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A note on fractional spaces generated by the positive operator with periodic conditions and applications

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Abstract
In this study, the second order differential operator A^x defined by the formula $A^x u = -u_{xx}(x) + \delta u(x)$, $\delta \geq 0$, with domain $D(A^x) = \{u(x) : u(x), u'(x), u''(x) \in C(R^1), u(x) = u(x + 2\pi), x \in R^1, \int_0^{2\pi} u(x)\,dx = 0\}$ is considered. The Green function of the differential operator A^x is constructed. The estimates for the Green function are obtained. It is proved that for any $\alpha \in (0, \frac{1}{2})$, the norms in the spaces $E_\alpha = E_\alpha(\mathring{C}(R^1), A^x)$ and $\mathring{C}^{2\alpha}(R^1)$ are equivalent.

The positivity of the operator A^x in Hölder spaces of $\mathring{C}^{2\alpha}(R^1)$, $\alpha \in (0, \frac{1}{2})$, is proved. In the applications, theorems on well-posedness of local and nonlocal boundary value problems for elliptic equations in the Hölder spaces are obtained.

Keywords: Green function; fractional spaces; boundary value problems; positive operator

1 Introduction

Various problems for partial differential equations can be considered as an abstract boundary value problem for an ordinary differential equation in a Banach space E with a densely defined unbounded space operator A. The role played by positivity of the differential and difference operators in a Banach space in the study of various properties of boundary value problems for partial differential equations, of stability of difference schemes for partial differential equations and summation of Fourier series is well known [1–5]. The positivity of a wider class of differential and difference operators in Banach spaces has been studied by many researchers [6–25].

Let E be a Banach space and $A : D(A) \subset E \to E$ be a linear unbounded operator densely defined in E. A is called a positive operator in the Banach space if the operator $(\lambda I + A)$ has a bounded inverse in E and for any $\lambda \geq 0$, the following estimate holds:

$$\left\|(\lambda I + A)^{-1}\right\|_{E\to E} \leq \frac{M}{\lambda + 1}.$$

Throughout the present paper, M denotes positive constants, which may differ in time and thus is not a subject of precision. However, we will use $M(\alpha, \beta, \ldots)$ to stress the fact that the constant depends only on $\alpha, \beta, \ldots$.

For a positive operator A in the Banach space E, let us introduce the fractional spaces $E_\alpha = E_\alpha(E, A)$ $(0 < \alpha < 1)$ consisting of those $v \in E$ for which the norm

$$\|v\|_{E_\alpha} = \sup_{\lambda>0} \lambda^\alpha \left\| A(\lambda + A)^{-1} v \right\|_E + \|v\|_E$$

is finite.

Let us introduce the Banach space $C^\beta(R^1)$, $\beta \in (0, \frac{1}{2})$, of all continuous 2π periodic functions $\varphi(x)$ defined on R^1 and satisfying a Hölder condition for which the following norm is finite:

$$\|\varphi\|_{C^\beta(R^1)} = \|\varphi\|_{\mathring{C}(R^1)} + \sup_{x,x+\tau\in[0,2\pi],\tau\neq 0} \frac{|\varphi(x+\tau) - \varphi(x)|}{|\tau|^\beta},$$

where $\mathring{C}(R^1)$ is the Banach space of all continuous 2π periodic functions $\varphi(x)$ defined on R^1 with the norm

$$\|\varphi\|_{\mathring{C}(R^1)} = \max_{x\in[0,2\pi]} \left|\varphi(x)\right|.$$

In [3], a new method of summations of Fourier series converging in

$$\mathring{C}\left(R^1\right) = \left\{ \varphi(x) \in C\left(R^1\right) : \int_0^{2\pi} \varphi(x)\,dx = 0 \right\}$$

is presented. It is based on the following result on the positivity of the differential operator A^x defined by the formula

$$A^x u = -u_{xx}(x) + \delta u(x), \quad \delta \geq 0, \tag{1}$$

with domain

$$D\left(A^x\right) = \left\{ u(x) : u(x), u'(x), u''(x) \in C\left(R^1\right), u(x) = u(x+2\pi), x \in R^1, \int_0^{2\pi} u(x)\,dx = 0 \right\}.$$

Theorem 1.1 [3] *The operator* $(A^x + \lambda)$ *has a bounded in* $\mathring{C}(R^1)$ *inverse for* $\delta = 0$, $\lambda \geq 0$ *and the following estimate holds*:

$$\left\| \left(A^x + \lambda\right)^{-1} \right\|_{\mathring{C}(R^1)\to\mathring{C}(R^1)} \leq \frac{1 + 16\pi^2}{1+\lambda}. \tag{2}$$

It is easy to see that (2) *is true for all* $\delta > 0$.

In the present study, the resolvent equation of the operator A^x

$$A^x u + \lambda u = \varphi \tag{3}$$

or

$$\begin{cases} -u''(x) + (\delta + \lambda)u(x) = \varphi(x), & 0 < x < 2\pi, \\ u(x) = u(x+2\pi), & x \in R^1, \int_0^{2\pi} u(x)\,dx = 0, \end{cases} \tag{4}$$

will be investigated. The Green function of A^x is constructed. The estimates for the Green function are obtained. It is proved that for any $\alpha \in (0, \frac{1}{2})$, the norms in the spaces

$E_\alpha = E_\alpha(\mathring{C}(R^1), A^x)$ and $\mathring{C}^{2\alpha}(R^1)$ are equivalent. Here, $\mathring{C}^{2\alpha}(R^1)$ is the subspace of $C^{2\alpha}(R^1)$ such that $\int_0^{2\pi} \varphi(x)\,dx = 0$. The positivity of the operator A^x in the Hölder spaces $\mathring{C}^{2\alpha}(R^1)$, $\alpha \in (0, \frac{1}{2})$, is proved. The structure of fractional spaces generated by this operator is investigated. In the applications, theorems on well-posedness of local and nonlocal boundary value problems for elliptic equations in Hölder spaces are obtained.

2 The Green function of A^x

Lemma 2.1 *Assume that* $\varphi \in C(R^1)$ *and* $\varphi(x) = \varphi(x + 2\pi)$, $x \in R^1$, $\int_0^{2\pi} \varphi(x)\,dx = 0$. *For any* $\lambda \geq 0$, *problem* (4) *is uniquely solvable and the following formula holds*:

$$u(x) = \left(A^x + \lambda I\right)^{-1}\varphi(x) = \int_0^{2\pi} G(x, s; \lambda)\varphi(s)\,ds, \tag{5}$$

where

$$G(x, s; \lambda) = \frac{1}{2\sqrt{\delta + \lambda}} \frac{1}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} \begin{cases} e^{-\sqrt{\delta+\lambda}(x-s)} + e^{-\sqrt{\delta+\lambda}(2\pi - x+s)}, & 0 \leq s \leq x, \\ e^{-\sqrt{\delta+\lambda}(s-x)} + e^{-\sqrt{\delta+\lambda}(2\pi + x-s)}, & x \leq s \leq 2\pi, \end{cases} \tag{6}$$

$$G(2\pi + x, s; \lambda) = G(x, s; \lambda), \quad x \in R^1.$$

Proof Let $x \in [0, 2\pi]$. From (3) there follows the problem

$$\begin{aligned} &-u''(x) + (\delta + \lambda)u(x) = \varphi(x), \quad 0 < x < 2\pi, \\ &u(0) = u(2\pi), \quad x \in R^1, \int_0^{2\pi} u(x)\,dx = 0. \end{aligned} \tag{7}$$

We will try to obtain a formula for (7). It is clearly

$$\begin{aligned} u(x) = {}& \left(1 - e^{-\sqrt{\delta+\lambda}4\pi}\right)^{-1}\Big\{\left(e^{-\sqrt{\delta+\lambda}x} - e^{-\sqrt{\delta+\lambda}(4\pi - x)}\right)u(0) \\ &+ \left(e^{-\sqrt{\delta+\lambda}(2\pi-x)} - e^{-\sqrt{\delta+\lambda}(2\pi+x)}\right)u(2\pi) - \left(e^{-\sqrt{\delta+\lambda}(2\pi-x)} - e^{-\sqrt{\delta+\lambda}(2\pi+x)}\right) \\ &\times (2\sqrt{\delta+\lambda})^{-1}\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}(2\pi-s)} - e^{-\sqrt{\delta+\lambda}(2\pi+s)}\right)\varphi(s)\,ds\Big\} \\ &+ (2\sqrt{\delta+\lambda})^{-1}\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)}\right)\varphi(s)\,ds \end{aligned} \tag{8}$$

is the solution of the following problem:

$$\begin{cases} -u''(x) + (\delta + \lambda)u(x) = \varphi(x), \quad 0 < x < 2\pi, \\ u(0), u(2\pi) \text{ are given.} \end{cases}$$

Using (8) and the conditions $\int_0^{2\pi} u(x)\,dx = 0$, $u(0) = u(2\pi)$, we get

$$\begin{aligned} \int_0^{2\pi} u(x)\,dx = {}& \left(1 - e^{-\sqrt{\delta+\lambda}4\pi}\right)^{-1}\int_0^{2\pi}\Big\{\left(e^{-\sqrt{\delta+\lambda}x} - e^{-\sqrt{\delta+\lambda}(4\pi-x)}\right)u(0) \\ &+ \left(e^{-\sqrt{\delta+\lambda}(2\pi-x)} - e^{-\sqrt{\delta+\lambda}(2\pi+x)}\right)u(0) - \left(e^{-\sqrt{\delta+\lambda}(2\pi-x)} - e^{-\sqrt{\delta+\lambda}(2\pi+x)}\right) \end{aligned}$$

$$\times (2\sqrt{\delta+\lambda})^{-1} \int_0^{2\pi} \left(e^{-\sqrt{\delta+\lambda}(2\pi-s)} - e^{-\sqrt{\delta+\lambda}(2\pi+s)}\right)\varphi(s)\,ds\Bigg\}\,dx$$

$$+ (2\sqrt{\delta+\lambda})^{-1} \int_0^{2\pi}\int_0^{2\pi} \left(e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)}\right)\varphi(s)\,ds\,dx = 0.$$

Then

$$u(0) = \left(1-e^{-\sqrt{\delta+\lambda}2\pi}\right)^{-1}(2\sqrt{\delta+\lambda})^{-1}\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}(2\pi-s)} + e^{-\sqrt{\delta+\lambda}s} - e^{-\sqrt{\delta+\lambda}2\pi} - 1\right)\varphi(s)\,ds.$$

By using the assumption $\int_0^{2\pi}\varphi(x)\,dx = 0$, we have

$$u(0) = \left(1-e^{-\sqrt{\delta+\lambda}2\pi}\right)^{-1}(2\sqrt{\delta+\lambda})^{-1}\int_0^{2\pi}\left(e^{-\sqrt{\delta+\lambda}(2\pi-s)} + e^{-\sqrt{\delta+\lambda}s}\right)\varphi(s)\,ds. \tag{9}$$

From (8) and (9) it follows that

$$u(x) = \int_0^{2\pi}\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\left(e^{-\sqrt{\delta+\lambda}(2\pi-|x-s|)} + e^{-\sqrt{\delta+\lambda}|x-s|}\right)\varphi(s)\,ds.$$

Therefore,

$$G(x,s;\lambda) = \frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\begin{cases} e^{-\sqrt{\delta+\lambda}(x-s)} + e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}, & 0\le s\le x,\\ e^{-\sqrt{\delta+\lambda}(s-x)} + e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}, & x\le s\le 2\pi.\end{cases}$$

Let $x\in R^1$. Then, from $u(x) = u(x+2\pi)$, $x\in R^1$ it follows that $G(2\pi+x,s;\lambda) = G(x,s;\lambda)$. Lemma 2.1 is proved. □

Note that the following pointwise estimates for $G(x,s;\lambda)$ and its first order derivatives hold:

$$\left|G(x,s;\lambda)\right| \le \frac{M(\delta)}{\sqrt{\delta+\lambda}}\begin{cases} e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}, & 0\le s\le x-\pi,\\ e^{-\sqrt{\delta+\lambda}(x-s)}, & x-\pi\le s\le x,\\ e^{-\sqrt{\delta+\lambda}(s-x)}, & x\le s\le \pi+x,\\ e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}, & \pi+x\le s\le 2\pi,\end{cases} \tag{10}$$

$$\left|G_x(x,s;\lambda)\right|, \left|G_s(x,s;\lambda)\right| \le M(\delta)\begin{cases} e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}, & 0< s< x-\pi,\\ e^{-\sqrt{\delta+\lambda}(x-s)}, & x-\pi< s< x,\\ e^{-\sqrt{\delta+\lambda}(s-x)}, & x< s< \pi+x,\\ e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}, & \pi+x< s< 2\pi,\end{cases} \tag{11}$$

$$\left|G(2\pi+x,s;\lambda)\right| = \left|G(x,s;\lambda)\right|,\quad x\in R^1.$$

Here, $M = \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}$.

3 The structure of fractional spaces generated by A^x and positivity of A^x in Hölder spaces

Clearly, the operator A^x and its resolvent $(A^x+\lambda)^{-1}$commute. By the definition of the norm in the fractional space $E_\alpha = E_\alpha(\mathring{C}(R^1), A^x)$, we get

$$\left\|\left(A^x+\lambda\right)^{-1}\right\|_{E_\alpha\to E_\alpha} \le \left\|\left(A^x+\lambda\right)^{-1}\right\|_{\mathring{C}(R^1)\to\mathring{C}(R^1)}.$$

Thus, from Theorem 1.1 it follows that A^x is a positive operator in the fractional spaces $E_\alpha(\mathring{C}(R^1), A^x)$. Moreover, we have the following result.

Theorem 3.1 *For $\alpha \in (0, \frac{1}{2})$, the norms of the spaces $E_\alpha(\mathring{C}(R^1), A^x)$ and the Hölder space $\mathring{C}^{2\alpha}(R^1)$ are equivalent.*

Proof For any $\lambda \geq 0$ we have the obvious equality

$$A^x\left(A^x+\lambda\right)^{-1}\varphi(x)=\varphi(x)-\lambda\left(A^x+\lambda\right)^{-1}\varphi(x).$$

By (5), we can write

$$\begin{aligned} A^x\left(A^x+\lambda\right)^{-1}\varphi(x) &= \varphi(x)-\lambda\int_0^{2\pi} G(x,s;\lambda)\varphi(s)\,ds \\ &= \frac{\delta}{\delta+\lambda}\varphi(x)+\frac{\lambda}{\delta+\lambda}\varphi(x)-\lambda\int_0^{2\pi} G(x,s;\lambda)\varphi(s)\,ds. \end{aligned} \tag{12}$$

By (5) and identity (12), we can write

$$A^x\left(A^x+\lambda\right)^{-1}\varphi(x)=\frac{\delta}{\delta+\lambda}\varphi(x)+\lambda\int_0^{2\pi} G(x,s;\lambda)\big(\varphi(x)-\varphi(s)\big)\,ds. \tag{13}$$

Then

$$\begin{aligned} \lambda^\alpha A^x\left(A^x+\lambda\right)^{-1}\varphi(x) &= \frac{\delta\lambda^\alpha}{\delta+\lambda}\varphi(x)+\lambda^{\alpha+1}\int_0^{2\pi} G(x,s;\lambda)\big(\varphi(x)-\varphi(s)\big)\,ds \\ &= P_1(x)+P_2(x), \end{aligned}$$

where

$$\begin{aligned} P_1(x) &= \frac{\delta\lambda^\alpha}{\delta+\lambda}\varphi(x), \\ P_2(x) &= \lambda^{\alpha+1}\int_0^{2\pi} G(x,s;\lambda)\big(\varphi(x)-\varphi(s)\big)\,ds. \end{aligned}$$

Using the definition of the norm of space $C^{2\alpha}(R^1)$ and $\frac{\lambda^\alpha\delta^{1-\alpha}}{\delta+\lambda} \leq 1$, we can write

$$\left|P_1(x)\right| \leq \frac{\delta\lambda^\alpha\delta^{1-\alpha}}{\delta+\lambda}\left|\varphi(x)\right| \leq \delta^\alpha \max_{x\in[0,2\pi]}\left|\varphi(x)\right| \leq \delta^\alpha \|\varphi\|_{C^{2\alpha}(R^1)}$$

for any $x \in [0, 2\pi]$. Then

$$\max_{x\in[0,2\pi]}\left|P_1(x)\right| \leq \delta^\alpha \|\varphi\|_{C^{2\alpha}(R^1)}$$

or

$$\|P_1\|_{C(R^1)} \leq \delta^\alpha \|\varphi\|_{C^{2\alpha}(R^1)}. \tag{14}$$

Then, using estimate (10), we get

$$\begin{aligned}
\left|P_2(x)\right| &\le \lambda^{\alpha+1}\int_0^{2\pi}\left|G(x,s;\lambda)\right|\left|\varphi(x)-\varphi(s)\right|ds \\
&\le \frac{M\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}}\left(\int_0^{x-\pi} e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}\left|\varphi(x)-\varphi(s)\right|ds\right. \\
&\quad+\int_{x-\pi}^{x} e^{-\sqrt{\delta+\lambda}(x-s)}\left|\varphi(x)-\varphi(s)\right|ds+\int_x^{x+\pi} e^{-\sqrt{\delta+\lambda}(s-x)}\left|\varphi(x)-\varphi(s)\right|ds \\
&\quad\left.+\int_{x+\pi}^{2\pi} e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}\left|\varphi(x)-\varphi(s)\right|ds\right) \\
&= P_{21}(x)+P_{22}(x)+P_{23}(x)+P_{24}(x),
\end{aligned}$$

where

$$\begin{aligned}
P_{21}(x) &= \frac{M(\delta)\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}}\int_0^{x-\pi} e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}\left|\varphi(x)-\varphi(s)\right|ds, \\
P_{22}(x) &= \frac{M(\delta)\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}}\int_{x-\pi}^{x} e^{-\sqrt{\delta+\lambda}(x-s)}\left|\varphi(x)-\varphi(s)\right|ds, \\
P_{23}(x) &= \frac{M(\delta)\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}}\int_x^{x+\pi} e^{-\sqrt{\delta+\lambda}(s-x)}\left|\varphi(x)-\varphi(s)\right|ds, \\
P_{24}(x) &= \frac{M(\delta)\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}}\int_{x+\pi}^{2\pi} e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}\left|\varphi(x)-\varphi(s)\right|ds.
\end{aligned}$$

Here $M(\delta)=\frac{1}{1-e^{-\sqrt{\delta}2\pi}}$.

Clearly, using the condition $\varphi(s)=\varphi(s+2\pi)$, $P_{21}(x)$ can be rewritten as

$$\begin{aligned}
P_{21}(x) &= \frac{M(\delta)\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}}\int_0^{x-\pi} e^{-\sqrt{\delta+\lambda}(2\pi-x+s)}\left|\varphi(x)-\varphi(2\pi+s)\right|ds \\
&\le \|\varphi\|_{C^{2\alpha}(R^1)}\frac{M(\delta)\lambda^{\alpha+1}}{2(\delta+\lambda)^{\alpha+1}}\Gamma(2\alpha+1)
\end{aligned}$$

for any $x\in[0,2\pi]$. Then

$$\max_{x\in[0,2\pi]} P_{21}(x)\le \|\varphi\|_{C^{2\alpha}(R^1)}\frac{M(\delta)}{\alpha}\Gamma(2\alpha+1). \tag{15}$$

Let us estimate $P_{22}(x)$.

$$\begin{aligned}
P_{22}(x) &\le \|\varphi\|_{C^{2\alpha}(R^1)}\frac{M(\delta)\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}}\int_{x-\pi}^{x} e^{-\sqrt{\delta+\lambda}(x-s)}(x-s)^{2\alpha}\,ds. \\
&\le \|\varphi\|_{C^{2\alpha}(R^1)}\frac{M(\delta)\lambda^{\alpha+1}}{2(\delta+\lambda)^{\alpha+1}}\Gamma(2\alpha+1)
\end{aligned}$$

for any $x\in[0,2\pi]$. Then

$$\max_{x\in[0,2\pi]} P_{22}(x)\le \|\varphi\|_{C^{2\alpha}(R^1)}\frac{M(\delta)}{\alpha}\Gamma(2\alpha+1). \tag{16}$$

Let us estimate $P_{23}(x)$.

$$P_{23}(x) \leq \|\varphi\|_{C^{2\alpha}(R^1)} \frac{M(\delta)\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int_x^{x+\pi} e^{-\sqrt{\delta+\lambda}(s-x)}(s-x)^{2\alpha}\, ds$$
$$\leq \|\varphi\|_{C^{2\alpha}(R^1)} \frac{M(\delta)\lambda^{\alpha+1}}{2(\delta+\lambda)^{\alpha+1}} \Gamma(2\alpha+1)$$

for any $x \in [0, 2\pi]$. Then

$$\max_{x\in[0,2\pi]} P_{23}(x) \leq \|\varphi\|_{C^{2\alpha}(R^1)} \frac{M(\delta)}{\alpha} \Gamma(2\alpha+1). \tag{17}$$

Clearly, using the condition $\varphi(x) = \varphi(x+2\pi)$, $P_{24}(x)$ can be rewritten as

$$P_{24}(x) \leq \|\varphi\|_{C^{2\alpha}(R^1)} \frac{M(\delta)\lambda^{\alpha+1}}{2\sqrt{\delta+\lambda}} \int_{x+\pi}^{2\pi} e^{-\sqrt{\delta+\lambda}(2\pi+x-s)}(2\pi+x-s)^{2\alpha}\, ds$$
$$\leq \|\varphi\|_{C^{2\alpha}(R^1)} \frac{M(\delta)\lambda^{\alpha+1}}{2(\delta+\lambda)^{\alpha+1}} \Gamma(2\alpha+1)$$

for any $x \in [0, 2\pi]$. Then

$$\max_{x\in[0,2\pi]} P_{24}(x) \leq \|\varphi\|_{C^{2\alpha}(R^1)} \frac{M(\delta)}{\alpha} \Gamma(2\alpha+1). \tag{18}$$

Combining estimates (15)-(18), we have

$$\max_{x\in[0,2\pi]} \left|P_2(x)\right| \leq \max_{x\in[0,2\pi]} P_{21}(x) + \max_{x\in[0,2\pi]} P_{22}(x) + \max_{x\in[0,2\pi]} P_{23}(x) + \max_{x\in[0,2\pi]} P_{24}(x)$$
$$\leq \|\varphi\|_{C^{2\alpha}(R^1)} \frac{M(\delta)}{\alpha} \Gamma(2\alpha+1). \tag{19}$$

Using estimate (14) and (19), we get

$$\max_{x\in[0,2\pi]} \left|\lambda^\alpha A^x \left(\lambda + A^x\right)^{-1} \varphi(x)\right| \leq \frac{M(\delta)}{\alpha} \|\varphi\|_{C^{2\alpha}(R^1)} + \frac{M(\delta)}{\alpha} \Gamma(2\alpha+1) \|\varphi\|_{C^{2\alpha}(R^1)}$$

for any $\lambda \geq 0$. Thus,

$$\|\varphi\|_{E_\alpha(C(R^1),A^x)} \leq \frac{M(\delta)}{\alpha} \left[1 + \Gamma(2\alpha+1)\right] \|\varphi\|_{C^{2\alpha}(R^1)}. \tag{20}$$

Now, let us prove the opposite inequality. For any positive operator A^x in the Banach space, we can write

$$\varphi(x) = \int_0^\infty A^x \left(\lambda + A^x\right)^{-2} \varphi(x)\, d\lambda.$$

From this relation and (5), it follows that

$$\varphi(x) = \int_0^\infty \left(A^x + \lambda\right)^{-1} A^x \left(A^x + \lambda\right)^{-1} \varphi(x)\, d\lambda$$
$$= \int_0^\infty \int_0^{2\pi} G(x, s; \lambda) A^x \left(A^x + \lambda\right)^{-1} \varphi(s)\, ds\, d\lambda.$$

Consequently,

$$\varphi(x_1)-\varphi(x_2)=\int_0^\infty\int_0^{2\pi}\big(G(x_1,s;\lambda)-G(x_2,s;\lambda)\big)A^x\big(A^x+\lambda\big)^{-1}\varphi(s)\,ds\,d\lambda$$
$$=\int_0^\infty\lambda^{-\alpha}\int_0^{2\pi}\big(G(x_1,s;\lambda)-G(x_2,s;\lambda)\big)\lambda^{\alpha}A^x\big(A^x+\lambda\big)^{-1}\varphi(s)\,ds\,d\lambda.$$

Therefore,

$$\big|\varphi(x_1)-\varphi(x_2)\big|\le\int_0^\infty\lambda^{-\alpha}\int_0^{2\pi}\big|G(x_1,s;\lambda)-G(x_2,s;\lambda)\big|\,ds\,d\lambda\,\|\varphi\|_{E_\alpha(C^{2\alpha}(R^1),A^x)}.$$

Let

$$T=|x_1-x_2|^{-2\alpha}\left(\int_0^\infty\lambda^{-\alpha}\int_0^{2\pi}\big|G(x_1,s;\lambda)-G(x_2,s;\lambda)\big|\,ds\,d\lambda\right).$$

Then, for any $x_1,x_2\in[0,2\pi]$ such that $x_2\ge x_1$, we have

$$\frac{|\varphi(x_1)-\varphi(x_2)|}{|x_1-x_2|^{2\alpha}}\le T\|\varphi\|_{E_\alpha(C^{2\alpha}(R^1),A^x)}.$$

Now, we will prove that

$$T\le\frac{M(\delta)}{2\alpha(1-2\alpha)}. \tag{21}$$

Using (6), we get

$$\big|G(x_1,s;\lambda)-G(x_2,s;\lambda)\big|\le\left|\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\right|$$
$$\times\begin{cases}|e^{-\sqrt{\delta+\lambda}(2\pi-x_1+s)}-e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}|, & 0\le s\le x_1-\pi,\\ |e^{-\sqrt{\delta+\lambda}(x_1-s)}-e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}|, & x_1-\pi\le s\le x_2-\pi,\\ |e^{-\sqrt{\delta+\lambda}(x_1-s)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}|, & x_2-\pi\le s\le x_1,\\ |e^{-\sqrt{\delta+\lambda}(s-x_1)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}|, & x_1\le s\le\frac{x_1+x_2}{2},\\ |e^{-\sqrt{\delta+\lambda}(s-x_1)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}|, & \frac{x_1+x_2}{2}\le s\le x_2,\\ |e^{-\sqrt{\delta+\lambda}(s-x_1)}-e^{-\sqrt{\delta+\lambda}(s-x_2)}|, & x_2\le s\le x_1+\pi,\\ |e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}-e^{-\sqrt{\delta+\lambda}(s-x_2)}|, & x_1+\pi\le s\le x_2+\pi,\\ |e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}-e^{-\sqrt{\delta+\lambda}(2\pi+x_2-s)}|, & x_2+\pi\le s\le 2\pi.\end{cases}$$

Then

$$T=|x_1-x_2|^{-2\alpha}\int_0^\infty\lambda^{-\alpha}$$
$$\times\left[\int_0^{x_1-\pi}\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\big|e^{-\sqrt{\delta+\lambda}(2\pi-x_1+s)}-e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}\big|\,ds\right.$$
$$+\int_{x_1-\pi}^{x_2-\pi}\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\big|e^{-\sqrt{\delta+\lambda}(x_1-s)}-e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}\big|\,ds$$
$$+\int_{x_2-\pi}^{x_1}\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\big|e^{-\sqrt{\delta+\lambda}(x_1-s)}-e^{-\sqrt{\delta+\lambda}(x_2-s)}\big|\,ds$$

$$
\begin{aligned}
&+\int_{x_1}^{\frac{x_1+x_2}{2}} \frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \left|e^{-\sqrt{\delta+\lambda}(s-x_1)} - e^{-\sqrt{\delta+\lambda}(x_2-s)}\right| ds \\
&+\int_{\frac{x_1+x_2}{2}}^{x_2} \frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \left|e^{-\sqrt{\delta+\lambda}(s-x_1)} - e^{-\sqrt{\delta+\lambda}(x_2-s)}\right| ds \\
&+\int_{x_2}^{x_1+\pi} \frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \left|e^{-\sqrt{\delta+\lambda}(s-x_1)} - e^{-\sqrt{\delta+\lambda}(s-x_2)}\right| ds \\
&+\int_{x_1+\pi}^{x_2+\pi} \frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \left|e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)} - e^{-\sqrt{\delta+\lambda}(s-x_2)}\right| ds \\
&+\int_{x_2+\pi}^{2\pi} \frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \left|e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)} - e^{-\sqrt{\delta+\lambda}(2\pi+x_2-s)}\right| ds\Bigg] d\lambda \\
&= T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8.
\end{aligned}
$$

Let us estimate the expression

$$
\begin{aligned}
T_1 = {}& |x_1-x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_0^{x_1-\pi} \left(\frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \right. \\
&\times \left. \left|e^{-\sqrt{\delta+\lambda}(2\pi-x_1+s)} - e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}\right| ds \right) d\lambda.
\end{aligned}
$$

For $0 \le s \le x_1 - \pi$, using the estimates

$$
\begin{aligned}
&\left|e^{-\sqrt{\delta+\lambda}(2\pi-x_1+s)} - e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}\right| \le 2e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}, \\
&\left|e^{-\sqrt{\delta+\lambda}(2\pi-x_1+s)} - e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}\right| \le \sqrt{\delta+\lambda}2e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}(x_2-x_1),
\end{aligned}
$$

we can write

$$
\begin{aligned}
T_1 \le{}& |x_1-x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \left(\int_0^{x_1-\pi} \frac{1}{\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \right. \\
&\times \left. \left[\sqrt{\delta+\lambda} e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)}(x_2-x_1)\, ds\right] \right) d\lambda \\
\le{}& M_2(\delta) \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} e^{-\sqrt{\delta+\lambda}(\pi-x_2+x_1)}\, d\lambda \\
\le{}& M_2(\delta) \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}(\pi-x_2+x_1)}\, d\lambda.
\end{aligned}
$$

Using the substitutions $\sqrt{\lambda} = \tau$ and $\pi - x_2 + x_1 = a$, we get

$$
T_1 \le M_2(\delta)\left(\int_0^1 \tau^{-2\alpha}\, d\tau + \int_1^\infty e^{-\tau a}\, d\tau \right) \le M_2(\delta)\left(\frac{1}{1-2\alpha} + \frac{1}{a} e^{-a} \right). \tag{22}
$$

Using (22), we get

$$
T_1 \le \frac{M_2(\delta, a)}{1-2\alpha}. \tag{23}
$$

Now, let us estimate the expression

$$T_2 = |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_{x_1-\pi}^{x_2-\pi} \left(\frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \times \left| e^{-\sqrt{\delta+\lambda}(x_1-s)} - e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)} \right| ds \right) d\lambda.$$

Using

$$\Delta_1 = \int_{x_1-\pi}^{x_2-\pi} \left| e^{-\sqrt{\delta+\lambda}(x_1-s)} - e^{-\sqrt{\delta+\lambda}(2\pi-x_2+s)} \right| ds$$

and the substitution

$$\frac{x_1 - s - \pi}{x_2 - x_1} = -y,$$

we have

$$\begin{aligned}\Delta_1 &= (x_2 - x_1) \int_0^1 \left(e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1)y)} - e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1)(1-y))} \right) dy \\ &\le (x_2 - x_1)^2 \sqrt{\delta+\lambda} \int_0^1 e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))z}\, dz.\end{aligned}$$

Then

$$\begin{aligned}T_2 &\le |x_1 - x_2|^{-2\alpha} \left(\int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \right. \\ &\quad \left. \times (x_2 - x_1)^2 \sqrt{\delta+\lambda} \int_0^1 e^{-\sqrt{\delta+\lambda}(\pi-(x_2-x_1))z}\, dz \right) d\lambda \\ &\le M_3(\delta) \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} e^{-\sqrt{\delta+\lambda}(\pi-x_2+x_1)}\, d\lambda \\ &\le M_3(\delta) \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\lambda}} e^{-\sqrt{\lambda}(\pi-x_2+x_1)}\, d\lambda.\end{aligned}$$

Using the substitutions $\sqrt{\lambda} = \tau$ and $\pi - x_2 + x_1 = a$, we get

$$T_2 \le M_3(\delta) \left(\int_0^1 \tau^{-2\alpha} e^{-\tau a}\, d\tau + \int_1^\infty \tau^{-2\alpha} e^{-\tau a}\, d\tau \right) \le M_4(\delta) \left(\frac{1}{1-2\alpha} + \frac{1}{a} e^{-a} \right). \tag{24}$$

Using (24), we obtain

$$T_2 \le \frac{M_4(\delta, a)}{1 - 2\alpha}. \tag{25}$$

Now, let us estimate the expression

$$T_3 = |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_{x_2-\pi}^{x_1} \left(\frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \times \left| e^{-\sqrt{\delta+\lambda}(x_1-s)} - e^{-\sqrt{\delta+\lambda}(x_2-s)} \right| ds \right) d\lambda.$$

We have

$$
\begin{aligned}
T_3 &= |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta+\lambda}} \\
&\quad \times \left(1 - e^{-\sqrt{\delta+\lambda}(x_2-x_1)}\right)\left(1 - e^{-\sqrt{\delta+\lambda}2\pi}\right)^{-1} \int_{x_2-\pi}^{x_1} e^{-\sqrt{\delta+\lambda}(x_1-s)}\,ds\,d\lambda \\
&\le M_5(\delta)|x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{1}{2(\delta+\lambda)}\left(1 - e^{-\sqrt{\delta+\lambda}(x_2-x_1)}\right) d\lambda \\
&\le M_5(\delta)|x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{1}{2\lambda}\left(1 - e^{-\sqrt{\lambda}(x_2-x_1)}\right) d\lambda.
\end{aligned}
$$

Using the substitution $\sqrt{\lambda}(x_2 - x_1) = \tau$, we get

$$
T_3 \le M_5(\delta) \int_0^\infty \frac{1 - e^{-\tau}}{\tau^{1+2\alpha}}\,d\tau = M_5(\delta) \int_0^\infty \frac{1}{\tau^{1+2\alpha}} \int_0^\tau e^{-s}\,ds\,d\tau.
$$

Let $0 \le s \le \infty$ and $s \le \tau \le \infty$. Then

$$
T_3 \le \frac{M_5(\delta)}{2\alpha} \int_0^\infty e^{-s} s^{-2\alpha}\,ds = \frac{M_5(\delta)}{2\alpha}\Gamma(2\alpha). \tag{26}
$$

Using (26), we obtain

$$
T_3 \le \frac{M_5(\delta)}{2\alpha(1-2\alpha)}. \tag{27}
$$

Let us estimate the expression

$$
\begin{aligned}
T_4 &= |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_{x_1}^{\frac{x_1+x_2}{2}} \left(\frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1 - e^{-\sqrt{\delta+\lambda}2\pi}} \right. \\
&\quad \left. \times \left| e^{-\sqrt{\delta+\lambda}(s-x_1)} - e^{-\sqrt{\delta+\lambda}(x_2-s)} \right| ds \right) d\lambda.
\end{aligned}
$$

We have

$$
\begin{aligned}
T_4 &\le |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{1}{2(\delta+\lambda)}\left(1 - e^{-\sqrt{\delta+\lambda}(\frac{x_2-x_1}{2})}\right) d\lambda \\
&\le |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{1}{2\lambda}\left(1 - e^{-\sqrt{\lambda}(\frac{x_2-x_1}{2})}\right) d\lambda.
\end{aligned}
$$

Using the substitution $\sqrt{\lambda}(\frac{x_2-x_1}{2}) = \tau$, we get

$$
T_4 \le \frac{M_6(\delta)}{4^\alpha} \int_0^\infty \frac{1 - e^{-\tau}}{\tau^{1+2\alpha}}\,d\tau = \frac{M_6(\delta)}{4^\alpha} \int_0^\infty \frac{1}{\tau^{1+2\alpha}} \left(\int_0^\tau e^{-s}\,ds \right) d\tau.
$$

Let $0 \le s \le \infty$ and $s \le \tau \le \infty$. Then

$$
T_4 \le \frac{M_6(\delta)}{4^\alpha 2\alpha} \int_0^\infty e^{-s} s^{-2\alpha}\,ds = \frac{M_6(\delta)}{4^\alpha \alpha}\Gamma(2\alpha) \le \frac{M_6(\delta)}{4^\alpha 2\alpha(1-2\alpha)}. \tag{28}
$$

Using (28), we have

$$T_4 \le \frac{M_6(\delta)}{2\alpha(1-2\alpha)}. \tag{29}$$

Let us estimate the expression

$$T_5 = |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_{\frac{x_1+x_2}{2}}^{x_2} \left(\frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \times \left| e^{-\sqrt{\delta+\lambda}(s-x_1)} - e^{-\sqrt{\delta+\lambda}(x_2-s)} \right| ds \right) d\lambda.$$

We have

$$\begin{aligned} T_5 &\le |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{1}{2(\delta+\lambda)} \left(1 - e^{-\sqrt{\delta+\lambda}(\frac{x_2-x_1}{2})}\right) d\lambda \\ &\le |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{1}{2\lambda} \left(1 - e^{-\sqrt{\lambda}(\frac{x_2-x_1}{2})}\right) d\lambda. \end{aligned}$$

Using the substitution $\sqrt{\lambda}(\frac{x_2-x_1}{2}) = \tau$, we get

$$T_5 \le \frac{M_7(\delta)}{4^\alpha} \int_0^\infty \frac{1-e^{-\tau}}{\tau^{1+2\alpha}} d\tau = \frac{M_7(\delta)}{4^\alpha} \int_0^\infty \frac{1}{\tau^{1+2\alpha}} \left(\int_0^\tau e^{-s} ds \right) d\tau.$$

Let $0 \le s \le \infty$ and $s \le \tau \le \infty$. Then

$$T_5 \le \frac{M_7(\delta)}{4^\alpha 2\alpha} \int_0^\infty e^{-s} s^{-2\alpha} ds = \frac{M_7(\delta)}{4^\alpha \alpha} \Gamma(2\alpha) \le \frac{M_7(\delta)}{4^\alpha 2\alpha(1-2\alpha)}. \tag{30}$$

From (30) it follows that

$$T_5 \le \frac{M_7(\delta)}{2\alpha(1-2\alpha)}. \tag{31}$$

Let us estimate the expression

$$T_6 = |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_{x_2}^{x_1+\pi} \left(\frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \times \left| e^{-\sqrt{\delta+\lambda}(s-x_1)} - e^{-\sqrt{\delta+\lambda}(s-x_2)} \right| ds \right) d\lambda.$$

We have

$$\begin{aligned} T_6 &= |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_{x_2}^{x_1+\pi} \left(\frac{1}{2\sqrt{\delta+\lambda}} \frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}} \times e^{-\sqrt{\delta+\lambda}(s-x_2)} \left(1 - e^{-\sqrt{\delta+\lambda}(x_2-x_1)}\right) ds \right) d\lambda \\ &\le |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{1}{2(\delta+\lambda)} \left(1 - e^{-\sqrt{\delta+\lambda}(x_2-x_1)}\right) d\lambda \\ &\le |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \frac{1}{2\lambda} \left(1 - e^{-\sqrt{\lambda}(x_2-x_1)}\right) d\lambda. \end{aligned}$$

Using the substitution $\sqrt{\lambda}(x_2 - x_1) = \tau$, we get

$$T_6 \le M_8(\delta) \int_0^\infty \frac{1 - e^{-\tau}}{\tau^{1+2\alpha}} \, d\tau = M_8(\delta) \int_0^\infty \frac{1}{\tau^{1+2\alpha}} \int_0^\tau e^{-s} \, ds \, d\tau.$$

Let $0 \le s \le \infty$ and $s \le \tau \le \infty$. Then

$$T_6 \le \frac{M_9(\delta)}{2\alpha} \int_0^\infty e^{-s} s^{-2\alpha} \, ds \le \frac{M_9(\delta)}{2\alpha} \Gamma(2\alpha). \tag{32}$$

Using (32), we have

$$T_6 \le \frac{M_9(\delta)}{2\alpha(1 - 2\alpha)}. \tag{33}$$

Let us estimate the expression

$$T_7 = |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_{x_1+\pi}^{x_2+\pi} \left(\frac{1}{2\sqrt{\delta + \lambda}} \frac{1}{1 - e^{-\sqrt{\delta+\lambda} 2\pi}} \right.$$
$$\left. \times \left| e^{-\sqrt{\delta+\lambda}(2\pi + x_1 - s)} - e^{-\sqrt{\delta+\lambda}(s - x_2)} \right| ds \right) d\lambda.$$

Using

$$\Delta_2 = \int_{x_1+\pi}^{x_2+\pi} \left| e^{-\sqrt{\delta+\lambda}(2\pi + x_1 - s)} - e^{-\sqrt{\delta+\lambda}(s - x_2)} \right| ds$$

and the substitution

$$\frac{\pi + x_2 - s}{x_2 - x_1} = y,$$

we obtain

$$\Delta_2 = \int_1^0 -(x_2 - x_1) \left(e^{-\sqrt{\delta+\lambda}(\pi - (x_2 - x_1)(1 - y))} - e^{-\sqrt{\delta+\lambda}(\pi - (x_2 - x_1) y)} \right) dy$$
$$\le (x_2 - x_1)^2 \sqrt{\delta + \lambda} \int_0^1 \left| -e^{-\sqrt{\delta+\lambda}(\pi - (x_2 - x_1)) z} \right| dz.$$

Let us estimate the expression

$$T_7 \le |x_1 - x_2|^{-2\alpha} \int_0^\infty \left(\lambda^{-\alpha} \frac{1}{2\sqrt{\delta + \lambda}} \frac{1}{1 - e^{-\sqrt{\delta+\lambda} 2\pi}} \right.$$
$$\left. \times (x_2 - x_1)^2 \sqrt{\delta + \lambda} \int_0^1 -e^{-\sqrt{\delta+\lambda}(\pi - (x_2 - x_1)) z} \, dz \right) d\lambda$$
$$\le M_{10}(\delta) \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\delta + \lambda}} \left(e^{-\sqrt{\delta+\lambda}(\pi - x_2 + x_1)} \right) d\lambda$$
$$\le M_{10}(\delta) \int_0^\infty \lambda^{-\alpha} \frac{1}{2\sqrt{\lambda}} \left(e^{-\sqrt{\lambda}(\pi - x_2 + x_1)} \right) d\lambda.$$

Using the substitutions $\sqrt{\lambda}=\tau$ and $(\pi - x_2 + x_1) = a$, we get

$$T_7 \le 2M_{10}(\delta)\left(\int_0^1 \tau^{-2\alpha}\,d\tau + \int_1^\infty e^{-\tau a}\,d\tau\right) = M_{11}(\delta)\left(\frac{1}{1-2\alpha}+\frac{1}{a}e^{-a}\right). \tag{34}$$

Using (34), we get

$$T_7 \le \frac{M_{11}(\delta,a)}{2\alpha(1-2\alpha)}. \tag{35}$$

Let us estimate the expression

$$\begin{aligned}T_8 = {}&|x_1-x_2|^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_{x_2+\pi}^{2\pi}\left(\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\right.\\ &\left.\times \left|e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}-e^{-\sqrt{\delta+\lambda}(2\pi+x_2-s)}\right|ds\right)d\lambda.\end{aligned}$$

For $x_2+\pi \le s \le 2\pi$, using the estimates

$$\begin{aligned}&\left|e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}-e^{-\sqrt{\delta+\lambda}(2\pi+x_2-s)}\right| \le 2e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)},\\ &\left|e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}-e^{-\sqrt{\delta+\lambda}(2\pi+x_2-s)}\right| \le \sqrt{\delta+\lambda}2e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}(x_2-x_1),\end{aligned}$$

we can write

$$\begin{aligned}T_8 \le {}&|x_1-x_2|^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_{x_2+\pi}^{2\pi}\left(\frac{1}{2\sqrt{\delta+\lambda}}\frac{1}{1-e^{-\sqrt{\delta+\lambda}2\pi}}\right.\\ &\left.\times \sqrt{\delta+\lambda}2e^{-\sqrt{\delta+\lambda}(2\pi+x_1-s)}(x_2-x_1)\,ds\right)d\lambda\\ \le{}& M_{12}(\delta)\int_0^\infty \lambda^{-\alpha}\frac{1}{\sqrt{\delta+\lambda}}e^{-\sqrt{\delta+\lambda}x_1}\,d\lambda\\ \le{}& 2M_{12}(\delta)\int_0^\infty \lambda^{-\alpha}\frac{1}{2\sqrt{\lambda}}e^{-\sqrt{\lambda}x_1}\,d\lambda.\end{aligned}$$

Using the substitutions $\sqrt{\lambda}=\tau$ and $x_1 = a$, we get

$$T_8 \le M_{13}(\delta)\left(\int_0^1 \tau^{-2\alpha}\,d\tau + \int_1^\infty e^{-\tau a}\,d\tau\right) \le M_{13}(\delta)\left(\frac{1}{1-2\alpha}+\frac{1}{a}e^{-a}\right). \tag{36}$$

Equation (36) yields

$$T_8 \le \frac{M_{13}(\delta,a)}{2\alpha(1-2\alpha)}. \tag{37}$$

Applying the triangle inequality and estimates (21), (23), (25), (27), (29), (31), (33), (35), and (37), we get

$$T \le \frac{M(\delta,a)}{2\alpha(1-2\alpha)}.$$

So, (21) is proved. Thus, for any $x_1, x_2 \in [0, 2\pi]$ we have

$$|x_1 - x_2|^{-2\alpha} \left|\varphi(x_1) - \varphi(x_2)\right| \leq \frac{M(\delta, a)}{2\alpha(1-2\alpha)} \|\varphi\|_{E_\alpha(C(R^1), A^x)}.$$

This means that the following inequality holds:

$$\|\varphi\|_{\mathring{C}^{2\alpha}(R^1)} \leq \frac{M(\delta, a)}{2\alpha(1-2\alpha)} \|\varphi\|_{E_\alpha(C(R^1), A^x).} \tag{38}$$

Estimates (20) and (38) finish the proof of Theorem 3.1. □

Since A^x is a positive operator in the fractional spaces $E_\alpha(\mathring{C}(R^1), A^x)$, from the result of Theorem 3.1 it follows also it is positive operator in the Hölder space $\mathring{C}^{2\alpha}(R^1)$. Namely, we have the following.

Theorem 3.2 *The operator* $(A^x + \lambda)$ *has an inverse bounded in* $\mathring{C}^{2\alpha}(R^1)$ *for any* $\lambda \geq 0$ *and the following estimate holds*:

$$\left\|\left(A^x + \lambda\right)^{-1}\right\|_{\mathring{C}^{2\alpha}(R^1) \to \mathring{C}^{2\alpha}(R^1)} \leq \frac{M(\delta, a)}{2\alpha(1-2\alpha)} \frac{M_{14}}{\delta + \lambda}.$$

4 Applications

First, we consider the boundary value problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) = f(t,x), & 0 < t < T, x \in R^1, \\ u(0,x) = \varphi(x), \qquad u(T,x) = \psi(x), & x \in R^1, \\ u(t,x) = u(t, x + 2\pi), \quad \int_0^{2\pi} u(t,s)\, ds = 0, 0 \leq t \leq T, x \in R^1. \end{cases} \tag{39}$$

Here, $\varphi(x)$, $\psi(x)$, and $f(t,x)$ are sufficiently smooth 2π-periodic functions in x and they satisfy any compatibility conditions which guarantee problem (39) has a smooth solution $u(t,x)$.

Theorem 4.1 *Let* $0 < 2\alpha < 1$. *Then, for the solution of the boundary value problem* (39), *we have the following coercive stability inequality*:

$$\begin{aligned} &\|u_{tt}\|_{C([0,T], \mathring{C}^{2\alpha}(R^1))} + \|u\|_{C([0,T], \mathring{C}^{2\alpha+2}(R^1))} \\ &\quad \leq M(\alpha)\left[\|\varphi\|_{\mathring{C}^{2\alpha+2}(R^1)} + \|\psi\|_{\mathring{C}^{2\alpha+2}(R^1)} + \|f\|_{C([0,T], \mathring{C}^{2\alpha}(R^1))}\right]. \end{aligned}$$

The proof of Theorem 4.1 is based on Theorem 3.1 on the structure of the fractional spaces $E_\alpha = E_\alpha(C(R^1), A^x)$, Theorem 1.1 on the positivity of the operator A^x, on the following theorems on coercive stability of boundary value for the abstract elliptic equation and on the structure of the fractional space $E'_\alpha = E_\alpha(E, A^{1/2})$ which is the Banach space consisting of those $v \in E$ for which the norm

$$\|v\|_{E'_\alpha} = \sup_{\lambda > 0} \lambda^\alpha \left\|A^{1/2}\left(\lambda + A^{1/2}\right)^{-1} v\right\|_{E_\alpha} + \|v\|_E$$

is finite.

Theorem 4.2 [5] *The spaces* $E_\alpha(E,A)$ *and* $E'_{2\alpha}(E,A^{1/2})$ *coincide for any* $0<\alpha<\frac{1}{2}$, *and their norms are equivalent.*

Theorem 4.3 [8] *Let* A *be positive operator in a Banach space* E *and* $f\in C([0,T],E'_\alpha)$, $0<\alpha<1$. *Then, for the solution of the nonlocal boundary value problem*

$$\begin{cases}-u''+Au(t)=f(t), & 0<t<T,\\ u(0)=\varphi, & u(T)=\psi,\end{cases} \tag{40}$$

in a Banach space E *with positive operator* A, *we have the coercive inequality*

$$\begin{aligned}&\|u''\|_{C([0,T],E'_\alpha)}+\|Au\|_{C([0,T],E'_\alpha)}\\ &\quad\le M\left[\|A\varphi\|_{E'_\alpha}+\|A\psi\|_{E'_\alpha}+\frac{M}{\alpha(1-\alpha)}\|f\|_{C([0,T],E'_\alpha)}\right].\end{aligned}$$

Second, we consider the nonlocal boundary value problem for the elliptic equation

$$\begin{cases}-\frac{\partial^2u(t,x)}{\partial t^2}-\frac{\partial^2u(t,x)}{\partial x^2}+\delta u(t,x)=f(t,x), & 0<t<T, x\in R^1,\\ u(0,x)=u(T,x), \quad u_t(0,x)=u_t(T,x), & x\in R^1,\\ u(t,x)=u(t,x+2\pi), \quad \int_0^{2\pi}u(t,s)\,ds=0, & 0\le t\le T, x\in R^1.\end{cases} \tag{41}$$

Here, $f(t,x)$ *is a sufficiently smooth* 2π*-periodic function in* x *and it satisfies any compatibility conditions which guarantee problem* (41) *has a smooth solution* $u(t,x)$.

Theorem 4.4 *Let* $0<2\alpha<1$. *Then, for the solution of boundary value problem* (39), *we have the following coercive stability inequality*:

$$\|u_{tt}\|_{C([0,T],\mathring{C}^{2\alpha}(R^1))}+\|u\|_{C([0,T],\mathring{C}^{2\alpha+2}(R^1))}\le M(\alpha)\|f\|_{C([0,T],\mathring{C}^{2\alpha}(R^1))}.$$

The proof of Theorem 4.4 is based on Theorem 3.1 on the structure of the fractional spaces $E_\alpha=E_\alpha(\mathring{C}(R^1),A^x)$, Theorem 1.1 on the positivity of the operator A^x, Theorem 3.2 on the structure of the fractional space $E'_\alpha=E_\alpha(E,A^{1/2})$ and on the following theorem on the coercive stability of the nonlocal boundary value for the abstract elliptic equation.

Theorem 4.5 [25] *Let* A *be a positive operator in a Banach space* E *and* $f\in C([0,T],E'_\alpha)$, $0<\alpha<1$. *Then, for the solution of the nonlocal boundary value problem*

$$\begin{cases}-u''+Au(t)=f(t), & 0<t<T,\\ u(0)=u(T), & u'(0)=u'(T),\end{cases} \tag{42}$$

in a Banach space E *with positive operator* A *the coercive inequality*

$$\|u''\|_{C([0,T],E'_\alpha)}+\|Au\|_{C([0,T],E'_\alpha)}\le\frac{M}{\alpha(1-\alpha)}\|f\|_{C([0,T],E'_\alpha)}$$

holds.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
FST has produced more than half of the results of the present paper. All authors read and approved the final manuscript.

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