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One class of generalized boundary value problem for analytic functions

Pingrun Li*

*Correspondence:
lipingrun@163.com
School of Mathematical Sciences,
Qufu Normal University, Jingxuanxi
Road 57, Qufu, Shandong 273165,
P.R. China

Abstract

In this paper, a boundary value problem for analytic functions with two unknown functions on two parallel straight lines is studied, the general solutions in the different domains as well as the conditions of solvability are obtained in class {1}, and the behaviors of solutions are discussed at $z = \infty$ and in the different domains, respectively. Therefore, the classic Riemann boundary value problem is extended further.

Keywords: boundary value problem for analytic functions; index; canonical function; the function class {1}

1 Introduction and preliminaries

Many mathematicians have studied the boundary value problems of analytic functions and formed a perfect theoretical system; see [1–7]. The boundary value problem of analytic functions on an infinite straight line has been studied in the literature, and there has been a brief description of boundary value problems of analytic function with an unknown function on several parallel lines. In this paper, we will put forward the boundary value problems of analytic functions with two unknown functions on two parallel lines and a general method different from the one in classical boundary value theory. Moreover, we will give and discuss the general solution and solvability conditions, which will generalize the classical theory of boundary value problems of analytic functions.

Let us describe the definitions of *Plemelj* formula and function class {1} on an infinite straight line.

Definition 1.1 Assume that $\omega(x)$ is a continuous complex function on the real axis X. We say that $\omega(x) \in \hat{H}$ if the following conditions hold:

(1) For any sufficiently large positive number M, $\omega(x)$ satisfies $\omega(x) \in H$ on $[-M, M]$ (see [8] for the definition of H).

(2) $|\omega(x_1) - \omega(x_2)| \le A|\frac{1}{x_1} - \frac{1}{x_2}|$, for any $|x_j| > M$ $(j = 1, 2)$ and some positive real number A.

Under condition (2), we say that $\omega(x)$ satisfies the Hölder condition on N_∞ and denote it $\omega(x) \in H(N_\infty)$, where $N_\infty = \{x : |x| > M\}$ is a neighborhood of ∞.

Definition 1.2 Assume that $\omega(x)$ is continuous on $(-\infty, \infty)$ and $\int_{-\infty}^{\infty} |\omega(x)|\, dx < +\infty$, then we say that $\omega(x) \in L_1(-\infty, \infty)$.

Definition 1.3 If $\omega(x)$ satisfies: (1) $\omega(x) \in \hat{H}$, (2) $\omega(x) \in L_1(-\infty,\infty)$, then we say that $\omega(x)$ belongs to the function class $\{1\}$.

Definition 1.4 Assume that $\omega(x) \in \{1\}$, then the integrals $\Omega^+(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \omega(t)e^{itz}\,dt$ and $\Omega^-(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \omega(t)e^{itz}\,dt$ are called the left and right one-sided Fourier integral, respectively.

Lemma 1.1 (see [1]) *If $\omega(z) \in H$ with respect to any finite part of some infinite domain D, and $\omega(z)$ is analytic in any neighborhood of infinity, then $\omega(z) \in \hat{H}$.*

Lemma 1.2 (see [2]) *If $\omega(t)$ belongs to the class* $\{1\}$, *then the left and right one-sided Fourier integrals defined in Definition* 1.4 *are analytic when* $\operatorname{Im} z > 0$ *and* $\operatorname{Im} z < 0$, *respectively.*

Lemma 1.3 (see [8]) *If $\omega(x) \in \hat{H}$, we have the Cauchy type integral $\Omega(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\omega(t)}{t-z}\,dt$, $z \notin (-\infty,\infty)$, then the following formula holds on the infinite straight line*:

$$\Omega^{\pm}(x) = \pm\frac{1}{2}\omega(x) + \Omega(x), \quad \textit{i.e.}, \quad \Omega^{\pm}(x) = \pm\frac{1}{2}\omega(x) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\omega(t)}{t-x}\,dt.$$

2 Problem presentation

Now, we put forward the boundary value problem of analytic functions on two parallel lines.

Without loss of generality, we assume that the two lines are parallel to the X-axis (otherwise, we can translate them into this case by a linear transformation), and denote them by L_1, L_2, where L_j can be expressed by $\zeta = x + il_j$ ($x \in (-\infty,\infty)$, $l_2 < l_1$ are real numbers) and take the direction from left to right as the positive direction. Let $L = L_1 + L_2$.

We want to get functions $\Phi(z)$ and $\Psi(z)$ such that $\Phi(z)$ is analytic in $\{\operatorname{Im} z > l_1\} \cup \{\operatorname{Im} z < l_2\}$, $\Psi(z)$ is analytic in $\{z : l_2 < \operatorname{Im} z < l_1\}$, and we have the following boundary value conditions:

$$\begin{cases} \Phi^+(\zeta) = D_1(\zeta)\Psi^-(\zeta) + G_1(\zeta), & \text{when } \zeta \in L_1, \\ \Phi^-(\zeta) = D_2(\zeta)\Psi^+(\zeta) + G_2(\zeta), & \text{when } \zeta \in L_2, \end{cases} \tag{2.1}$$

where L_j: $\zeta = x + il_j$ ($j = 1,2$), $x \in (-\infty,\infty)$.

Actually, (2.1) is a boundary value problem on two parallel straight lines $\operatorname{Im} z = l_1$, $\operatorname{Im} z = l_2$ with ∞ as a pole. Here $\Phi^+(\zeta)$ is the boundary value of analytic function $\Phi^+(z)$ which is analytic in $\{z : \operatorname{Im} z > l_1\}$ and belongs to the class $\{1\}$ on L_1, $\Phi^-(\zeta)$ is the boundary value of analytic function $\Phi^-(z)$ which is analytic in $\{z : \operatorname{Im} z < l_2\}$ and belongs to the class $\{1\}$ on L_2, and $\Psi^{\pm}(\zeta)$ is the boundary value of analytic function $\Psi(z)$ which is analytic in $\{z : l_2 < \operatorname{Im} z < l_1\}$ and belongs to the class $\{1\}$ on L_1, L_2, respectively. The functions $D_1(\zeta)$ and $D_2(\zeta)$ belong to $\hat{H}$ on L_1, L_2, respectively. The functions $G_1(\zeta)$ and $G_2(\zeta)$ belong to the class $\{1\}$ on L_1, L_2, respectively. Hence, for the functions appearing in (2.1) the one-sided limits exist when $x \to \infty$ on L_1, L_2.

It can be seen from (2.1) that the order of $\Phi(z)$ is equal to that of $\Psi(z)$ at infinity. Therefore, if the orders of $\Phi(z)$ and $\Psi(z)$ are m at infinity, then such a problem can be denoted as R_m. Actually, problem R_0 and problem R_{-1} are often discussed. On the problem R_0, both $\Phi(\infty)$ and $\Psi(\infty)$ are supposed to be finite and nonzero. On R_{-1}, both $\Phi(\infty)$ and $\Psi(\infty)$ are assumed to be zero. Such a problem R is called regular if $D_j(\zeta)$ is not zero on L; otherwise, it is called irregular or of exception type.

Remark 2.1 Since the positive direction of L_j is the direction from left to right, when the observer moves from left to right on L_j, the boundary values of left region of L_j is positive boundary value, *i.e.*, the positive boundary value of $\Phi(z)$ is the boundary value above L_1, and the negative boundary value of $\Phi(z)$ is ones below L_2. The positive or negative boundary values of $\Psi(z)$ can be defined in a similar way.

3 Resolution

We only consider problem R_0 in this paper. Hence, we assume $\Phi(\infty)$ and $\Psi(\infty)$ are finite and nonzero. For problem R_m, similar arguments can be used. In this paper, we only consider the normal case, that is, $D_j(\zeta)$ $(j = 1, 2)$ does not have zeroes and poles on L_j. For the irregular case, similar discussions also work (see Section 2.5 of Chapter 2 in [1]). Equation (2.1) can be written as

$$\begin{cases} \Phi^+(\zeta) = D_1(\zeta)\Psi^-(\zeta) + G_1(\zeta), & \text{when } \zeta \in L_1, \\ \Psi^+(\zeta) = \frac{1}{D_2(\zeta)}\Phi^-(\zeta) - \frac{G_2(\zeta)}{D_2(\zeta)}, & \text{when } \zeta \in L_2. \end{cases} \tag{3.1}$$

In order to unify, let $C_1(\zeta) = G_1(\zeta)$, $C_2(\zeta) = G_2(\zeta)/D_2(\zeta)$, and the above equation can be transformed into

$$\begin{cases} \Phi^+(\zeta) = D_1(\zeta)\Psi^-(\zeta) + C_1(\zeta), & \text{when } \zeta \in L_1, \\ \Psi^+(\zeta) = \frac{1}{D_2(\zeta)}\Phi^-(\zeta) - C_2(\zeta), & \text{when } \zeta \in L_2. \end{cases} \tag{3.2}$$

By putting $\kappa_1 = \operatorname{Ind}_{L_1} D_1(\zeta)$, $\kappa_2 = \operatorname{Ind}_{L_2} D_2(\zeta)$, and $\kappa = \sum_{j=1}^{2} \kappa_j$, we call κ as the index of problem (2.1). Without loss of generality, we take three points z_0, z_1, z_2 on the Z plane such that $l_1 < \operatorname{Im} z_1$, $l_2 < \operatorname{Im} z_0 < l_1$, $\operatorname{Im} z_2 < l_2$. Then we take the following piecewise function:

$$Y_1(z) = \begin{cases} e^{\Omega_1(z)}, & \operatorname{Im} z > l_1, \\ (\frac{z-z_0}{z-z_1})^{k_1} e^{\Omega_1(z)}, & \operatorname{Im} z < l_1, \end{cases} \qquad Y_2(z) = \begin{cases} (\frac{z-z_0}{z-z_2})^{k_2} e^{\Omega_2(z)}, & \operatorname{Im} z > l_2, \\ e^{\Omega_2(z)}, & \operatorname{Im} z < l_2, \end{cases} \tag{3.3}$$

here $\Omega_j(z)$ $(j = 1, 2)$ is defined as follows:

$$\Omega_j(z) = \frac{1}{\sqrt{2\pi}} \int_{il_j}^{+\infty+il_j} r_j(t) e^{itz}\, dt, \quad \text{when } l_j < \operatorname{Im} z_j$$

and

$$\Omega_j(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+il_j}^{il_j} r_j(t) e^{itz}\, dt, \quad \text{when } \operatorname{Im} z_j < l_j,$$

where

$$r_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+il_1}^{+\infty+il_1} \log \tilde{D}_1(\tau) \cdot e^{-i\tau t}\, d\tau, \quad \tilde{D}_1(\tau) = \left(\frac{\tau - z_0}{\tau - z_1}\right)^{\kappa_1} D_1(\tau),$$

$$r_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+il_2}^{+\infty+il_2} \log \tilde{D}_2(\tau) \cdot e^{-i\tau t}\, d\tau, \quad \tilde{D}_2(\tau) = \left(\frac{\tau - z_0}{\tau - z_2}\right)^{\kappa_2} D_2^{-1}(\tau).$$

The function $\Omega_j(z)$ defined above is analytic on the complex plane except L_1 and L_2. The logarithmic function of the integrand has a certain analytic branch such that $\log \frac{t-z_0}{t-z_j}|_{t=\infty} = 0$; then $Y_j^+(z)$ and $Y_j^-(z)$ are analytic in $\{z : \operatorname{Im} z > l_j\}$ and $\{z : \operatorname{Im} z < l_j\}$, respectively. Moreover,

$$\frac{Y_1^+(t)}{Y_1^-(t)} = \left(\frac{t-z_0}{t-z_1}\right)^{-k_1} e^{\Omega_1^+(t)-\Omega_1^-(t)} = \left(\frac{t-z_0}{t-z_1}\right)^{-k_1} \exp\left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty+il_1}^{+\infty+il_1} r_1(\zeta) e^{it\zeta}\, d\zeta\right\},$$

owing to

$$V\big[r_1(t)\big] = \frac{1}{\sqrt{2\pi}} \int_{-\infty+il_1}^{+\infty+il_1} r_1(\zeta) e^{it\zeta}\, d\zeta,$$

by the representative of $r_1(t)$ as well as the relationship between Fourier transform and inverse Fourier transform, we have $V[r_1(t)] = \log[(\frac{t-z_0}{t-z_1})^{k_1} D_1(t)]$, therefore

$$\frac{Y_1^+(t)}{Y_1^-(t)} = \left(\frac{t-z_0}{t-z_1}\right)^{-k_1} \exp\left\{\log\left[\left(\frac{t-z_0}{t-z_1}\right)^{k_1} D_1(t)\right]\right\} = D_1(t). \tag{3.4}$$

Similarly, one has

$$\frac{Y_2^+(t)}{Y_2^-(t)} = D_2^{-1}(t). \tag{3.5}$$

Putting (3.4), (3.5) into (3.2), we can obtain

$$\begin{cases} \Phi^+(t)[Y_1^+(t)]^{-1} = \Psi^-(t)[Y_1^-(t)]^{-1} + C_1(t)[Y_1^+(t)]^{-1}, & t \in l_1, \\ \Psi^+(t)[Y_2^+(t)]^{-1} = \Phi^-(t)[Y_2^-(t)]^{-1} - C_2(t)[Y_2^+(t)]^{-1}, & t \in l_2. \end{cases} \tag{3.6}$$

In (3.6), the first equality is multiplied by $[Y_2^+(t)]^{-1}$, the second one is multiplied by $[Y_1^-(t)]^{-1}$, then

$$\begin{cases} \Phi^+(t)[Y_1^+(t)]^{-1}[Y_2^+(t)]^{-1} \\ \quad = \Psi^-(t)[Y_1^+(t)]^{-1}[Y_2^+(t)]^{-1} + C_1(t)[Y_1^+(t)]^{-1}[Y_2^+(t)]^{-1}, & t \in l_1, \\ \Psi^+(t)[Y_2^+(t)]^{-1}[Y_1^-(t)]^{-1} \\ \quad = \Phi^-(t)[Y_2^-(t)]^{-1}[Y_1^-(t)]^{-1} - C_2(t)[Y_2^+(t)]^{-1}[Y_1^-(t)]^{-1}, & t \in l_2, \end{cases} \tag{3.7}$$

denoting

$$F_1^+(z) = \frac{1}{\sqrt{2\pi}} \int_{il_1}^{+\infty+il_1} f_1(\tau) e^{i\tau z}\, d\tau, \qquad F_1^-(z) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty+il_1}^{il_1} f_1(\tau) e^{i\tau z}\, d\tau, \tag{3.8}$$

where

$$f_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+il_1}^{+\infty+il_1} \frac{C_1(\tau)}{Y_1^+(\tau) Y_2^+(\tau)} e^{-i\tau t}\, d\tau.$$

Using Lemma 1.2, we know that $F_1^+(z)$, $F_1^-(z)$ are analytic in $\operatorname{Im} z > l_1$, $\operatorname{Im} z < l_1$, respectively. On L_1, we have

$$F_1^+(t) - F_1^-(t) = \frac{C_1(t)}{Y_1^+(t)Y_2^+(t)}.$$

Again we denote

$$F_2^+(z) = \frac{1}{\sqrt{2\pi}} \int_{il_2}^{+\infty+il_2} f_2(\tau)e^{i\tau z}\, d\tau, \qquad F_2^-(z) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty+il_2}^{il_2} f_2(\tau)e^{i\tau z}\, d\tau, \tag{3.9}$$

where

$$f_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+il_2}^{+\infty+il_2} \frac{C_2(\tau)}{Y_1^-(\tau)Y_2^+(\tau)} e^{-i\tau t}\, d\tau.$$

Similarly, $F_2^+(z)$, $F_2^-(z)$ are analytic in $\operatorname{Im} z > l_2$, $\operatorname{Im} z < l_2$, respectively. On L_2, we obtain

$$F_2^+(t) - F_2^-(t) = \frac{C_2(t)}{Y_1^-(t)Y_2^+(t)}.$$

Then (3.7) may be reduced to

$$\begin{cases} \Phi^+(t)[Y_1^+(t)Y_2^+(t)]^{-1} - F_1^+(t) \\ \quad = \Psi^-(t)[Y_1^-(t)Y_2^+(t)]^{-1} - F_1^-(t), \quad t \in l_1, \\ \Psi^+(t)[Y_1^-(t)Y_2^+(t)]^{-1} + F_2^-(t) \\ \quad = \Phi^-(t)[Y_1^-(t)Y_2^-(t)]^{-1} + F_2^-(t), \quad t \in l_2, \end{cases} \tag{3.10}$$

in the two sides of the first equation of (3.10), by adding $F_2^+(t)$; in the two sides of the second one by subtraction of $F_1^-(t)$, we have

$$\begin{cases} \Phi^+(t)[Y_1^+(t)Y_2^+(t)]^{-1} - F_1^+(t) + F_2^+(t) \\ \quad = \Psi^-(t)[Y_1^-(t)Y_2^+(t)]^{-1} - F_1^-(t) + F_2^+(t), \quad t \in l_1, \\ \Psi^+(t)[Y_1^-(t)Y_2^+(t)]^{-1} + F_2^-(t) - F_1^-(t) \\ \quad = \Phi^-(t)[Y_1^-(t)Y_2^-(t)]^{-1} + F_2^-(t) - F_1^-(t), \quad t \in l_2, \end{cases} \tag{3.11}$$

the left side of the first equation of (3.11) is denoted by $M_1^+(t)$, the right side of one is denoted by $M_1^-(t)$; the left side of the second equation of (3.11) is denoted by $M_2^+(t)$, the right side of this one is denoted by $M_2^-(t)$. Let

$$M_1(z) = \begin{cases} M_1^+(z), & \operatorname{Im} z > l_1, \\ M_1^-(z), & \operatorname{Im} z < l_1, \end{cases} \tag{3.12}$$

where

$$M_1^+(z) = \Phi^+(z)\big[Y_1^+(z)Y_2^+(z)\big]^{-1} - F_1^+(z) + F_2^+(z),$$

$$M_1^-(z) = \Psi^-(z)\big[Y_1^-(z)Y_2^+(z)\big]^{-1} - F_1^-(z) + F_2^+(z).$$

(1) We firstly consider the solutions of $M_1^+(z)$ and $M_1^-(z)$, respectively in $\operatorname{Im} z > l_1$ and $\operatorname{Im} z < l_1$.

Case: $k \geq 0$.

Since $[Y_1^+(z)]^{-1}$, $[Y_2^+(z)]^{-1}$ are analytic in $\{z : \operatorname{Im} z > l_1\}$, $[Y_1^+(z)Y_2^+(z)]^{-1}$ is analytic. It follows from $F_j^+(z)$ $(j = 1, 2)$ is analytic that $M_1^+(z)$ is analytic. Hence, the $\lim_{z\to\infty\ (\operatorname{Im} z>l_1)} M_1^+(z)$ exists. Suppose that $M_1^+(z) = H_1(z)$, then

$$\Phi^+(z) = Y_1^+(z)Y_2^+(z)\bigl[F_1^+(z) - F_2^+(z) + H_1(z)\bigr], \quad \operatorname{Im} z > l_1, \tag{3.13}$$

where $H_1(z)$ is analytic in $\{z : \operatorname{Im} z > l_1\}$ and the $\lim_{z\to\infty\ (\operatorname{Im} z>l_1)} H_1(z)$ exists.

When $\operatorname{Im} z < l_1$, since $\Psi^-(z)$ is defined in $\{z : l_2 < \operatorname{Im} z < l_1\}$, z_0 is a k-order pole of $[Y_1^-(z)Y_2^+(z)]^{-1}$ $(k > 0)$. In order to ensure that $M_1^-(z)$ is bounded at a pole z_0, we can multiply by a factor $(z - z_0)^k$. Thus, it has a k-order at $z = \infty$, *i.e.*, we have a polynomial with k degree. Let $(z - z_0)^k M_1^-(z) = p_k(z)$, hence, $M_1^-(z) = \frac{p_k(z)}{(z-z_0)^k}$, *i.e.*,

$$\Psi^-(z)\bigl[Y_1^-(z)Y_2^+(z)\bigr]^{-1} - F_1^-(z) + F_2^+(z) = \frac{p_k(z)}{(z - z_0)^k}, \quad \operatorname{Im} z < l_1, \tag{3.14}$$

where $p_k(z) = C_0 + C_1(z - z_0) + \cdots + C_k(z - z_0)^k$ is a polynomial with degree no more than κ.

Case: $k < 0$.

It follows from similar arguments as above that $M_1^+(z)$ is analytic in $\{z : \operatorname{Im} z > l_1\}$.

Because $[Y_1^-(z)Y_2^+(z)]^{-1}$, $F_1^-(z)$ and $F_2^+(z)$ are analytic in $\{z : \operatorname{Im} z < l_1\}$, so is $M_1^-(z)$. Moreover, $M_1^+(t) = M_1^-(t)$ on L_1. Hence, $M_1(z)$ is holomorphic in the whole complex plane and the $\lim_{z\to\infty} M_1(z)$ exists. By the Liouville theorem and the principle of analytic continuation, there is a constant C such that $M_1(z) = C$, and one has

$$\Psi^-(z) = Y_1^-(z)Y_2^+(z)\bigl[F_1^-(z) - F_2^+(z) + C\bigr], \quad \operatorname{Im} z < l_1, \tag{3.15}$$

$$\Phi^-(z) = Y_1^+(z)Y_2^+(z)\bigl[F_1^+(z) - F_2^+(z) + C\bigr], \quad \operatorname{Im} z > l_1. \tag{3.16}$$

Noticing that $z = z_0$ is a pole of $Y_1^-(z)Y_2^+(z)$ with order $-k$, $\Psi^-(z)$ has a singularity at z_0. In order to ensure that $\Psi^-(z)$ is analytic in $\{z : \operatorname{Im} z < l_1\}$ (in fact, for solving $\Psi^-(z)$ in the range that $\operatorname{Im} z < l_1$, we should only consider the case that $l_2 < \operatorname{Im} z < l_1$). For the case that $k = -1$, it is sufficient to eliminate the singularity by putting $C = F_2^+(z_0) - F_1^-(z_0)$, then one can define $\Psi^-(z)$ in the following way:

$$\Psi^-(z) = Y_1^-(z)Y_2^+(z)\bigl[F_1^-(z) - F_2^+(z) + F_2^+(z_0) - F_1^-(z_0)\bigr]. \tag{3.17}$$

When $\kappa \leq -2$, such a C still cannot eliminate the singularity of $\Psi^-(z)$ at $z = z_0$. But the following condition should be satisfied:

$$F_2^{+(q)}(z_0) - F_1^{-(q)}(z_0) = 0, \quad q = 1, 2, \ldots, -\kappa - 1,$$

i.e.,

$$\int_{-\infty+il_1}^{il_1} \tau^q f_1(\tau)e^{i\tau z_0}\,d\tau + \int_{il_2}^{+\infty+il_2} \tau^q f_2(\tau)e^{i\tau z_0}\,d\tau = 0, \quad q = 1, 2, \ldots, -\kappa - 1; \tag{3.18}$$

(3.15) is a solution of (2.1) if and only if (3.18) holds. Therefore,

$$\text{as } k>0,\quad \Psi^-(z)=Y_1^-(z)Y_2^+(z)\left[F_1^-(z)-F_2^+(z)+\frac{p_k(z)}{(z-z_0)^k}\right],\quad \operatorname{Im} z<l_1, \tag{3.19}$$

$$\Phi^+(z)=Y_1^+(z)Y_2^+(z)\left[F_1^+(z)-F_2^+(z)+H_1(z)\right],\quad \operatorname{Im} z>l_1, \tag{3.20}$$

$$\text{as } k<0,\quad \Psi^-(z)=Y_1^-(z)Y_2^+(z)\left[F_1^-(z)-F_2^+(z)+C\right],\quad \operatorname{Im} z<l_1, \tag{3.21}$$

$$\Phi^+(z)=Y_1^+(z)Y_2^+(z)\left[F_1^+(z)-F_2^+(z)+C\right],\quad \operatorname{Im} z>l_1. \tag{3.22}$$

However, the solvability conditions should be satisfied (3.18) for $k<-1$. Similarly, we can define the following piecewise function:

$$M_2(z)=\begin{cases} M_2^+(z), & \operatorname{Im} z>l_2,\\ M_2^-(z), & \operatorname{Im} z<l_2,\end{cases} \tag{3.23}$$

where

$$M_2^+(z)=\Psi^+(z)\left[Y_1^-(z)Y_2^+(z)\right]^{-1}-F_1^-(z)+F_2^-(z), \tag{3.24}$$

$$M_2^-(z)=\Phi^-(z)\left[Y_1^-(z)Y_2^-(z)\right]^{-1}-F_1^-(z)+F_2^-(z). \tag{3.25}$$

(2) We secondly consider the solutions of $M_2^+(z)$ and $M_2^-(z)$, respectively, in $\operatorname{Im} z>l_2$ and $\operatorname{Im} z<l_2$.

Case: $k\geq 0$.

Since $Y_1^-(z)$ and $Y_2^-(z)$ are analytic in $\{z:\operatorname{Im} z<l_2\}$, so is $[Y_1^-(z)Y_2^-(z)]^{-1}$. It follows from $F_j^-(z)$ $(j=1,2)$ being analytic that $M_2^-(z)$ is analytic. Hence, the $\lim_{z\to\infty\ (\operatorname{Im} z<l_2)} M_2^-(z)$ exists. Suppose that $M_2^-(z)=H_2(z)$, then

$$\Phi^-(z)=Y_1^-(z)Y_2^-(z)\left[F_1^-(z)-F_2^-(z)+H_2(z)\right], \tag{3.26}$$

where $H_2(z)$ is analytic in $\{z:\operatorname{Im} z<l_2\}$ and the $\lim_{z\to\infty\ (\operatorname{Im} z<l_2)} H_2(z)$ exists.

When $\operatorname{Im} z>l_2$, $z=z_0$ is a k-order pole of $[Y_1^-(z)Y_2^+(z)]^{-1}$ $(k>0)$, then $z=z_0$ is a k-order pole of $M_2^+(z)$. By similar arguments to above, one has

$$\Psi^+(z)\left[Y_1^-(z)Y_2^+(z)\right]^{-1}-F_1^-(z)+F_2^+(z)=\frac{p_k(z)}{(z-z_0)^k}. \tag{3.27}$$

Case: $k<0$.

Since $[Y_1^-(z)Y_2^-(z)]^{-1}$ has no singularity in $\{z:\operatorname{Im} z<l_2\}$, $M_2^-(z)$ is analytic in $\{z:\operatorname{Im} z<l_2\}$. Noticing that $[Y_1^-(z)Y_2^+(z)]^{-1}$ is analytic in $\{z:\operatorname{Im} z>l_2\}$, one finds that $M_2^+(z)$ is analytic in $\{z:\operatorname{Im} z>l_2\}$ and $M_2^+(t)=M_2^-(t)$ on L_2. Therefore, $M_2(z)$ is holomorphic in the whole complex plane and the $\lim_{z\to\infty} M_2(z)$ exists. By similar arguments to (1), there exists a constant C, such that $M_2(z)=C$ and

$$\begin{aligned}\Psi^+(z)&=Y_1^-(z)Y_2^+(z)\left[F_1^-(z)-F_2^+(z)+C\right],\quad \operatorname{Im} z>l_2,\\ \Phi^-(z)&=Y_1^+(z)Y_2^+(z)\left[F_1^+(z)-F_2^+(z)+C\right],\quad \operatorname{Im} z<l_2.\end{aligned} \tag{3.28}$$

Moreover, (3.18) is also a necessary condition for solvability. Hence,

$$\text{as } k > 0, \quad \Psi^+(z) = Y_1^-(z)Y_2^+(z)\left[F_1^-(z) - F_2^+(z) + \frac{p_k(z)}{(z-z_0)^k}\right], \tag{3.29}$$

$$\Phi^-(z) = Y_1^-(z)Y_2^-(z)\left[F_1^-(z) - F_2^-(z) + H_2(z)\right], \tag{3.30}$$

$$\text{as } k < 0, \quad \Psi^+(z) = Y_1^-(z)Y_2^+(z)\left[F_1^-(z) - F_2^+(z) + C\right], \tag{3.31}$$

$$\Phi^-(z) = Y_1^+(z)Y_2^+(z)\left[F_1^+(z) - F_2^+(z) + C\right]. \tag{3.32}$$

Collecting results, for $k \geq 0$ and $l_2 < \operatorname{Im} z < l_1$, one has

$$\Psi(z) = Y_1^-(z)Y_2^+(z)\left[F_1^-(z) - F_2^+(z) + \frac{p_k(z)}{(z-z_0)^k}\right], \tag{3.33}$$

and, for $k < 0$ and $l_2 < \operatorname{Im} z < l_1$,

$$\Psi(z) = Y_1^-(z)Y_2^+(z)\left[F_1^-(z) - F_2^+(z) + C\right], \tag{3.34}$$

where $p_k(z)$ and C as above. For $z \in \{z : \operatorname{Im} z > l_1\}$,

$$\Phi^+(z) = Y_1^+(z)Y_2^+(z)\left[F_1^+(z) - F_2^+(z) + H_1(z)\right], \tag{3.35}$$

where $H_1(z)$ is analytic in $\{z : \operatorname{Im} z > l_1\}$ as $k \geq 0$ and the $\lim_{z\to\infty\ (\operatorname{Im} z > l_1)} H_1(z)$ exists. When $k < 0$, $H_1(z) \equiv C$ (constant). For $z \in \{z : \operatorname{Im} z < l_2\}$,

$$\Phi^-(z) = Y_1^-(z)Y_2^-(z)\left[F_1^-(z) - F_2^-(z) + H_2(z)\right], \tag{3.36}$$

where $H_2(z)$ is analytic in $\{z : \operatorname{Im} z < l_2\}$ when $k \geq 0$ and $\lim_{z\to\infty\ (\operatorname{Im} z < l_2)} H_2(z)$ exists. $H_2(z) \equiv C$ (constant) when $k < 0$.

Hence we get the solution of the boundary value problem (2.1).

Theorem 3.1 *The boundary value problem* (3.1) *with two unknown functions* $\Psi(z)$ *and* $\Phi(z)$ *on two parallel lines has a solution in* $\{z : l_2 < \operatorname{Im} z < l_1\}$ *and* $\{\operatorname{Im} z > l_1\} \cup \{\operatorname{Im} z < l_2\}$, *respectively. Moreover, the general solution can be expressed by* (3.33)-(3.36), *where* $Y_j^{\pm}(z)$ $(j = 1, 2)$ *is defined by* (3.3) *and* $F_j(z)$ $(j = 1, 2)$ *are defined by* (3.8) *and* (3.9). *When* $\kappa > -1$, $p_k(z)$ *is a polynomial with* κ *order, and when* $\kappa \leq -1$, *the necessary conditions for solvability still are* (3.18). *In all, the degree of freedom of the solution is* $\kappa + 1$.

4 Further discussion on solution and solvability conditions

In this section, we say more about the solution (3.33)-(3.36) of (2.1) and the solvability conditions.

(1) The case that the solution lies in $\operatorname{Im} z > l_1$ and $\operatorname{Im} z < l_2$.

As in (3.27) and (3.28), $\Phi^+(z)$ is analytic in $\{z : \operatorname{Im} z > l_1\}$ and $\Phi^-(z)$ is analytic in $\{z : \operatorname{Im} z < l_2\}$. No matter how we choose κ, the boundary value problem (2.1) is solvable and its solution can be expressed by (3.35)-(3.36).

(2) The case that the solution lies in $l_2 < \operatorname{Im} z < l_1$.

It can be seen from the expression of $\Psi(z)$ that z_0 is a $|\kappa|$-order pole of $Y_1^-(z)Y_2^+(z)$ when $\kappa < 0$. In order to ensure (2.1) is solvable, one has $C = F_2^+(z_0) - F_1^-(z_0)$ when $k = -1$, *i.e.*,

$$C = \frac{1}{\sqrt{2\pi}} \int_{-\infty+il_1}^{il_1} f_1(\tau)e^{i\tau z_0}\,d\tau + \frac{1}{\sqrt{2\pi}} \int_{il_2}^{+\infty+il_2} f_2(\tau)e^{i\tau z_0}\,d\tau. \tag{4.1}$$

When $k < -1$, the following $|\kappa| - 1$ conditions are required:

$$\int_{-\infty+il_1}^{il_1} \tau^q f_1(\tau)e^{i\tau z_0}\,d\tau + \int_{il_2}^{+\infty+il_2} \tau^q f_2(\tau)e^{i\tau z_0}\,d\tau = 0, \quad q = 1, 2, \ldots, -\kappa - 1. \tag{4.2}$$

Then $\Psi(z)$ is analytic in $\{z : l_2 < \operatorname{Im} z < l_1\}$ and has a bounded solution. When $\kappa > 0$, z_0 is a κ-order pole of $Y_1^-(z)Y_2^+(z)$, and therefore $Y_1^-(z)Y_2^+(z)\frac{p_\kappa(z)}{(z-z_0)^\kappa}$ is analytic in $\{z : l_2 < \operatorname{Im} z < l_1\}$. Hence, $\Psi(z)$ is analytic in $\{z : l_2 < \operatorname{Im} z < l_1\}$ and $\Psi(z)$ is a constant while $z = \infty$.

For $z \in \{z : l_2 < \operatorname{Im} z < l_1\}$, $\Psi(z)$ can be defined by (3.33), (3.34) if $D_1(z)$ and $D_2^{-1}(z)$ are not zero. Otherwise, if $z_1^*, z_2^*, \ldots, z_n^*$ are common zero-points of $D_1(z)$ and $D_2^{-1}(z)$ with the orders $s_1, s_2, \ldots, s_n$, respectively, then $\Psi^{(j)}(z_q^*) = 0$ $(1 \le q \le n, 1 \le j \le s_q)$. Let $s = \sum_{q=1}^n s_q$. Then the following solvability conditions must be augmented.

As $\kappa \ge 0$, the following $\kappa + 1$ element equations with unknown numbers $c_0, c_1, \ldots, c_\kappa$:

$$\left[\frac{p_\kappa(z)}{(z-z_0)^\kappa}\right]_{z=z_q^*}^{(j)} = \frac{j!i^j}{\sqrt{2\pi}}\left[\int_{-\infty+il_1}^{il_1} \tau^j f_1(\tau)e^{i\tau z}\,d\tau + \int_{il_2}^{+\infty+il_2} \tau^j f_2(\tau)e^{i\tau z}\,d\tau\right]. \tag{4.3}$$

As $\kappa < 0$, the following condition is required:

$$\int_{-\infty+il_1}^{il_1} \tau^j f_1(\tau)e^{i\tau z}\,d\tau + \int_{il_2}^{+\infty+il_2} \tau^j f_2(\tau)e^{i\tau z}\,d\tau = 0 \tag{4.4}$$

$(j = 0, 1, 2, \ldots, s_q;\ q = 1, 2, \ldots, n)$, where $c_0, c_1, \ldots, c_\kappa$ are the coefficients of $p_k(z)$.

(3) The case of solutions at $z = \infty$.

In order to discuss the solution at $z = \infty$, we denote

$$\gamma_\infty^{(1)} = \mu_\infty^{(1)} + i\nu_\infty^{(1)} = \frac{1}{2\pi i}\bigl\{\log D_1(il_1 + \infty) - \log D_1(il_1 - \infty)\bigr\},$$
$$\gamma_\infty^{(2)} = \mu_\infty^{(2)} + i\nu_\infty^{(2)} = \frac{1}{2\pi i}\bigl\{\log D_2(il_2 + \infty) - \log D_2(il_2 - \infty)\bigr\},$$
$$\mu_\infty = \mu_\infty^{(1)} + \mu_\infty^{(2)},$$

where the logarithm function $\log D_j(\tau)$ takes some certain continuous branch when $\operatorname{Re}\zeta > 0$ or $\operatorname{Re}\zeta < 0$ such that $0 \le \mu_\infty < 1$.

If $z = \infty$ is a common node, it follows from $F_j(\infty) = 0$ $(j = 1, 2)$ that $F_j(\zeta) = \frac{F_j^*(\zeta)}{|\zeta|^{\mu_j^*}}$, $\mu_j^* < \mu_\infty^{(j)}$ and $F_j^*(\zeta) \in H$ near $z = \infty$. By the conditions of (2.1), one has $\Psi(\zeta) \in \{1\}$. Therefore, $\Psi(\infty)$ exists and is finite. Denote $F(\zeta) = F_1(\zeta) - F_2(\zeta)$. Note that $0 \le \mu_\infty < 1$. If $\mu_\infty > \frac{1}{2}$, it is clear that $Y_1^-(\zeta)Y_2^+(\zeta)F(\zeta) = O(1/|\zeta|^{\mu_\infty - \varepsilon})$ (where ε is a positive number sufficiently small and $|\zeta|$ is large enough) and $Y_1^-(\zeta)Y_2^+(\zeta)\frac{p_\kappa(\zeta)}{(\zeta-z_0)^\kappa} = O(1)$. If $\mu_\infty \le \frac{1}{2}$, in order to ensure that $\Psi(\zeta) \in \{1\}$, the coefficient e_k of $p_k(z)$ should be taken as

$$e_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty+il_1}^{il_1} f_1(\tau)e^{i\tau z}\,d\tau - \frac{1}{\sqrt{2\pi}} \int_{il_2}^{+\infty+il_2} f_2(\tau)e^{i\tau z}\,d\tau, \quad \text{while } \kappa \ge 0. \tag{4.5}$$

For $\kappa < 0$, the condition (4.2) should hold and $j = 1, 2, \dots, |\kappa|$.

If $z = \infty$ is a special node, *i.e.*, $\mu_\infty = 0$, one can translate it into the case that $\mu_\infty \le \frac{1}{2}$ as a common node. For the rest, similar arguments can be used [9].

As for the boundary value problem with n unknown functions on n ($n > 2$) parallel lines, there is no essential difference for the solving method with the case $n = 2$. We will not elaborate.

5 Example

In this section we consider one important example in practice. In (3.2), suppose

$$D_1(\zeta) = D_2(\zeta) = 1, \qquad C_1(\zeta) = \frac{1}{1+\zeta^2}, \qquad C_2(\zeta) = \frac{1}{2+\zeta^2},$$
$$L_1\colon \quad \zeta = 0, \qquad L_2\colon \quad \zeta = x + i \quad (-\infty < x < +\infty).$$

Without loss of generality, we assume that $z_1 = \frac{3i}{2}$, $z_2 = \frac{-i}{2}$, $z_0 = \frac{i}{2}$. Then we have $\kappa_1 = \kappa_2 = 0$ and hence $\kappa = 0$. Therefore, $\gamma_j(t) = 0$, $\Omega_j(t) = 0$, $Y_j(z) = 1$ ($j = 1, 2$). In this case, by (3.8) and (3.9), we obtain

$$f_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{1+\tau^2} e^{-i\tau t}\, d\tau = \frac{\sqrt{\pi} e^{-t}}{2},$$
$$f_2(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i}^{+\infty+i} \frac{1}{2+\tau^2} e^{-i\tau t}\, d\tau = \frac{\sqrt{\pi} e^{-\sqrt{2}t}}{2},$$

and then

$$F_1^+(z) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f_1(\tau) e^{i\tau z}\, d\tau = \frac{i}{2\sqrt{2}(z+i)}, \tag{5.1}$$
$$F_1^-(z) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f_1(\tau) e^{i\tau z}\, d\tau = \frac{i}{2\sqrt{2}(z-i)}. \tag{5.2}$$

Similarly, we have

$$F_2^+(z) = \frac{i}{2\sqrt{2}(z+\sqrt{2}i)}, \qquad F_2^-(z) = \frac{i}{2\sqrt{2}(z-\sqrt{2}i)}. \tag{5.3}$$

Then we obtain the solutions of (3.2):

$$\begin{aligned}
\Psi(z) &= \frac{i}{2\sqrt{2}(z-i)} - \frac{i}{2\sqrt{2}(z+\sqrt{2}i)} + C, \quad \text{when } 0 < \mathrm{Im} < 1, \\
\Phi^+(z) &= \frac{i}{2\sqrt{2}(z+i)} - \frac{i}{2\sqrt{2}(z+\sqrt{2}i)} + H_1(z), \quad \text{when } \mathrm{Im}\, z > 1, \\
\Phi^-(z) &= \frac{i}{2\sqrt{2}(z-i)} - \frac{i}{2\sqrt{2}(z-\sqrt{2}i)} + H_2(z), \quad \text{when } \mathrm{Im}\, z < 0,
\end{aligned} \tag{5.4}$$

where C is a constant, $H_1(z) = (1 - \frac{\sqrt{2}}{2}) \frac{1}{1+z^2}$, $H_2(z) = 0$.

Competing interests

The author declares that they have no competing interests.

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