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The existence of solutions for impulsive *p*-Laplacian boundary value problems at resonance on the half-line

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Abstract

By using the continuous theorem of Ge and Ren and constructing suitable Banach spaces and operators, we investigate the existence of solutions for an impulsive *p*-Laplacian boundary value problem with integral boundary condition at resonance on the half-line. An example is given to illustrate our main results.

MSC: 34B40

Keywords: impulsive; *p*-Laplacian operator; boundary value problem; integral boundary condition; resonance

1 Introduction

Boundary value problems on the half-line arise in various applications such as in the study of the unsteady flow of a gas through semi-infinite porous medium, in analyzing the heat transfer in radial flow between circular disks, in the study of plasma physics, in an analysis of the mass transfer on a rotating disk in a non-Newtonian fluid, *etc.* [1]

Many dynamical systems have an impulsive dynamical behavior due to abrupt changes at certain instants during the evolution process. The mathematical description of these phenomena leads to impulsive differential equations. For some general and recent works on the theory of impulsive differential equations we refer the reader to [2–4]. Impulsive differential equations occur in biology, medicine, mechanics, engineering, chaos theory, *etc.* [5–9]. Impulsive boundary value problems have been studied by many papers; see [10–15]. For example, in [14], the authors studied the existence of solutions for the problem

$$\begin{cases} (p(t)u'(t))' = f(t,u(t)), & t \in (0,\infty)\backslash\{t_1,t_2,\ldots,t_n\}, \\ \triangle u'(t_k) = I_k(u(t_k)), & k = 1,2,\ldots,n, \\ \alpha u(0) - \beta \lim_{t\to 0^+} p(t)u'(t) = 0, \\ \gamma \lim_{t\to\infty} u(t) + \delta \lim_{t\to\infty} p(t)u'(t) = 0. \end{cases}$$

In [15], the impulsive boundary value problem on the half-line

$$\begin{cases} \frac{1}{p(t)}(p(t)x'(t))' = f(t,x_t), & t \in (0,\infty)\backslash\{t_1,t_2,\ldots,t_n\}, \\ \triangle x'(t_k) = I_k(x_{t_k}), & k = 1,2,\ldots,m, \\ \lambda x(0) - \beta \lim_{t\to 0^+} p(t)x'(t) = a, \\ \gamma x(\infty) + \delta \lim_{t\to\infty} p(t)x'(t) = b \end{cases}$$

was studied.

A boundary value problem is said to be a resonance one if the corresponding homogeneous boundary value problem has a non-trivial solution. The boundary value problems at resonance have been studied by many papers; see [16–22]. In [22], the author gave the existence of solutions for the p-Laplacian boundary value problem at resonance on the half-line

$$\begin{cases} (\varphi_p(u'))'(t) = \psi(t)f(t,u(t),u'(t)), \quad t \in [0,+\infty), \\ u'(+\infty) = 0, \qquad u(0) = \int_0^{+\infty} h(t)u(t)\,dt, \end{cases}$$

where $\varphi_p(s) = |s|^{p-2}s$, $p > 1$.

As far as we know, the impulsive p-Laplacian boundary value problems at resonance on the half-line have not been investigated. In this paper, we will discuss the existence of solutions for the problem

$$\begin{cases} (\varphi_p(u'(t)))' + f(t,u(t),u'(t)) = 0, \quad t \in [0,\infty)\backslash\{t_1,t_2,\ldots,t_k\}, \\ \triangle\varphi_p(u'(t_i)) = I_i(u(t_i),u'(t_i)), \quad i = 1,2,\ldots,k, \\ u(0) = 0, \qquad \varphi_p(u'(+\infty)) = \int_0^{+\infty} h(t)\varphi_p(u'(t))\,dt, \end{cases} \tag{1.1}$$

where $0 < t_1 < t_2 < \cdots < t_k < +\infty$, $\triangle\varphi_p(u'(t_i)) = \varphi_p(u'(t_i+0)) - \varphi_p(u'(t_i-0))$.

In this paper, we will always suppose that the following conditions hold.

(H_1) $h(t) \geq 0$, $t \in [0,+\infty)$, $\int_0^{+\infty} h(t)\,dt = 1$, $f : [0,+\infty) \times \mathbb{R}^2 \to \mathbb{R}$, and $I_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1,2,\ldots,k$ are continuous.

(H_2) For any constant $r > 0$, there exist a function $h_r \in L[0,+\infty)$ and a constant $M_r > 0$, such that $|f(t,(1+t)u,v)| \leq h_r(t)$, $t \in [0,+\infty)$, $|u| < r$, $|v| < r$, $|I_i(u,v)| \leq M_r$, $i = 1,2,\ldots,k$, $|u| \leq r(1+t_k)$, $|v| \leq r$.

2 Preliminaries

For convenience, we introduce some notations and a theorem. For more details see [23].

Definition 2.1 [23] Let X and Y be two Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively. A continuous operator $M : X \cap \operatorname{dom} M \to Y$ is said to be quasi-linear if

(i) $\operatorname{Im} M := M(X \cap \operatorname{dom} M)$ is a closed subset of Y,

(ii) $\operatorname{Ker} M := \{x \in X \cap \operatorname{dom} M : Mx = 0\}$ is linearly homeomorphic to $\mathbb{R}^n$, $n < \infty$,

where $\operatorname{dom} M$ denote the domain of the operator M.

Let $X_1 = \operatorname{Ker} M$ and X_2 be the complement space of X_1 in X, then $X = X_1 \oplus X_2$. On the other hand, suppose Y_1 is a subspace of Y and that Y_2 is the complement of Y_1 in Y, *i.e.* $Y = Y_1 \oplus Y_2$. Let $P : X \to X_1$ and $Q : Y \to Y_1$ be two projectors and $\Omega \subset X$ an open and bounded set with the origin $\theta \in \Omega$.

Definition 2.2 [23] Suppose that $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is a continuous operator. Denote N_1 by N. Let $\Sigma_\lambda = \{x \in \overline{\Omega} : Mx = N_\lambda x\}$. N_λ is said to be M-compact in $\overline{\Omega}$ if there exist a vector subspace Y_1 of Y satisfying $\dim Y_1 = \dim X_1$ and an operator $R : \overline{\Omega} \times [0,1] \to X_2$ being continuous and compact such that for $\lambda \in [0,1]$,

(a) $(I-Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Y$,

(b) $QN_\lambda x = \theta, \lambda \in (0,1) \Leftrightarrow QNx = \theta$,

(c) $R(\cdot,0)$ is the zero operator and $R(\cdot,\lambda)|_{\Sigma_\lambda} = (I-P)|_{\Sigma_\lambda}$,
(d) $M[P+R(\cdot,\lambda)] = (I-Q)N_\lambda$.

Theorem 2.1 [23] *Let X and Y be two Banach spaces with the norms $\|\cdot\|_X$, $\|\cdot\|_Y$, respectively, and $\Omega \subset X$ an open and bounded nonempty set. Suppose that*

$$M : X \cap \operatorname{dom} M \to Y$$

is a quasi-linear operator and $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is M-compact. In addition, if the following conditions hold:

(C_1) $Mx \neq N_\lambda x$, $\forall x \in \partial\Omega \cap \operatorname{dom} M$, $\lambda \in (0,1)$,
(C_2) $\deg\{JQN, \Omega \cap \operatorname{Ker} M, 0\} \neq 0$,

then the abstract equation $Mx = Nx$ has at least one solution in $\operatorname{dom} M \cap \overline{\Omega}$, where $N = N_1$, $J : \operatorname{Im} Q \to \operatorname{Ker} M$ is a homeomorphism with $J(\theta) = \theta$.

3 Main results

In the following, we will always suppose that q satisfies $1/p + 1/q = 1$.

Let $\mathbb{R}^+ = [0,+\infty)$, $J' = \mathbb{R}^+ \backslash \{t_1, t_2, \ldots, t_k\}$, $Y = L(\mathbb{R}^+)$ with norm $\|y\|_1 = \int_0^{+\infty} |y(t)|\,dt$,

$$\begin{aligned} PC^1(\mathbb{R}^+) = \bigl\{ u : u \in C^1(J'), u'(t_i - 0), u'(t_i + 0) \text{ exist and} \\ u'(t_i - 0) = u'(t_i), i = 1,2,\ldots,k \bigr\}, \end{aligned}$$

$$X = \left\{ u : u(0) = 0, u \in C(\mathbb{R}^+) \cap PC^1(\mathbb{R}^+), \sup_{t\in\mathbb{R}^+} \frac{|u(t)|}{1+t} < +\infty, \lim_{t\to+\infty} u'(t) \text{ exists} \right\}$$

with norm $\|u\| = \max\{\|\frac{u}{1+t}\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty = \sup_{t\in\mathbb{R}^+} |u(t)|$.

Let $Z = Y \times \mathbb{R}^k$, with norm $\|(y, c_1, c_2, \ldots, c_k)\| = \max\{\|y\|_1, |c_1|, |c_2|, \ldots, |c_k|\}$. Then $(X, \|\cdot\|)$ and $(Z, \|\cdot\|)$ are Banach spaces.

Define the operators $M : X \cap \operatorname{dom} M \to Z$, $N_\lambda : X \to Z$ as follows:

$$Mu = \begin{bmatrix} (\varphi_p(u'))'(t) \\ \triangle\varphi_p(u'(t_1)) \\ \cdots \\ \triangle\varphi_p(u'(t_k)) \end{bmatrix}, \qquad N_\lambda u = \begin{bmatrix} -\lambda f(t, u(t), u'(t)) \\ \lambda I_1(u(t_1), u'(t_1)) \\ \cdots \\ \lambda I_k(u(t_k), u'(t_k)) \end{bmatrix},$$

where $\operatorname{dom} M = \{u \in X : (\varphi_p(u'))' \in Y, \varphi_p(u'(+\infty)) = \int_0^{+\infty} h(t)\varphi_p(u'(t))\,dt\}$.

It is clear that $u \in \operatorname{dom} M$ is a solution of the problem (1.1) if it satisfies $Mu = Nu$, where $N = N_1$. For convenience, let $(a,b)^T := \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$, denote $J_0 = [0, t_1]$, $J_i = (t_i, t_{i+1}]$, $i = 1,2,\ldots,k-1$, $J_k = (t_k, +\infty)$.

Lemma 3.1 *M is a quasi-linear operator.*

Proof It is easy to get $\operatorname{Ker} M = \{at \mid a \in \mathbb{R}\} := X_1$.

For $u \in X \cap \operatorname{dom} M$, if $Mu = (y, c_1, c_2, \ldots, c_k)^T$, then

$$(\varphi_p(u'))'(t) = y(t), \qquad \triangle\varphi_p(u'(t_i)) = c_i, \quad i = 1,2,\ldots,k.$$

For $t \in J_0$, we get

$$\varphi_p\big(u'(t)\big) = \int_0^t y(s)\,ds + a.$$

For $t \in J_1$, considering $\triangle\varphi_p(u'(t_1)) = c_1$, we get

$$\varphi_p\big(u'(t)\big) = \int_0^t y(s)\,ds + a + c_1.$$

For $t \in J_i$, $i = 2,3,\ldots,k$, considering $\triangle\varphi_p(u'(t_i)) = c_i$, we get

$$\varphi_p\big(u'(t)\big) = \int_0^t y(s)\,ds + a + \sum_{t_i<t} c_i.$$

By $\varphi_p(u'(+\infty)) = \int_0^{+\infty} h(t)\varphi_p(u'(t))\,dt$ and $\int_0^{+\infty} h(t)\,dt = 1$, we find that $(y, c_1, c_2, \ldots, c_k)^T$ satisfies

$$\int_0^{+\infty} h(t)\int_t^{+\infty} y(s)\,ds + \int_0^{t_k}\sum_{t_i\geq t} c_i h(t)\,dt = 0. \tag{3.1}$$

On the other hand, if $(y, c_1, c_2, \ldots, c_k)^T$ satisfies (3.1), take

$$u(t) = \int_0^t \varphi_q\left(\int_0^s y(r)\,dr + \sum_{t_i<s} c_i\right) ds.$$

By a simple calculation, we get $u \in X \cap \operatorname{dom} M$ and $Mu = (y, c_1, c_2, \ldots, c_k)^T$. Thus

$$\operatorname{Im} M = \big\{(y, c_1, c_2, \ldots, c_k)^T \mid y \in Y, c_1, c_2, \ldots, c_k \text{ satisfies (3.1)}\big\}.$$

Obviously, $\operatorname{Im} M \subset Z$ is closed. So, M is quasi-linear. The proof is completed. □

Take projectors $P : X \to X_1$ and $Q : Z \to Z_1$ as follows:

$$(Pu)(t) = u'(+\infty)t,$$
$$Q(y, c_1, c_2, \ldots, c_k)^T = \left(\frac{\int_0^{+\infty} h(t)\int_t^{+\infty} y(s)\,ds\,dt + \int_0^{t_k}\sum_{t_i\geq t} c_i h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt} e^{-t}, 0, \ldots, 0\right)^T,$$

where $Z_1 = \{(ce^{-t}, 0, \ldots, 0)^T \mid c \in \mathbb{R}\}$. Obviously, $QZ = Z_1$ and $\dim Z_1 = \dim X_1$.

Define an operator $R : X \times [0,1] \to X_2$ as

$$\begin{aligned} R(u,\lambda)(t) = \int_0^t \varphi_q\bigg(\int_s^{+\infty} \lambda\bigg[f\big(r, u(r), u'(r)\big) \\ - \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s, u(s), u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\geq t} I_i(u(t_i), u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt} e^{-r}\bigg]\,dr \\ + \varphi_p\big(u'(+\infty)\big) - \lambda\sum_{t_j\geq s} I_j\big(u(t_j), u'(t_j)\big)\bigg)\,ds - u'(+\infty)t, \quad t \in J_i, i = 0,1,\ldots,k, \end{aligned}$$

where $X_1 \oplus X_2 = X$.

By [1, 24], we get the following lemma.

Lemma 3.2 *Assume that $V \subset X$ is bounded. V is compact if $\{\frac{u(t)}{1+t} : u \in V\}$ and $\{u'(t) : u \in V\}$ are both equicontinuous on J_i, $i = 0,1,\ldots,k-1$, and $J_T = (t_k, T]$, for any given $T > t_k$, respectively, and equiconvergent at infinity.*

Lemma 3.3 *$R : \overline{\Omega} \times [0,1] \to X_2$ is continuous and compact, where $\Omega \subset X$ is an open bounded set.*

Proof By (H_1), (H_2), the continuity of φ_q and Lebesgue's dominated convergence theorem, we find that R is continuous and $\{R(u,\lambda) \mid u \in \overline{\Omega}, \lambda \in [0,1]\}$ is bounded. We will prove that $R(\overline{\Omega} \times [0,1])$ is compact.

Since $\Omega \subset X$ is bounded, there exists a constant $r > 0$ such that $\|u\| \le r$, $u \in \overline{\Omega}$. It follows from (H_2) that there exist a function $h_r \in L(\mathbb{R}^+)$ and a constant $M_r > 0$ such that $|f(t,u(t),u'(t))| \le h_r(t)$, $|I_i(u(t_i),u'(t_i))| \le M_r$, $i = 1,2,\ldots,k$, $t \in \mathbb{R}^+$, $u \in \overline{\Omega}$. For any given $T > t_k$, $x_1, x_2 \in J_i$, $i = 0,1,\ldots,k-1,T$, $x_1 < x_2$, we have

$$\begin{aligned}
&\left|\frac{R(u,\lambda)(x_2)}{1+x_2} - \frac{R(u,\lambda)(x_1)}{1+x_1}\right| \\
&\quad\le \left|\frac{1}{1+x_2}\int_0^{x_2}\varphi_q\left(\int_s^{+\infty}\lambda\left[f\bigl(x,u(x),u'(x)\bigr)\right.\right.\right. \\
&\qquad - \left.\frac{\int_0^{+\infty}h(t)\int_t^{+\infty}f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\ge t}I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty}h(t)e^{-t}\,dt}e^{-x}\right]dx \\
&\qquad + \left.\varphi_p\bigl(u'(+\infty)\bigr) - \lambda\sum_{t_j\ge s}I_j\bigl(u(t_j),u'(t_j)\bigr)\right)ds \\
&\qquad - \frac{1}{1+x_1}\int_0^{x_1}\varphi_q\left(\int_s^{+\infty}\lambda\left[f\bigl(x,u(x),u'(x)\bigr)\right.\right. \\
&\qquad - \left.\frac{\int_0^{+\infty}h(t)\int_t^{+\infty}f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\ge t}I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty}h(t)e^{-t}\,dt}e^{-x}\right]dx \\
&\qquad + \left.\left.\varphi_p\bigl(u'(+\infty)\bigr) - \lambda\sum_{t_j\ge s}I_j\bigl(u(t_j),u'(t_j)\bigr)\right)ds\right| + \left|\frac{x_2}{1+x_2} - \frac{x_1}{1+x_1}\right|\left|u'(+\infty)\right| \\
&\quad\le \left|\frac{1}{1+x_2} - \frac{1}{1+x_1}\right|T\varphi_q\left(\int_0^{+\infty}\left[h_r(x) + \frac{\int_0^{+\infty}h_r(t)\,dt + \sum_{i=1}^k|I_i(u(t_i),u'(t_i))|}{\int_0^{+\infty}h(t)e^{-t}\,dt}e^{-x}\right]dx\right. \\
&\qquad + \left.\varphi_p(r) + kM_r\right) \\
&\qquad + \frac{x_2-x_1}{1+x_2}\varphi_q\left(\int_0^{+\infty}\left[h_r(x) + \frac{\int_0^{+\infty}h_r(t)\,dt + \sum_{i=1}^k|I_i(u(t_i),u'(t_i))|}{\int_0^{+\infty}h(t)e^{-t}\,dt}e^{-x}\right]dx\right. \\
&\qquad + \left.\varphi_p(r) + kM_r\right) + \left|\frac{x_2}{1+x_2} - \frac{x_1}{1+x_1}\right|r \\
&\quad\le \left[\left|\frac{1}{1+x_2} - \frac{1}{1+x_1}\right|T + (x_2-x_1)\right]\varphi_q\left(\|h_r\|_1 + \frac{\|h_r\|_1 + kM_r}{\int_0^{+\infty}h(t)e^{-t}\,dt} + \varphi_p(r) + kM_r\right) \\
&\qquad + \left|\frac{x_2}{1+x_2} - \frac{x_1}{1+x_1}\right|r.
\end{aligned}$$

Since t, $\frac{1}{1+t}$, and $\frac{t}{1+t}$ are equicontinuous on J_i, $i = 1, 2, \ldots, k-1, T$, we find that $\{\frac{R(u,\lambda)(t)}{1+t}, u \in \overline{\Omega}, \lambda \in [0,1]\}$ are equicontinuous on J_i, $i = 1, 2, \ldots, k-1, T$. We have

$$\begin{aligned}
&\bigl|R(u,\lambda)'(x_1) - R(u,\lambda)'(x_2)\bigr| \\
&\quad = \Biggl|\varphi_q\Biggl(\int_{x_1}^{+\infty} \lambda\biggl[f\bigl(s, u(s), u'(s)\bigr) \\
&\qquad - \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt}e^{-s}\biggr]\,ds \\
&\qquad + \varphi_p\bigl(u'(+\infty)\bigr) - \lambda\sum_{t_j\geq x_1} I_j\bigl(u(t_j),u'(t_j)\bigr)\Biggr) \\
&\qquad - \varphi_q\Biggl(\int_{x_2}^{+\infty} \lambda\biggl[f\bigl(s, u(s), u'(s)\bigr) \\
&\qquad - \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt}e^{-s}\biggr]\,ds \\
&\qquad + \varphi_p\bigl(u'(+\infty)\bigr) - \lambda\sum_{t_j\geq x_2} I_j\bigl(u(t_j),u'(t_j)\bigr)\Biggr)\Biggr|.
\end{aligned}$$

For $u \in \overline{\Omega}$, $\lambda \in [0,1]$, define

$$\begin{aligned}
F(u,\lambda)(t) = {}& \int_t^{+\infty} \lambda\biggl[f\bigl(s, u(s), u'(s)\bigr) \\
& - \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt}e^{-s}\biggr]\,ds \\
& + \varphi_p\bigl(u'(+\infty)\bigr) - \lambda\sum_{t_j\geq t} I_j\bigl(u(t_j),u'(t_j)\bigr).
\end{aligned}$$

Obviously,

$$\begin{aligned}
&\bigl|F(u,\lambda)(t)\bigr| \leq \|h_r\|_1 + \frac{\|h_r\|_1 + kM_r}{\int_0^{+\infty} h(t)e^{-t}\,dt} + \varphi_p(r) + kM_r := K, \quad u \in \overline{\Omega}, \lambda \in [0,1], t \in \mathbb{R}^+, \\
&\bigl|F(u,\lambda)(x_1) - F(u,\lambda)(x_2)\bigr| \\
&\quad = \biggl|\int_{x_1}^{x_2} \lambda\biggl[f\bigl(s, u(s), u'(s)\bigr) \\
&\qquad - \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt}e^{-s}\biggr]\,ds\biggr| \\
&\quad \leq \int_{x_1}^{x_2} h_r(t)\,dt + \frac{\|h_r\|_1 + kM_r}{\int_0^{+\infty} h(t)e^{-t}\,dt}\bigl(e^{-x_1} - e^{-x_2}\bigr), \quad u \in \overline{\Omega}, \lambda \in [0,1].
\end{aligned}$$

It follows from the absolute continuity of integral and the equicontinuity of e^{-t} that $\{F(u,\lambda)(t), u \in \overline{\Omega}, \lambda \in [0,1]\}$ are equicontinuous on J_i, $i = 1, 2, \ldots, k-1, T$. By the uniform continuity of $\varphi_q(t)$ in $[-K, K]$, we find that $\{R(u,\lambda)'(t), u \in \overline{\Omega}, \lambda \in [0,1]\}$ are equicontinuous on J_i, $i = 1, 2, \ldots, k-1, T$.

For any $u \in \overline{\Omega}$, $\lambda \in [0,1]$, since

$$\begin{aligned}&\left|\int_t^{+\infty} \lambda\Bigg[f\big(s,u(s),u'(s)\big)\\&\quad - \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt}e^{-s}\Bigg]\,ds\right|\\&\quad\leq \int_t^{+\infty} h_r(s) + \frac{\|h_r\|_1 + kM_r}{\int_0^{+\infty} h(t)e^{-t}\,dt}e^{-s}\,ds \to 0 \quad (t\to\infty)\end{aligned}$$

and $\varphi_q(u)$ is uniform continuous on $[-K-\varphi_p(r), K+\varphi_p(r)]$, for any $\varepsilon > 0$, there exists a constant $T_1 > t_k$ such that

$$\begin{aligned}&\Bigg|\varphi_q\Bigg(\int_s^{+\infty} \lambda\Bigg[f\big(r,u(r),u'(r)\big)\\&\quad - \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt}e^{-r}\Bigg]\,dr\\&\quad + \varphi_p\big(u'(+\infty)\big)\Bigg) - u'(+\infty)\Bigg| \leq \frac{\varepsilon}{4}, \quad s > T_1, u\in\overline{\Omega}, \lambda\in[0,1].\end{aligned}$$

Obviously, there exists a constant $T > T_1$ such that, for any $t > T$,

$$\frac{1}{1+t}\big(\varphi_q(K) + r\big)T_1 < \frac{\varepsilon}{4}.$$

Thus, for any $x_1, x_2 > T$, we have

$$\begin{aligned}&\left|\frac{R(u,\lambda)(x_1)}{1+x_1} - \frac{R(u,\lambda)(x_2)}{1+x_2}\right|\\&\quad = \Bigg|\frac{1}{1+x_1}\Bigg\{\int_0^{x_1}\varphi_q\Bigg(\int_s^{+\infty}\lambda\Bigg[f\big(r,u(r),u'(r)\big)\\&\qquad - \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt}e^{-r}\Bigg]\,dr\\&\qquad + \varphi_p\big(u'(+\infty)\big) - \lambda\sum_{t_j\geq s} I_j\big(u(t_j),u'(t_j)\big)\Bigg)\,ds - u'(+\infty)x_1\Bigg\}\\&\qquad - \frac{1}{1+x_2}\Bigg\{\int_0^{x_2}\varphi_q\Bigg(\int_s^{+\infty}\lambda\Bigg[f\big(r,u(r),u'(r)\big)\\&\qquad - \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt}e^{-r}\Bigg]\,dr\\&\qquad + \varphi_p\big(u'(+\infty)\big) - \lambda\sum_{t_j\geq s} I_j\big(u(t_j),u'(t_j)\big)\Bigg)\,ds - u'(+\infty)x_2\Bigg\}\Bigg|\\&\quad \leq \Bigg|\frac{1}{1+x_1}\Bigg\{\int_0^{T_1}\varphi_q\Bigg(\int_s^{+\infty}\lambda\Bigg[f\big(r,u(r),u'(r)\big)\\&\qquad - \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt}e^{-r}\Bigg]\,dr\end{aligned}$$

$$
\begin{aligned}
&+ \varphi_p\bigl(u'(+\infty)\bigr) - \lambda \sum_{t_j \geq s} I_j\bigl(u(t_j), u'(t_j)\bigr)\biggr) ds - u'(+\infty)T_1\biggr\}\biggr| \\
&+ \biggl|\frac{1}{1+x_1}\biggl\{\int_{T_1}^{x_1} \varphi_q\biggl(\int_s^{+\infty} \lambda\biggl[f\bigl(r, u(r), u'(r)\bigr) \\
&- \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i \geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt} e^{-r}\biggr]\,dr \\
&+ \varphi_p\bigl(u'(+\infty)\bigr)\biggr) ds - u'(+\infty)(x_1 - T_1)\biggr\}\biggr| \\
&+ \biggl|\frac{1}{1+x_2}\biggl\{\int_0^{T_1} \varphi_q\biggl(\int_s^{+\infty} \lambda\biggl[f\bigl(r, u(r), u'(r)\bigr) \\
&- \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i \geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt} e^{-r}\biggr]\,dr \\
&+ \varphi_p\bigl(u'(+\infty)\bigr) - \lambda \sum_{t_j \geq s} I_j\bigl(u(t_j), u'(t_j)\bigr)\biggr) ds - u'(+\infty)T_1\biggr\}\biggr| \\
&+ \biggl|\frac{1}{1+x_2}\biggl\{\int_{T_1}^{x_2} \varphi_q\biggl(\int_s^{+\infty} \lambda\biggl[f\bigl(r, u(r), u'(r)\bigr) \\
&- \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i \geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt} e^{-r}\biggr]\,dr \\
&+ \varphi_p\bigl(u'(+\infty)\bigr)\biggr) ds - u'(+\infty)(x_2 - T_1)\biggr\}\biggr| \\
\leq{}& \frac{1}{1+x_1}\bigl(\varphi_q(K) + r\bigr)T_1 + \frac{x_1 - T_1}{1+x_1}\frac{\varepsilon}{4} + \frac{1}{1+x_2}\bigl(\varphi_q(K) + r\bigr)T_1 + \frac{x_2 - T_1}{1+x_2}\frac{\varepsilon}{4} < \varepsilon,
\end{aligned}
$$

$$
\begin{aligned}
&\bigl|R(u,\lambda)'(x_1) - R(u,\lambda)'(x_2)\bigr| \\
&\quad\leq \biggl|\varphi_q\biggl(\int_{x_1}^{+\infty} \lambda\biggl[f\bigl(s, u(s), u'(s)\bigr) \\
&\qquad- \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i \geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt} e^{-s}\biggr]\,ds \\
&\qquad+ \varphi_p\bigl(u'(+\infty)\bigr)\biggr) - u'(+\infty)\biggr| + \biggl|\varphi_q\biggl(\int_{x_2}^{+\infty} \lambda\biggl[f\bigl(s, u(s), u'(s)\bigr) \\
&\qquad- \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i \geq t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt} e^{-s}\biggr]\,ds \\
&\qquad+ \varphi_p\bigl(u'(+\infty)\bigr)\biggr) - u'(+\infty)\biggr| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.
\end{aligned}
$$

By Lemma 3.2, we find that $\{R(u,\lambda) \mid u \in \overline{\Omega}, \lambda \in [0,1]\}$ is compact. The proof is completed. □

Lemma 3.4 *Assume that $\Omega \subset X$ is an open bounded set. Then N_λ is M-compact in $\overline{\Omega}$.*

Proof By (H_1), we get $N_\lambda : \overline{\Omega} \to Y$, $\lambda \in [0,1]$ is continuous. It is clear that $\operatorname{Im} P = \operatorname{Ker} M$, $QN_\lambda x = \theta, \lambda \in (0,1) \Leftrightarrow QNx = \theta$, *i.e.* Definition 2.2(b) holds.

For $u \in \overline{\Omega}$, it follows from $Q(I-Q)N_\lambda u = \theta$ that $(I-Q)N_\lambda u$ satisfies (3.1). So, $(I-Q)N_\lambda u \in \operatorname{Im} M$, *i.e.* $(I-Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M$. Furthermore, by $\operatorname{Im} M = \operatorname{Ker} Q$ and $z = Qz + (I-Q)z$ we find that $z \in \operatorname{Im} M$ implies $z = (I-Q)z \in (I-Q)Z$, *i.e.* $\operatorname{Im} M \subset (I-Q)Z$. Thus, $(I-Q)N_\lambda(\overline{\Omega}) \subset \operatorname{Im} M \subset (I-Q)Z$, *i.e.* Definition 2.2(a) holds.

Obviously, $R(\cdot,0) = 0$. For $u \in \Sigma_\lambda = \{u \in \overline{\Omega} \cap \operatorname{dom} M : Mu = N_\lambda u\}$, we get $QN_\lambda u = \theta$ and

$$\varphi_p\big(u'(t)\big) = \int_t^{+\infty} \lambda f\big(s,u(s),u'(s)\big)\,ds + \varphi_p\big(u'(+\infty)\big) - \lambda \sum_{t_i \ge t} I_i\big(u(t_i),u'(t_i)\big).$$

So, we have

$$R(u,\lambda) = \int_0^t \varphi_q\big(\varphi_p\big(u'(s)\big)\big)\,ds - u'(+\infty)t = (I-P)u,$$

i.e. Definition 2.2(c) holds.

For $u \in \overline{\Omega}$, $\lambda \in [0,1]$, $t \in J_i$, $i = 0,1,2,\ldots,k$, we have

$$\begin{aligned}
&\big(\varphi_p\big(Pu + R(u,\lambda)\big)'\big)'(t) \\
&\quad = -\lambda f\big(t,u(t),u'(t)\big) \\
&\qquad - \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} -\lambda f(s,u(s),u'(s))\,ds\,dt + \lambda\int_0^{t_k}\sum_{t_i\ge t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt} e^{-t}
\end{aligned}$$

and

$$\begin{aligned}
&\varphi_p\big(\big(Pu + R(u,\lambda)\big)'(t)\big) \\
&\quad = \int_t^{+\infty} \lambda\bigg[f\big(r,u(r),u'(r)\big) \\
&\qquad - \frac{\int_0^{+\infty} h(t)\int_t^{+\infty} f(s,u(s),u'(s))\,ds\,dt - \int_0^{t_k}\sum_{t_i\ge t} I_i(u(t_i),u'(t_i))h(t)\,dt}{\int_0^{+\infty} h(t)e^{-t}\,dt} e^{-r}\bigg]\,dr \\
&\qquad + \varphi_p\big(u'(+\infty)\big) - \lambda\sum_{t_j\ge t} I_j\big(u(t_j),u'(t_j)\big).
\end{aligned}$$

By a simple calculation, we can get

$$M\big[Pu + R(u,\lambda)\big] = (I-Q)N_\lambda u.$$

So, Definition 2.2(d) holds. These, together with Lemma 3.3, mean that N_λ is M-compact in $\overline{\Omega}$. The proof is completed. □

Theorem 3.1 *Assume that* (H_1), (H_2), *and the following conditions hold*:

(H_3) *There exist nonnegative functions* $a(t)$, $b(t)$, $c(t)$, *and nonnegative constants* d_i, g_i, e_i, $i = 1,2,\ldots,k$ *with* $(1+t)^{p-1}a(t), b(t), c(t) \in Y$, *and* $\|a(t)(1+t)^{p-1}\|_1 + \|b\|_1 + \sum_{i=1}^k [d_i(1+t_i)^{p-1} + g_i] < 1$ *such that*

$$\begin{aligned}
&\big|f(t,x,y)\big| \le a(t)\big|\varphi_p(x)\big| + b(t)\big|\varphi_p(y)\big| + c(t), \quad \text{a.e. } t \in [0,+\infty), x,y \in \mathbb{R}, \\
&\big|I_i(x,y)\big| \le d_i\big|\varphi_p(x)\big| + g_i\big|\varphi_p(y)\big| + e_i, \quad i = 1,2,\ldots,k, x,y \in \mathbb{R}.
\end{aligned}$$

(H_4) *There exists a constant $e_0 > 0$ such that if $\inf_{t\in\mathbb{R}^+} |u'(t)| > e_0$, then one of the following inequalities holds*:

$$(1)\quad u'(t)\int_0^{+\infty} h(t)\left(\int_t^{+\infty} f\big(s,u(s),u'(s)\big)\,ds - \sum_{t_i\geq t} I_i\big(u(t_i),u'(t_i)\big)\right)dt > 0;$$

$$(2)\quad u'(t)\int_0^{+\infty} h(t)\left(\int_t^{+\infty} f\big(s,u(s),u'(s)\big)\,ds - \sum_{t_i\geq t} I_i\big(u(t_i),u'(t_i)\big)\right)dt < 0,$$

where $t \in [0,+\infty)$. Then boundary value problem (1.1) *has at least one solution.*

In order to prove Theorem 3.1, we show two lemmas.

Lemma 3.5 *Suppose that* (H_1)-(H_4) *hold. Then the set*

$$\Omega_1 = \big\{u \in \operatorname{dom} M \mid Mu = N_\lambda u, \lambda \in (0,1)\big\}$$

is bounded in X.

Proof For $u \in \Omega_1$, we have $QN_\lambda u = 0$, *i.e.*

$$\begin{aligned}&\int_0^{+\infty} h(t)\int_t^{+\infty} f\big(s,u(s),u'(s)\big)\,ds\,dt - \int_0^{t_k}\sum_{t_i\geq t} I_i\big(u(t_i),u'(t_i)\big)h(t)\,dt\\ &\quad= \int_0^{+\infty} h(t)\left[\int_t^{+\infty} f\big(s,u(s),u'(s)\big)\,ds - \sum_{t_i\geq t} I_i\big(u(t_i),u'(t_i)\big)\right]dt = 0.\end{aligned}$$

By (H_4), there exists a constant $t_0 \in \mathbb{R}^+$ such that $|u'(t_0)| \leq e_0$. Assume $t_0 \in J_m$, $m = 0,1,\dots,k$. It follows from $Mu = N_\lambda u$ that

$$\varphi_p\big(u'(t)\big) = \begin{cases} \int_t^{t_0} \lambda f(s,u(s),u'(s))\,ds + \varphi_p(u'(t_0)) - \lambda\sum_{j=i+1}^m I_j(u(t_j),u'(t_j)),\\ \quad t\in J_i, i = 0,1,\dots,m-1,\\ \int_t^{t_0} \lambda f(s,u(s),u'(s))\,ds + \varphi_p(u'(t_0)), \quad t \in J_m,\\ \int_t^{t_0} \lambda f(s,u(s),u'(s))\,ds + \varphi_p(u'(t_0)) + \lambda\sum_{j=m+1}^i I_j(u(t_j),u'(t_j)),\\ \quad t\in J_i, i = m+1,m+2,\dots,k.\end{cases} \tag{3.2}$$

Since $u(t) = \int_0^t u'(s)\,ds$,

$$\frac{|u(t)|}{1+t} \leq \|u'\|_\infty, \quad t\in[0,+\infty). \tag{3.3}$$

By (3.2), (H_3), and (3.3), we obtain

$$\begin{aligned}\big|\varphi_p\big(u'(t)\big)\big| &\leq \int_0^{+\infty}\big[a(t)\big|\varphi_p\big(u(t)\big)\big| + b(t)\big|\varphi_p\big(u'(t)\big)\big| + c(t)\big]\,dt + \varphi_p(e_0)\\ &\quad+ \sum_{i=1}^k\big(d_i\big|\varphi_p\big(u(t_i)\big)\big| + g_i\big|\varphi_p\big(u'(t_i)\big)\big| + e_i\big)\\ &\leq \left(\big\|a(t)(1+t)^{p-1}\big\|_1 + \sum_{i=1}^k d_i(1+t_i)^{p-1}\right)\varphi_p\left(\left\|\frac{u}{1+t}\right\|_\infty\right)\end{aligned}$$

$$+\left(\|b\|_1+\sum_{i=1}^{k}g_i\right)\varphi_p\left(\left\|u'\right\|_\infty\right)+\|c\|_1+\varphi_p(e_0)+\sum_{i=1}^{k}e_i$$

$$\le\left(\left\|a(t)(1+t)^{p-1}\right\|_1+\|b\|_1+\sum_{i=1}^{k}\left[d_i(1+t_i)^{p-1}+g_i\right]\right)\varphi_p\left(\left\|u'\right\|_\infty\right)$$

$$+\|c\|_1+\varphi_p(e_0)+\sum_{i=1}^{k}e_i.$$

Thus

$$\left\|u'\right\|_\infty\le\varphi_q\left(\frac{\|c\|_1+\varphi_p(e_0)+\sum_{i=1}^{k}e_i}{1-(\|a(t)(1+t)^{p-1}\|_1+\|b\|_1+\sum_{i=1}^{k}[d_i(1+t_i)^{p-1}+g_i])}\right).$$

This, together with (3.3), means that Ω_1 is bounded in X. □

Lemma 3.6 *Assume that* (H_1), (H_2), *and* (H_4) *hold. Then*

$$\Omega_2=\{u\in\operatorname{Ker}M\mid QNu=0\}$$

is bounded in X, *where* $N=N_1$.

Proof For $u\in\Omega_2$, we have $u=at$, $a\in\mathbb{R}$, and $Q(Nu)=0$, *i.e.*

$$\int_0^{+\infty}h(t)\int_t^{+\infty}f(s,as,a)\,ds\,dt-\int_0^{t_k}\sum_{t_i\ge t}I_i(at_i,a)h(t)\,dt$$
$$=\int_0^{+\infty}h(t)\left[\int_t^{+\infty}f(s,as,a)\,ds-\sum_{t_i\ge t}I_i(at_i,a)\right]dt=0.$$

By (H_4), we get $\|u\|=|a|=|u'(t)|\le e_0$. So, Ω_2 is bounded. The proof is completed. □

Proof of Theorem 3.1 Let $\Omega=\{u\in X\mid\|u\|<r\}$, where $r>e_0$ is large enough such that $\Omega\supset\overline{\Omega}_1\cup\overline{\Omega}_2$.

By Lemmas 3.5 and 3.6, we have $Mu\ne N_\lambda u$, $u\in\operatorname{dom}M\cap\partial\Omega$, and $QNu\ne0$, $u\in\operatorname{Ker}M\cap\partial\Omega$.

Let $H(u,\delta)=\rho\delta u+(1-\delta)JQNu$, $\delta\in[0,1]$, $u\in\operatorname{Ker}M\cap\overline{\Omega}$, where $J:\operatorname{Im}Q\to\operatorname{Ker}M$ is a homeomorphism with $J(ae^{-t},0,\dots,0)^T=at$, $\rho=\begin{cases}-1, & \text{if }(\mathrm{H}_4)(1)\text{ holds},\\ 1, & \text{if }(\mathrm{H}_4)(2)\text{ holds}.\end{cases}$

For $u\in\operatorname{Ker}M\cap\partial\Omega$, we have $u=at\ne0$. Thus

$$H(u,\delta)=\rho\delta at-(1-\delta)\frac{\int_0^{+\infty}h(t)\int_t^{+\infty}f(s,as,a)\,ds\,dt-\int_0^{t_k}\sum_{t_i\ge t}I_i(at_i,a)h(t)\,dt}{\int_0^{+\infty}h(t)e^{-t}\,dt}t.$$

If $\delta=1$, $H(u,1)=\rho at\ne0$. If $\delta=0$, by $QNu\ne0$, we get $H(u,0)=JQN(at)\ne0$. For $0<\delta<1$, we now prove that $H(u,\delta)\ne0$. Otherwise, if $H(u,\delta)=0$, then

$$\int_0^{+\infty}h(t)\int_t^{+\infty}f(s,as,a)\,ds\,dt-\int_0^{t_k}\sum_{t_i\ge t}I_i(at_i,a)h(t)\,dt=\frac{\rho\delta a}{1-\delta}\int_0^{+\infty}h(t)e^{-t}\,dt.$$

Thus

$$a\int_0^{+\infty} h(t)\left[\int_t^{+\infty} f(s,as,a)\,ds - \sum_{t_i\geq t} I_i(at_i,a)\right]dt = \frac{\rho\delta a^2}{1-\delta}\int_0^{+\infty} h(t)e^{-t}\,dt.$$

Since $|u'(t)| = |a| = \|u\| = r > e_0$, this is a contradiction with (H_4) and the definition of ρ. So, $H(u,\delta) \neq 0$, $u \in \operatorname{Ker} M \cap \partial\Omega$, $\delta \in [0,1]$.

By the homotopy of degree, we get

$$\begin{aligned}\deg\Big(JQN, \Omega\cap\operatorname{Ker}M, 0\Big) &= \deg\Big(H(\cdot,0), \Omega\cap\operatorname{Ker}M, 0\Big)\\ &= \deg\Big(H(\cdot,1), \Omega\cap\operatorname{Ker}M, 0\Big)\\ &= \deg\Big(\rho I, \Omega\cap\operatorname{Ker}M, 0\Big) \neq 0.\end{aligned}$$

By Theorem 2.1, we can find that $Mu = Nu$ has at least one solution in $\overline{\Omega}$. The proof is completed. □

4 Example

Let us consider the following impulsive p-Laplacian boundary value problems at resonance on the half-line

$$\begin{cases}(\varphi_p(u'(t)))' + f(t,u(t),u'(t)) = 0, & t \in [0,\infty)\backslash\{t_1,t_2,\ldots,t_k\},\\ \triangle\varphi_p(u'(t_i)) = c_i, & i = 1,2,\ldots,k,\\ u(0) = 0, & \varphi_p(u'(+\infty)) = \int_0^{+\infty} e^{-t}\varphi_p(u'(t))\,dt,\end{cases} \tag{4.1}$$

where $0 < t_1 < t_2 < \cdots < t_k < +\infty$, $p = \frac{4}{3}$, $f(t,x,y) = \frac{e^{-4t}}{(1+t)^{\frac{1}{3}}}\sqrt[3]{\sin x} + e^{-4t}\sqrt[3]{y} + e^{-4t}$.

Corresponding to the problem (1.1), we have $h(t) = e^{-t}$, $I_i(u,v) = c_i$, $i = 1,2,\ldots,k$. Take $h_r(t) = ((1+t)^{-\frac{1}{3}} + r^{\frac{1}{3}} + 1)e^{-4t}$, $a(t) = \frac{e^{-4t}}{(1+t)^{\frac{1}{3}}}$, $b(t) = c(t) = e^{-4t}$, $d_i = g_i = 0$, $e_i = c_i$, $i = 1,2,\ldots,k$, $e_0 = e^{12(1+t_k)}(1 + 20\sum_{i=1}^{k}|c_i|)^3$, $M_r = \max_{1\leq i\leq k}\{|c_i|\}$.

By a simple calculation, we find that (H_1)-(H_3) and (H_4)(1) hold.

By Theorem 3.1, we find that the problem (4.1) has at least one solution.

Competing interests
The author declares that she has no competing interests.

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References

1. Agarwal, RP, O'Regan, D: Infinite Interval Problems for Differential, Difference and Integral Equations. Kluwer Academic, Dordrecht (2001)
2. Benchohra, M, Henderson, J, Ntouyas, S: Impulsive Equations and Inclusions. Contemporary Mathematics and Its Applications, vol. 2. Hindawi Publishing Corporation, New York (2006)
3. Lakshmikantham, V, Bainov, DD, Simeonov, PS: Theory of Impulsive Differential Equations. Series in Modern Applied Mathematics, vol. 6. World Scientific, Hackensack (1994)
4. Samoilenko, AM, Perestyuk, NA: Impulsive Differential Equations. World Scientific, Singapore (1995)
5. Zhang, X, Shuai, Z, Wang, K: Optimal impulsive harvesting policy for single population. Nonlinear Anal., Real World Appl. **4**, 639-651 (2003)

6. Carter, TE: Necessary and sufficient conditions for optimal impulsive rendezvous with linear equations of motion. Dyn. Control **10**, 219-227 (2000)
7. Chang, CC, Liu, CQ, Huston, RL: Dynamics of multibody systems subjected to impulsive constraints. Multibody Syst. Dyn. **8**, 161-184 (2002)
8. Li, C, Liao, X, Yang, X: Impulsive stabilization and synchronization of a class of chaotic delay systems. Chaos **15**, 043103 (2005)
9. Shen, JH, Li, JL: Existence and global attractivity of positive periodic solutions for impulsive predator-prey model with dispersion and time delays. Nonlinear Anal., Real World Appl. **10**, 227-243 (2009)
10. Agarwal, RP, Franco, D, O'Regan, D: Singular boundary value problems for first and second order impulsive differential equations. Aequ. Math. **69**, 83-96 (2005)
11. Tian, Y, Bai, Z: Existence results for the three-point impulsive boundary value problem involving fractional differential equations. Comput. Math. Appl. **59**, 2601-2609 (2010)
12. Jankowski, T: Positive solutions for second order impulsive differential equations involving Stieltjes integral conditions. Nonlinear Anal. **74**, 3775-3785 (2011)
13. Nieto, JJ, O'Regan, D: Variational approach to impulsive differential equations. Nonlinear Anal., Real World Appl. **10**, 680-690 (2009)
14. Kaufmann, ER, Kosmatov, N, Raffoul, YN: A second-order boundary value problem with impulsive effects on an unbounded domain. Nonlinear Anal. **69**, 2924-2929 (2008)
15. Yan, B: Boundary value problems on the half-line with impulsive and infinite delay. J. Math. Anal. Appl. **259**, 94-114 (2001)
16. Kosmatov, N: Multi-point boundary value problems on an unbounded domain at resonance. Nonlinear Anal. **68**, 2158-2171 (2008)
17. Zhang, X, Feng, M, Ge, W: Existence result of second-order differential equations with integral boundary conditions at resonance. J. Math. Anal. Appl. **353**, 311-319 (2009)
18. Liu, Y, Ge, W: Solvability of nonlocal boundary value problems for ordinary differential equations of higher order. Nonlinear Anal. **57**, 435-458 (2004)
19. Du, Z, Lin, X, Ge, W: Some higher-order multi-point boundary value problem at resonance. J. Comput. Appl. Math. **177**, 55-65 (2005)
20. Liu, B: Solvability of multi-point boundary value problem at resonance (II). Appl. Math. Comput. **136**, 353-377 (2003)
21. Jiang, W: Solvability of boundary value problem with *p*-Laplacian at resonance. Bound. Value Probl. **2014**, 36 (2014). doi:10.1186/1687-2770-2014-36
22. Jiang, W: Solvability for *p*-Laplacian boundary value problem at resonance on the half-line. Bound. Value Probl. **2013**, 207 (2013). doi:10.1186/1687-2770-2013-207
23. Ge, W, Ren, J: An extension of Mawhin's continuation theorem and its application to boundary value problems with a *p*-Laplacian. Nonlinear Anal. TMA **58**, 477-488 (2004)
24. Agarwal, RP, Franco, D, O'Regan, D: Singular boundary value problems for first and second order impulsive differential equations. Aequ. Math. **69**, 83-96 (2005)