overline{x}\big\rangle\, dz \\ &\quad + \int_{\partial A} b(z)\big\langle |\overline{x}|^{p-2}\overline{x} - |x_0|^{p-2}x_0, x_0 - \overline{x}\big\rangle\, ds \leq 0. \end{aligned}$$

From the definition of a sup solution of (1.1), we have

$$\begin{aligned} &-\int_A |D\overline{x}|^{p-2}\big\langle D\overline{x}, D(x_0+\varepsilon v-\overline{x})\big\rangle\,dz-\alpha\int_A |\overline{x}|^{p-2}\langle \overline{x}, x_0+\varepsilon v-\overline{x}\rangle\,dz+\int_A \overline{u}(x_0+\varepsilon v-\overline{x})\,dz \\ &\quad -\int_{\partial A} b(z)|\overline{x}|^{p-2}\langle \overline{x}, x_0+\varepsilon v-\overline{x}\rangle\,ds\le 0. \end{aligned}$$

From $A(j)$(vi), we have $\int_C u(x_0+\varepsilon v)\,dz\le 0$. Furthermore,

$$\int_C(\overline{u}-u)(\overline{x}-x_0-\varepsilon v)\,dz\le \varepsilon c\int_{\{x_0+\varepsilon v\ge\overline{x}>x_0\}} v\,dz.$$

Also,

$$m\{x_0+\varepsilon v\ge\overline{x}>x_0\}\to 0,\quad \text{as } \varepsilon\to 0,$$

and

$$\begin{aligned} &Dx_0(z)=0\quad \text{a.e. on } \{x_0=0\}, \\ &Dx_0(z)=D\overline{x}(z)\quad \text{a.e. on } \{x_0=\overline{x}\}. \end{aligned}$$

So, we have

$$\begin{aligned} 0\le{}& \varepsilon\int_Z |Dx_0|^{p-2}\langle Dx_0, Dv\rangle\,dz+\varepsilon\alpha\int_Z |x_0|^{p-2}\langle x_0, v\rangle\,dz \\ &+\varepsilon\int_{\partial Z} b(z)|x_0|^{p-2}\langle x_0, v\rangle\,ds-\varepsilon\int_Z uv\,dz \\ &-\varepsilon\int_C |Dx_0|^{p-2}\langle Dx_0, Dv\rangle\,dz+\varepsilon c\int_{\{x_0+\varepsilon v\ge\overline{x}>x_0\}} v\,dz \\ &+\varepsilon\int_C\big\langle |D\overline{x}|^{p-2}D\overline{x}-|Dx_0|^{p-2}Dx_0, Dv\big\rangle\,dz. \end{aligned}$$

As $\varepsilon\to 0$, for all $v\in W_n^{1,p}(Z)$, we obtain

$$0\le\big\langle I(x_0), v\big\rangle-\int_Z uv\,dz=\big\langle I(x_0)-u, v\big\rangle.$$

That is,

$$I(x_0)=u.$$

Then $x_0\in W_n^{1,p}(Z)$ is a solution of (1.1).

Step 3: In a similar way, we introduce another truncation function $v_-:\mathbb{R}\to\mathbb{R}_-$ by

$$v_-(x)=\begin{cases} x, & x<0, \\ 0, & x\ge 0, \end{cases}$$

then define the locally Lipschitz functional $\varphi_- : W_n^{1,p}(Z) \to \mathbb{R}$ by

$$\varphi_-(x) = \frac{1}{p}\|Dx\|_p^p + \frac{\alpha}{p}\|x\|_p^p + \frac{1}{p}\int_{\partial Z} b(z)|x|^p\,ds - \int_Z j_-\big(z,x(z)\big)\,dz, \quad \forall x \in W_n^{1,p}(Z),$$

where $j_-(z,x) = j(z,\nu_-(x))$ for all $z \in \mathbb{R}$, $x \in \mathbb{R}$ which is locally Lipschitz. Then we have another nontrivial solution $x_* \in W_n^{1,p}(Z)$ which is a local minimum of φ_- and of φ too. □

4 Existence of the third nontrivial solution

In this section, we prove the existence of the third solution. Then we give the new assumptions which differ slightly from $A(j)$(v):

$A'(j)$
- (i) $z \mapsto j(z,x)$ is measurable for all $x \in \mathbb{R}$;
- (ii) $x \mapsto j(z,x)$ is locally Lipschitz for a.e. $z \in Z$;
- (iii) $|u| \le \gamma(z) + C|x|^{p-1}$ for a.e. $z \in Z$, all $x \in \mathbb{R}$, $u \in \partial j(z,x)$, with $\gamma \in L^\infty(Z)_+$ and $C > 0$;
- (iv) $\limsup_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \le \omega(z)$ for a.e. $z \in Z$, all $u \in \partial j(z,x)$, with $\omega \in L^\infty(Z)_+$ satisfying $\omega(z) \le \alpha$ a.e. in Z and $\omega(z) < \alpha$ in some set of positive measure;
- (v) $\eta(z) \le \liminf_{x\to 0} \frac{u}{|x|^{p-2}x}$ for a.e. $z \in Z$, all $u \in \partial j(z,x)$, with $\eta \in L^\infty(Z)_+$ satisfying $\lambda_1 + \alpha \le \eta(z)$ a.e. in Z and $\lambda_1 + \alpha < \eta(z)$ in some set of positive measure, λ_1 is the first eigenvalue of $-\Delta_p$ with the Robin boundary condition;
- (vi) $ux \ge 0$ for a.e. $z \in Z$, all $x \in \mathbb{R}$, $u \in \partial j(z,x)$.

Theorem 4.1 *Let assumptions* $A'(j)$(i)-(vi) *hold. Then we can find three nontrivial solutions* $x_0 \in \operatorname{int}(C_+)$, $x_* \in -\operatorname{int}(C_+)$, *and* $y_0 \in C_n^1(\overline{Z})$ *of* (1.1).

Proof From Theorem 3.1, we have two constant sign solutions $x_0 \in \operatorname{int}(C_+)$ and $x_* \in -\operatorname{int}(C_+)$ which are the local minimizers of φ_+ and of φ_-, also of φ. We may assume that x_0 is the only nontrivial critical point of φ_+ and x_* is the only nontrivial critical point of φ_-. In fact, if there exists another nontrivial critical point x_1 of φ_+, $x_1 \neq x_0$. Then, by a similar discussion, $x_1 \in \operatorname{int}(C_+)$ and it solves (1.1). Thus we have a third nontrivial solution, a.e. $y_0 = x_1$.

Moreover, as for φ, we see that φ is coercive and so we can easily prove the Palais-Smale condition. In fact, as in the proof of Theorem 3.1, for a.e. $z \in Z$, all $x \in \mathbb{R}$, we have

$$j(z,x) < \frac{1}{p}\big(\omega(z) + \varepsilon\big)|x|^p + \delta_\varepsilon(z)|x|,$$

where ω satisfies (iii), $\delta_\varepsilon \in L^\infty(Z)_+$, $\delta_\varepsilon \neq 0$.

Then, using Lemma 3.5, we have

$$\begin{aligned}\varphi(x) &\ge \frac{1}{p}\|Dx\|_p^p + \frac{\alpha}{p}\|x\|_p^p + \frac{1}{p}\int_{\partial Z} b(z)|x|^p\,ds - \frac{1}{p}\int_Z \omega(z)\big|x(z)\big|^p\,dz - \frac{\varepsilon}{p}\|x\|_p^p - C\|x\|_p \\ &\ge \frac{1}{p}(\varepsilon_0 - \varepsilon)\|x\|_p^p - C\|x\|_p.\end{aligned}$$

It follows that φ is coercive.

We set $S = \partial B_\delta(x_0) = \{x \in W_n^{1,p}(Z) : \|x - x_0\|_{W_n^{1,p}(Z)} = \delta\}$, $Q = [x_*, x_0]$, with relative boundary $\partial Q = \{x_*, x_0\}$. If we choose $0 < \delta < \|x_* - x_0\|_{W_n^{1,p}(Z)}$. Then S and ∂Q link. In fact, $S \cap \partial Q = \emptyset$, and for any map $h \in C^0(Q, W_n^{1,p}(Z))$ such that $h|_{\partial Q} = \mathrm{Id}$, we can choose some $t \in (0,1)$ satisfying

$$\left\| h\bigl(tx_* + (1-t)x_0\bigr) - x_0 \right\|_{W_n^{1,p}(Z)} = \delta,$$

so $h(Q) \cap S \neq \emptyset$, S and ∂Q link.

When we choose δ, we can also assume δ satisfy $\inf_{x\in S} \varphi > \varphi(x_0)$ and $\inf_{x\in S} \varphi > \varphi(x_*)$ (x_0, x_* are local minimizers of φ), we may assume that $\varphi(x_*) < \varphi(x_0)$. Therefore, we can apply Proposition 2.3; let $\Gamma = \{h \in C^0(Q, W_n^{1,p}(Z)) : h|_{\partial Q} = \mathrm{Id}\}$, produce $y_0 \in W_n^{1,p}(Z)$, a critical point of φ, such that

$$\begin{aligned} &0 \in \partial\varphi(y_0), \\ &\varphi(x_*) < \varphi(x_0) < \inf_{x\in S} \varphi \le \varphi(y_0) = \inf_{h\in\Gamma} \sup_{x\in Q} \varphi\bigl(h(x)\bigr). \end{aligned}$$

From the above inequality, we have $y_0 \neq x_0$, $y_0 \neq x_*$.

From $0 \in \partial\varphi(y_0)$, we know that

$$\begin{cases} -\Delta_p y_0(z) + \alpha|y_0(z)|^{p-2}y_0(z) \in \partial j(z, y_0(z)), & z \in Z, \\ |\nabla y_0(z)|^{p-2}\frac{\partial y_0(z)}{\partial n} + b(z)|y_0(z)|^{p-2}y_0(z) = 0, & z \in \partial Z, \end{cases} \tag{4.1}$$

and from the regularity theory (see [24]), we have $y_0 \in C_n^1(\overline{Z})$, hence (4.1) holds in all $z \in Z$, we get $y_0 \in C_n^1(\overline{Z})$.

Finally, we prove that $y_0 \neq 0$. It is equivalent to proving that there is a path $h \in \Gamma$ such that for all $x \in Q$,

$$\varphi\bigl(h(x)\bigr) < 0 = \varphi(0).$$

From Proposition 2.1, recall that $S = W_n^{1,p}(Z) \cap \partial B_1$, $\partial B_1 = \{x \in L^p(Z) : \|x\|_p = 1\}$ endowed with the $W_n^{1,p}(Z)$-topology. Furthermore, set $S_c = S \cap C_n^1(\overline{Z})$ equipped with the $C_n^1(\overline{Z})$-topology. Then we can find $h_0 \in S_c$ by virtue of the density of S_c in S in the $W_n^{1,p}(Z)$-topology, so $C(Q, S_c)$ is dense in $C(Q, S)$, and

$$\max\left\{ \|Dx\|_p^p + \int_{\partial Z} b(z)|x|^p\, ds, x \in h_0(Q) \right\} \le \lambda_1 + \delta. \tag{4.2}$$

From assumption $A'(j)$(v), we can find $\delta_0 > 0$, such that for a.e. $z \in Z$, all $0 < |x| < \delta_0$, $u \in \partial j(z, x)$, we get

$$\eta(z) \le \frac{u}{|x|^{p-2}x}.$$

So for a.e. $z \in Z$, all $0 < |x| < \delta_0$,

$$\frac{\eta}{p}|x|^p < j(z, x). \tag{4.3}$$

Since $h_0(Q) \in S_c$, for the δ_0, we can find $\varepsilon > 0$, such that for a.e. $z \in \overline{Z}$, $x \in h_0(Q)$, we have

$$\varepsilon\left|x(z)\right| \le \delta_0. \tag{4.4}$$

Then let $\delta > 0$ be such that $\lambda_1 + \alpha + \delta < \eta$, from (4.2), (4.3), (4.4), and $\|x\|_p = 1$, we have

$$\begin{aligned}
\varphi(\varepsilon x) &= \frac{\varepsilon^p}{p}\|Dx\|_p^p + \frac{\alpha\varepsilon^p}{p}\|x\|_p^p + \frac{\varepsilon^p}{p}\int_{\partial Z} b(z)|x|^p\,ds - \int_Z j\big(z, \varepsilon x(z)\big)\,dz \\
&\le \frac{\varepsilon^p}{p}\|Dx\|_p^p + \frac{\alpha\varepsilon^p}{p}\|x\|_p^p + \frac{\varepsilon^p}{p}\int_{\partial Z} b(z)|x|^p\,ds - \frac{\eta\varepsilon^p}{p}\|x\|_p^p \\
&= \frac{\varepsilon^p}{p}\left(\|Dx\|_p^p + \int_{\partial Z} b(z)|x|^p\,ds\right) + \frac{\alpha - \eta}{p}\varepsilon^p \\
&\le \frac{\lambda_1 + \delta + \alpha - \eta}{p}\varepsilon^p < 0.
\end{aligned}$$

We consider the continuous path $h_\varepsilon = \varepsilon h_0$, then for all $x \in Q$,

$$\varphi\big(h_\varepsilon(x)\big) < 0, \quad \forall x \in Q.$$

Next recall that φ is coercive and satisfies the Palais-Smale condition. From the discussion, we set $a = \varphi(x_0) = \inf\varphi < 0$, $b = 0$, φ has no critical points in $\varphi^{-1}(a,b)$, $K_a = \{x_0\}$. Then with the help of Proposition 2.5, there exists a deformation $\Phi : [0,1] \times \varphi^b \to \varphi^b$ such that

$$\begin{aligned}
&\Phi(t,\cdot)|_{K_a} = \mathrm{Id}, \quad \forall t \in [0,1], \\
&\Phi\big(1, \varphi^b\big) \subseteq \varphi^a \cup K_a, \\
&\varphi\big(\Phi(t,x)\big) \le \varphi(x), \quad \forall (t,x) \in [0,1] \times \varphi^b.
\end{aligned} \tag{4.5}$$

In fact, the continuous path Γ can be seen as $\Gamma = \{h \in C^0([0,1], W_n^{1,p}(Z)) : h(0) = x_*, h(1) = x_0\}$. Then we define $h_1 : [0,1] \to \varphi^b$ by

$$h_1(t) = \Phi(t, \varepsilon u_1), \quad \forall t \in [0,1].$$

Then it is a continuous path, so from (4.5), we have

$$\begin{aligned}
&h_1(0) = \Phi(0, \varepsilon u_1) = \varepsilon u_1, \\
&h_1(1) = \Phi(1, \varepsilon u_1) = x_0 \quad \big(\varphi^a = \emptyset, K_a = \{x_0\}\big), \\
&\varphi\big(h_1(t)\big) = \varphi\big(\Phi(t, \varepsilon u_1)\big) \le \varphi(\varepsilon u_1) < 0, \quad \forall t \in [0,1]\ (\varphi|_{h_\varepsilon} < 0).
\end{aligned}$$

Thus, we construct a continuous path h_1 joining εu_1 and x_0 such that

$$\varphi|_{h_1} < 0.$$

Similarly, we construct a continuous path h_2 joining $-\varepsilon u_1$ and x_* such that

$$\varphi|_{h_2} < 0.$$

Then we join h_2, h_ε, h_1, and we construct a continuous path $h \in \Gamma$ such that

$$\varphi|_h < 0.$$

It follows that $\varphi(y_0) < 0 = \varphi(0)$ and so $y_0 \neq 0$.

Therefore, we find the third nontrivial solution of (1.1). □

5 Open related questions

Consider the problem

$$\begin{cases} -\Delta_p x + \alpha(z)|x|^{p-2}x \in \partial j(z,x), & z \in Z, \\ |\nabla x|^{p-2}\frac{\partial x}{\partial n} + b(z)|x|^{p-2}x = 0, & z \in \partial Z, \end{cases} \tag{5.1}$$

where $Z \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary ∂Z, $\Delta_p x = \operatorname{div}(|\nabla x|^{p-2}\nabla x)$ $(1 < p < \infty)$ is the p-Laplacian operator, $\alpha(z), b(z) \in L^\infty(\partial Z)$, $b(z) \geq 0$, and $b(z) \neq 0$ on ∂Z. $j(z,x)$ is a measurable potential function on $Z \times \mathbb{R}$, which is locally Lipschitz in the $x \in \mathbb{R}$, $\partial j(z,x)$ stands for the generalized subdifferential of $x \mapsto j(z,x)$. Also $\frac{\partial x}{\partial n}$ denotes the outer normal derivative of x with respect to ∂Z.

Whether problem (5.1) has more solutions and whether it has oscillating solutions, we will discuss in the future.

Competing interests
The author declares that she has no competing interests.

Author's contributions
The author wrote, read, and approved the final manuscript.

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