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Infinite first order differential systems with nonlocal initial conditions

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Abstract
We discuss the solvability of an infinite system of first order ordinary differential equations on the half line, subject to nonlocal initial conditions. The main result states that if the nonlinearities possess a suitable 'sub-linear' growth then the system has at least one solution. The approach relies on the application, in a suitable Fréchet space, of the classical Schauder-Tychonoff fixed point theorem. We show that, as a special case, our approach covers the case of a system of a finite number of differential equations. An illustrative example of an application is also provided.

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Keywords: infinite systems; nonlocal initial condition; Fréchet space; fixed point

1 Introduction

In the recent paper [1] Bolojan-Nica and co-authors developed a technique that can be used to study the solvability of a system of N first order differential equations of the form

$$\begin{cases} x_1'(t) = g_1(t, x_1(t), x_2(t), \dots, x_N(t)), \\ \dots \\ x_N'(t) = g_N(t, x_1(t), x_2(t), \dots, x_N(t)), \end{cases} \quad t \in [0,1], \tag{1.1}$$

subject to the coupled *nonlocal* conditions

$$\begin{cases} x_1(0) = \sum_{j=1}^N \langle \eta_{1j}, x_j \rangle, \\ \dots \\ x_N(0) = \sum_{j=1}^N \langle \eta_{Nj}, x_j \rangle, \end{cases} \tag{1.2}$$

where $\eta_{ij} : C[0,1] \to \mathbb{R}$ are continuous linear functionals. These functionals involve a suitable support and can be written in a form involving Stieltjes integrals, namely,

$$\langle \eta_{ij}, v \rangle = \int_0^{\hat{t}} v(s)\, dA_{ij}(s) \quad \bigl(v \in C[0,1]\bigr), \tag{1.3}$$

where $\hat{t} \in [0,1]$ is given. Formula (1.3) covers, as special cases, initial conditions of the type

$$\langle \eta_{ij}, v \rangle = \sum_{k=1}^{m} \eta_{ijk} v(t_k), \tag{1.4}$$

where $\eta_{ijk} \in \mathbb{R}$ and $0 \leq t_1 < t_2 < \cdots < t_m \leq \hat{t}$. Note that, if all the functionals η_{ij} have discrete expressions, (1.2) gives a multi-point condition. The approach in [1] relies on the fixed point theorems of Perov, Schauder, and Schaefer and on a vector method for treating systems which uses matrices with spectral radius less than one.

We mention that there exists a wide literature on differential equations subject to nonlocal conditions; we refer here to the pioneering work of Picone [2], the reviews by Whyburn [3], Conti [4], Ma [5], Ntouyas [6] and Štikonas [7], the papers by Karakostas and Tsamatos [8, 9] and by Webb and Infante [10, 11].

Note that on non-compact intervals the problem of finding solutions of differential systems becomes more complicated, since the associated integral formulation may display a lack of compactness. For systems of a finite number of differential equations on non-compact intervals, subject to linear (or more general) conditions, we refer to the papers by Andres *et al.* [12], Cecchi *et al.* [13, 14], De Pascale *et al.* [15], Marino and Pietramala [16] and Marino and Volpe [17].

On the other hand, infinite systems of differential equations often occur in applications, for example in stochastic processes and quantum mechanics [18] (also see Chapter XXIII in [19]) and in the physical chemistry of macromolecules [20]. For some recent work on infinite systems on compact intervals we refer the reader to the papers by Frigon [21], by Banaś *et al.* [22–24] and references therein.

The methodology to treat initial value problems for infinite systems often relies on using the theory of differential equations in Banach spaces, but it is also known that this approach can reduce the set of solutions and some of their specific properties. A fruitful alternative is to deal with such systems in the framework of locally convex spaces. For a discussion emphasizing these aspects we refer the reader to the paper by Jebelean and Reghiş [25] and references therein.

The case of initial value problems for infinite systems on non-compact intervals has been investigated in [25], where the authors extended a comparison theorem due to Stokes [26], valid for the scalar case. Here we utilize the framework developed in [25], in order to deal with infinite systems of nonlocal initial value problems on the half line. To be more precise, let us denote by J the half line $[0,+\infty)$ and by $\mathcal{S}$ the space of all real valued sequences, that is

$$\mathcal{S} := \mathbb{R}^{\mathbb{N}} = \{x = (x_1, x_2, \dots, x_n, \dots) : x_n \in \mathbb{R}, \forall n \in \mathbb{N}\},$$

which is assumed to be endowed with the product topology. We fix $t_0 \in (0,+\infty)$ and consider the space $C[0,t_0]$ of all continuous real valued functions defined on $[0,t_0]$ with the usual supremum norm. For each $n \in \mathbb{N}$, let $f_n : J \times \mathcal{S} \to \mathbb{R}$ be a continuous function and $\alpha_n : C[0,t_0] \to \mathbb{R}$ be a continuous linear functional.

Here we deal with the solvability of the nonlocal initial value problem

$$\begin{cases} x_n'(t) = f_n(t, x_1(t), x_2(t), \dots, x_n(t), \dots), & t \in J, \\ x_n(0) = \langle \alpha_n, x_n|_{[0,t_0]} \rangle \end{cases} \quad (n \in \mathbb{N}), \tag{1.5}$$

where, in a similar way to above, $\langle \alpha_n, v \rangle$ denotes the value of the functional α_n at $v \in C[0,t_0]$. By a *solution* of problem (1.5) we mean a sequence of functions $x_n : J \to \mathbb{R}$, $n \in \mathbb{N}$, such that each x_n is derivable on J and satisfies

$$x_n'(t) = f_n\bigl(t, x_1(t), x_2(t), \ldots, x_n(t), \ldots\bigr), \quad \forall t \in J,$$

together with the nonlocal initial condition $x_n(0) = \langle \alpha_n, x_n|_{[0,t_0]}\rangle$.

The rest of the paper is organized as follows. In Section 2 we transform the system (1.5) into a fixed point problem within a suitable Fréchet space. In Section 3 we present the main existence result, we show how this approach can be utilized in the special case of a finite number of differential equations and, finally, we provide an example that illustrates our theory. Our results are new in the context of nonlocal problems for infinite systems and complement earlier results in the literature.

2 An equivalent fixed point problem

We firstly introduce some notations that are used in the sequel. For $x, y \in S$, the product element $x \cdot y \in \mathcal{S}$ is defined by $x \cdot y = (x_1y_1, x_2y_2, \ldots, x_ny_n, \ldots)$; if x is such that $x_n \neq 0$ for all $n \in \mathbb{N}$, then we denote by $x^{-1} = (x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}, \ldots)$. We also set

$$[x]_n = \max_{1 \le i \le n} |x_i|, \quad \forall n \in \mathbb{N},$$

and notice that the family of seminorms $([\cdot]_n)_{n\in\mathbb{N}}$ on the space $\mathcal{S}$ generates a product topology. Endowed with this topology, $\mathcal{S}$ is a Fréchet space, that is, a locally convex space which is complete with respect to a translation invariant metric.

For the sake of simplicity, we rewrite (1.5) in a more compact form. In order to do this, we define the map $f : J \times \mathcal{S} \to \mathcal{S}$ by

$$f(t,y) = \bigl(f_1(t,y), f_2(t,y), \ldots, f_n(t,y), \ldots\bigr), \quad \forall (t,y) \in J \times \mathcal{S}$$

and the operator $\alpha : C([0,t_0], \mathcal{S}) = \prod_{n=1}^{\infty} C[0,t_0] \to \mathcal{S}$ by setting

$$\langle \alpha, v\rangle = \bigl(\langle \alpha_1 v_1\rangle, \langle \alpha_2, v_2\rangle, \ldots, \langle \alpha_n, v_n\rangle, \ldots\bigr), \quad \forall v = (v_1, v_2, \ldots, v_n, \ldots) \in C\bigl([0,t_0], \mathcal{S}\bigr). \quad (2.1)$$

Notice that f is continuous and α is linear and continuous. Therefore the problem (1.5) can be re-written as

$$\begin{cases} x'(t) = f(t, x(t)), & t \in J, \\ x(0) = \langle \alpha, x|_{[0,t_0]}\rangle \end{cases} \quad (2.2)$$

the derivation $'$ being understood component-wise. Integrating (2.2) we get

$$x(t) = c + \int_0^t f\bigl(s, x(s)\bigr)\, ds,$$

with some $c \in \mathcal{S}$. The nonlocal initial condition yields

$$c = c \cdot \langle \alpha, \mathbb{I}\rangle + \left\langle \alpha, \left(\int_0^{\cdot} f\bigl(s, x(s)\bigr)\, ds\right)\Big|_{[0,t_0]}\right\rangle,$$

where $\mathbb{I} = (1, 1, \dots, 1, \dots)$. Thus, assuming that

$$\langle \alpha_n, 1 \rangle \neq 1, \quad \forall n \in \mathbb{N}, \tag{2.3}$$

we obtain

$$x(t) = \bigl(\mathbb{I} - \langle \alpha, \mathbb{I} \rangle\bigr)^{-1} \cdot \left\langle \alpha, \left(\int_0^{\cdot} f\bigl(s, x(s)\bigr)\, ds \right)\Big|_{[0,t_0]} \right\rangle + \int_0^t f\bigl(s, x(s)\bigr)\, ds. \tag{2.4}$$

We consider the space $C(J)$ of all continuous real valued functions defined on J with the topology of the uniform convergence on compacta. Given $(t_k)_{k\in\mathbb{N}}$ a strictly increasing sequence in $(0, +\infty)$ with $t_k \to +\infty$ as $k \to \infty$, this topology is generated by the (countable) family of seminorms

$$\nu_k(h) = \max_{t\in[0,t_k]} \bigl|h(t)\bigr|, \quad h \in C(J), k \in \mathbb{N}.$$

Note that $C(J)$ is a Fréchet space. We denote by $C(J, \mathcal{S})$ the space of all $\mathcal{S}$-valued continuous functions on J. Since

$$C(J, \mathcal{S}) = \prod_{n=1}^{\infty} C(J),$$

the space $C(J, \mathcal{S})$ equipped with the product topology also becomes a Fréchet space.

Proposition 2.1 *Under the assumption* (2.3) *we have*:

(i) $x \in C(J, \mathcal{S})$ *is a solution of problem* (1.5) *iff it is a fixed point of the operator* $T : C(J, \mathcal{S}) \to C(J, \mathcal{S})$ *defined by*

$$T(\nu)(t) = \bigl(\mathbb{I} - \langle \alpha, \mathbb{I} \rangle\bigr)^{-1} \cdot \left\langle \alpha, \left(\int_0^{\cdot} f\bigl(s, \nu(s)\bigr)\, ds \right)\Big|_{[0,t_0]} \right\rangle + \int_0^t f\bigl(s, \nu(s)\bigr)\, ds; \tag{2.5}$$

(ii) *T is continuous and maps bounded sets into relatively compact sets.*

Proof (i) This fact follows by virtue of (2.4) and (2.5).

(ii) It is a standard matter to check that, for all $k \in \mathbb{N}$, the operator $R_k : C([0, t_k], \mathcal{S}) \to C([0, t_k], \mathcal{S})$ defined by

$$R_k(\nu)(t) = \int_0^t f\bigl(s, \nu(s)\bigr)\, ds, \quad \nu \in C\bigl([0, t_k], \mathcal{S}\bigr), t \in [0, t_k]$$

is continuous. We now show that $R : C(J, \mathcal{S}) \to C(J, \mathcal{S})$ given by

$$R(\nu)(t) = \int_0^t f\bigl(s, \nu(s)\bigr)\, ds, \quad \nu \in C(J, \mathcal{S}), t \in J,$$

inherits the same property. Indeed let $\nu \in C(J, \mathcal{S})$ and $(\nu^m)_{m\in\mathbb{N}} \subset C(J, \mathcal{S})$ be with $\nu^m \to \nu$, as $m \to \infty$. Then one has $\nu^m|_{[0,t_k]} \to \nu|_{[0,t_k]}$ in $C([0, t_k], \mathcal{S})$, for all $k \in \mathbb{N}$. Using the continuity of R_k, it follows that

$$R\bigl(\nu^m\bigr)|_{[0,t_k]} = R_k\bigl(\nu^m|_{[0,t_k]}\bigr) \to R_k(\nu|_{[0,t_k]}) = R(\nu)|_{[0,t_k]}, \quad \text{for all } k \in \mathbb{N},$$

which means that $R(\nu^m) \to R(\nu)$ in $C(J, \mathcal{S})$.

We also have that R maps bounded sets into relatively compact sets. To see this, let $M \subset C(J, \mathcal{S})$ be bounded. Using the Arzelà-Ascoli theorem, it is routine to show that for any fixed $n \in \mathbb{N}$, the set

$$\left\{R(v)_n|_{[0,t_k]} : v \in M\right\} = \left\{R_k(v|_{[0,t_k]})_n : v \in M\right\}$$

is relatively compact in $C[0, t_k]$ for each $k \in \mathbb{N}$. Hence, the set $\{R(v)_n : v \in M\}$ is relatively compact in $C(J)$. By virtue of the Tychonoff theorem, we have that $R(M)$ is relatively compact in the product space $C(J, \mathcal{S})$.

Now, the conclusion follows from the equality

$$T(v) = \left(\mathbb{I} - \langle \alpha, \mathbb{I}\rangle\right)^{-1} \cdot \left\langle \alpha, R(v)|_{[0,t_0]}\right\rangle + R(v), \quad \forall v \in C(J, \mathcal{S}),$$

and the above properties of the operator R. □

We shall make use of the following consequence of the classical Schauder-Tychonoff fixed point theorem; see for example Corollary 2, p.107 of [27].

Theorem 2.1 *Let K be a closed convex set in a Hausdorff, complete locally convex vector space and $F : K \mapsto K$ be continuous. If $F(K)$ is relatively compact then F has a fixed point in K.*

3 Main result

We present now the main result of the paper. In order to do this, we fix a strictly increasing sequence $(n_p)_{p\in\mathbb{N}} \subset \mathbb{N}$ and $(t_p)_{p\in\mathbb{N}}$ a sequence of real numbers, such that

$$t_0 < t_1 < \cdots < t_p \to +\infty, \quad \text{as } p \to \infty, \tag{3.1}$$

and, for α as in (2.1), we denote

$$\|\alpha\|_* = \left(\|\alpha_1\|, \|\alpha_2\|, \dots, \|\alpha_n\|, \dots\right), \tag{3.2}$$

where $\|\alpha_n\|$ stands for the norm of the functional α_n in the dual space of $C[0, t_0]$ ($n \in \mathbb{N}$).

Theorem 3.1 *Assume that* (2.3) *holds and that f satisfies the condition*

$$\left[f(t,x)\right]_{n_p} \leq \begin{cases} A_p(t)[x]_{n_p} + B_p, & t \in [0, t_0], \\ C_p([x]_{n_p} + 1), & t \in [t_0, t_p] \end{cases} \quad (x \in \mathcal{S}, p \in \mathbb{N}) \tag{3.3}$$

with $A_p \in L^1_+(0, t_0)$ and $B_p, C_p \in \mathbb{R}^+$. If the inequality

$$\left\{\left[\left(\mathbb{I} - \langle \alpha, \mathbb{I}\rangle\right)^{-1}\right]_{n_p}\left[\|\alpha\|_*\right]_{n_p} + 1\right\}\|A_p\|_{L^1} < 1 \tag{3.4}$$

holds for every $p \in \mathbb{N}$, then the problem (2.2) (*hence* (1.5)) *has at least one solution $x \in C(J, \mathcal{S})$.*

Proof Taking into account Proposition 2.1, it suffices to show that T has a fixed point in $C(J,\mathcal{S})$. In order to do this, we shall apply Theorem 2.1. Firstly, note that

$$\begin{aligned}\left[\left\langle \alpha, \left(\int_0^{\cdot} u(s)\,ds\right)\Big|_{[0,t_0]}\right\rangle\right]_n &= \max_{1\le i\le n}\left|\left\langle \alpha_i, \left(\int_0^{\cdot} u_i(s)\,ds\right)\Big|_{[0,t_0]}\right\rangle\right| \\ &\le \big[\|\alpha\|_*\big]_n \int_0^{t_0} \big[u(s)\big]_n\,ds,\end{aligned} \tag{3.5}$$

for all $u \in C(J,\mathcal{S})$ and $n \in \mathbb{N}$.

For each $p \in \mathbb{N}$, let

$$G_p := \big[\big(\mathbb{I} - \langle\alpha,\mathbb{I}\rangle\big)^{-1}\big]_{n_p}\big[\|\alpha\|_*\big]_{n_p} + 1$$

and $\theta_p > 0$ be such that

$$M_p := G_p\|A_p\|_{L^1} + \frac{C_p}{\theta_p} < 1;$$

note that such a θ_p exists from (3.4).

We introduce the (continuous) seminorms

$$\begin{aligned}&\mathcal{P}_p(x) = \max_{0\le t\le t_0}\big[x(t)\big]_{n_p}, \qquad \mathcal{Q}_p(x) = \max_{t_0\le t\le t_p} e^{-\theta_p(t-t_0)}\big[x(t)\big]_{n_p}, \\ &\mathcal{R}_p(x) = \max\big\{\mathcal{P}_p(x), \mathcal{Q}_p(x)\big\}, \quad x \in C(J,\mathcal{S}), p \in \mathbb{N}.\end{aligned}$$

We denote

$$K_p := G_p t_0 B_p + C_p(t_p - t_0),$$

we take ρ_p such that

$$\rho_p \ge (1 - M_p)^{-1}K_p, \tag{3.6}$$

and define

$$\Omega_p := \big\{x \in C(J,\mathcal{S}) : \mathcal{R}_p(x) \le \rho_p\big\}, \quad p \in \mathbb{N}.$$

Since each Ω_p is closed and convex, the (nonempty) set

$$\Omega := \bigcap_{p=1}^{\infty}\Omega_p$$

has the same properties. Next, we prove that $T(\Omega) \subset \Omega$.

Let $x \in \Omega$ and $p \in \mathbb{N}$. If $t \in [0,t_0]$, using (3.5), we have

$$\begin{aligned}\big[T(x)(t)\big]_{n_p} &\le \left[\big(\mathbb{I} - \langle\alpha,\mathbb{I}\rangle\big)^{-1}\cdot\left\langle \alpha, \left(\int_0^{\cdot} f\big(s,x(s)\big)\,ds\right)\Big|_{[0,t_0]}\right\rangle\right]_{n_p} + \left[\int_0^t f\big(s,x(s)\big)\,ds\right]_{n_p} \\ &\le \big[\big(\mathbb{I} - \langle\alpha,\mathbb{I}\rangle\big)^{-1}\big]_{n_p}\big[\|\alpha\|_*\big]_{n_p}\int_0^{t_0}\big[f\big(s,x(s)\big)\big]_{n_p}\,ds + \int_0^{t_0}\big[f\big(s,x(s)\big)\big]_{n_p}\,ds \\ &= G_p\int_0^{t_0}\big[f\big(s,x(s)\big)\big]_{n_p}\,ds.\end{aligned}$$

The growth condition (3.3) gives

$$\big[T(x)(t)\big]_{n_p} \le G_p\big\{\|A_p\|_{L^1}\mathcal{P}_p(x) + t_0 B_p\big\},$$

which, taking the maximum over $[0, t_0]$, yields

$$\mathcal{P}_p\big(T(x)\big) \le G_p\big\{\|A_p\|_{L^1}\mathcal{P}_p(x) + t_0 B_p\big\}. \tag{3.7}$$

If $t \in [t_0, t_p]$, then arguing as above we get

$$\begin{aligned}
\big[T(x)(t)\big]_{n_p} &\le G_p\big\{\|A_p\|_{L^1}\mathcal{P}_p(x) + t_0 B_p\big\} + \int_{t_0}^{t}\big[f\big(s, x(s)\big)\big]_{n_p}\,ds \\
&\le G_p\big\{\|A_p\|_{L^1}\mathcal{P}_p(x) + t_0 B_p\big\} + C_p\int_{t_0}^{t}\big(\big[x(s)\big]_{n_p} + 1\big)\,ds \\
&= G_p\big\{\|A_p\|_{L^1}\mathcal{P}_p(x) + t_0 B_p\big\} \\
&\quad + C_p\int_{t_0}^{t} e^{\theta_p(s-t_0)}e^{-\theta_p(s-t_0)}\big[x(s)\big]_{n_p}\,ds + C_p(t_p - t_0) \\
&\le G_p\big\{\|A_p\|_{L^1}\mathcal{P}_p(x) + t_0 B_p\big\} + C_p\left\{\mathcal{Q}_p(x)\frac{e^{\theta_p(t-t_0)}}{\theta_p} + t_p - t_0\right\}.
\end{aligned} \tag{3.8}$$

Multiplying the inequality (3.8) by $e^{-\theta_p(t-t_0)}$ (≤ 1), we obtain

$$\big[T(x)(t)\big]_{n_p}e^{-\theta_p(t-t_0)} \le G_p\big\{\|A_p\|_{L^1}\mathcal{P}_p(x) + t_0 B_p\big\} + C_p\left\{\frac{\mathcal{Q}_p(x)}{\theta_p} + t_p - t_0\right\},$$

and, taking the maximum over $[t_0, t_p]$, we obtain

$$\mathcal{Q}_p\big(T(x)\big) \le G_p\big\{\|A_p\|_{L^1}\mathcal{P}_p(x) + t_0 B_p\big\} + C_p\left\{\frac{\mathcal{Q}_p(x)}{\theta_p} + t_p - t_0\right\}. \tag{3.9}$$

From (3.7), (3.9), and (3.6), we obtain

$$\begin{aligned}
\mathcal{R}_p\big(T(x)\big) &\le G_p\big\{\|A_p\|_{L^1}\mathcal{R}_p(x) + t_0 B_p\big\} + C_p\left\{\frac{\mathcal{R}_p(x)}{\theta_p} + t_p - t_0\right\} \\
&= M_p\mathcal{R}_p(x) + K_p \\
&\le M_p\rho_p + \rho_p(1 - M_p) = \rho_p,
\end{aligned}$$

showing that $x \in \Omega_p$, for all $p \in \mathbb{N}$; hence, $x \in \Omega$. Finally, note that Ω is a bounded subset of $C(J, \mathcal{S})$. Then, from Proposition 2.1, the continuous operator T maps Ω into a relatively compact set. Therefore, by virtue of Theorem 2.1, T has a fixed point in Ω and the proof is complete. □

Remark 3.1 Note that the result above is also valid in the case of a non-compact interval $J_b = [0, b)$ with $t_0 \in (0, b)$, $b \in (0, +\infty)$, instead of J. The only modification is to consider $t_p \to b$ in (3.1), instead of $t_p \to +\infty$, as $p \to \infty$.

We now illustrate how this methodology can be applied in the case of a finite system of differential equations. The assumptions are relatively easy to check and we include this result for completeness.

We fix $N \in \mathbb{N}$ and discuss the solvability of the nonlocal initial value problem

$$\begin{cases} x_n'(t) = g_n(t, x_1(t), x_2(t), \ldots, x_N(t)), & t \in J, \\ x_n(0) = \langle \eta_n, x_n|_{[0,t_0]} \rangle \end{cases} \quad (n = 1, \ldots, N), \tag{3.10}$$

where $g_n : J \times \mathbb{R}^N \to \mathbb{R}$ are continuous functions and $\eta_n : C[0, t_0] \to \mathbb{R}$ are continuous linear functionals, satisfying

$$\langle \eta_n, 1 \rangle \neq 1, \quad n = 1, \ldots, N. \tag{3.11}$$

We denote

$$\|y\|_\infty := \max_{1 \le n \le N} |y_n|, \quad \forall y = (y_1, \ldots, y_N) \in \mathbb{R}^N$$

and

$$\eta_* := \bigl(\|\eta_1\|, \ldots, \|\eta_N\|\bigr), \qquad \underline{\eta} := \left(\frac{1}{1 - \langle \eta_1, 1 \rangle}, \ldots, \frac{1}{1 - \langle \eta_N, 1 \rangle} \right).$$

With the notation above we can state the following existence result.

Theorem 3.2 *Let $(t_p)_{p \in \mathbb{N}}$ be a sequence satisfying* (3.1). *Assume the condition* (3.11) *holds and that $g := (g_1, \ldots, g_N)$ satisfies*

$$\bigl\|g(t, x)\bigr\|_\infty \le \begin{cases} A(t)\|x\|_\infty + B, & t \in [0, t_0], \\ C_p(\|x\|_\infty + 1), & t \in [t_0, t_p] \end{cases} \quad \bigl(x \in \mathbb{R}^N, p \in \mathbb{N}\bigr) \tag{3.12}$$

with $A \in L^1_+(0, t_0)$ and $B, C_p \in \mathbb{R}^+$. If the inequality

$$\bigl\{\|\underline{\eta}\|_\infty \|\eta_*\|_\infty + 1\bigr\} \|A\|_{L^1} < 1 \tag{3.13}$$

holds, then the nonlocal initial value problem (3.10) *has at least one solution.*

Proof Consider the problem (2.2) with

$$f = (g_1, g_2, \ldots, g_N, g_N, \ldots), \qquad \alpha = (\eta_1, \eta_2, \ldots, \eta_N, \eta_N, \ldots),$$

and note that if $x = (x_1, x_2, \ldots, x_N, \ldots) \in C(J, \mathcal{S})$ is a solution, then $(x_1, x_2, \ldots, x_N)$ solves (3.10). To prove the solvability of (2.2) with the above choices of f and α, we apply Theorem 3.1 with $n_p = N + p - 1$, $\forall p \in \mathbb{N}$. In this view, it follows from (3.12) that the growth condition (3.3) is fulfilled with $A_p = A$ and $B_p = B$. Also we have

$$\bigl\{\bigl[\bigl(\mathbb{I} - \langle \alpha, \mathbb{I} \rangle\bigr)^{-1}\bigr]_{n_p} \bigl[\|\alpha\|_*\bigr]_{n_p} + 1\bigr\} \|A_p\|_{L^1} = \bigl\{\|\underline{\eta}\|_\infty \|\eta_*\|_\infty + 1\bigr\} \|A\|_{L^1},$$

for all $p \in \mathbb{N}$. Hence, the condition (3.4) is satisfied from (3.13) and the proof is complete. □

We conclude the paper with an example, where we notice that the constants that occur in Theorem 3.1 can be effectively computed.

Example 3.1 Consider the nonlocal initial value problem:

$$\begin{cases} x_n'(t) = \frac{k_n}{1+t^2} x_n + t\cos x_{n+1}, & t \in J, \\ x_n(0) = \frac{1}{n+t_0}\int_0^{t_0} x_n(s)\,ds \end{cases} \quad (n \in \mathbb{N}). \tag{3.14}$$

The system (3.14) has at least one solution, provided that $k = (k_1, k_2, \dots, k_n, \dots) \in \mathcal{S}$ satisfies

$$|k_n| < \frac{1}{(1+t_0)\arctan t_0}, \quad \forall n \in \mathbb{N}. \tag{3.15}$$

To see this, we apply Theorem 3.1 with

$$f_n(t, x_1, x_2, \dots, x_n, \dots) = \frac{k_n}{1+t^2} x_n + t\cos x_{n+1},$$
$$\langle \alpha_n, v\rangle = \frac{1}{n+t_0}\int_0^{t_0} v(s)\,ds, \quad \forall v \in C[0, t_0],$$

$n_p = p$ and $(t_p)_{p\in\mathbb{N}}$ an arbitrary sequence satisfying (3.1). We have

$$\|\alpha_n\| = \langle \alpha_n, 1\rangle = \frac{t_0}{n+t_0} < 1, \quad \forall n \in \mathbb{N}, \tag{3.16}$$

and therefore (2.3) is satisfied. Also the growth condition (3.3) is fulfilled with

$$A_p(t) = \frac{[k]_p}{1+t^2}, \qquad B_p = t_0, \qquad C_p = [k]_p + t_p, \quad \forall p \in \mathbb{N}. \tag{3.17}$$

Using (3.16) and (3.17) we can compute

$$\left\{\left[\left(\mathbb{I} - \langle \alpha, \mathbb{I}\rangle\right)^{-1}\right]_p \left[\|\alpha\|_*\right]_p + 1\right\} \|A_p\|_{L^1} = [k]_p (1+t_0)\arctan t_0$$

and (3.4) holds from (3.15).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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