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Exponential decay rate for a quasilinear von Karman equation of memory type with acoustic boundary conditions

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Abstract

In this paper, we show the exponential decay result of the quasilinear von Karman equation of memory type with acoustic boundary conditions. This work is devoted to investigating the influence of kernel function g and the effect of the nonlinear term $|u'|^\rho u''$ and to proving exponential decay rates of solutions when g does not necessarily decay exponentially. This result improves on earlier ones concerning the exponential decay.

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Keywords: von Karman equation; acoustic boundary condition; exponential decay rate

1 Introduction

Let $\Omega \subset R^2$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, $\Gamma_0 \cup \Gamma_1 = \partial\Omega$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, Γ_0 and Γ_1 have positive measure and $\nu = (\nu_1, \nu_2)$ be the outward unit normal vector, and by $\tau = (-\nu_2, \nu_1)$ we denote the corresponding unit tangent vector on $\partial\Omega$. We define $u' = \frac{\partial u}{\partial t}$, $\Delta u = \sum_{i=1}^2 \frac{\partial^2 u}{\partial x_i^2}$, $\Delta^2 u = \sum_{i=1}^2 \frac{\partial^4 u}{\partial x_i^4}$, where $x = (x_1, x_2) \in \Omega$. In this paper, we consider the quasilinear von Karman equation of memory type with acoustic boundary conditions:

$$|u'|^\rho u'' - \alpha\Delta u'' + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\,ds = [u, v] \quad \text{in } \Omega \times (0,\infty), \tag{1.1}$$

$$\Delta^2 v = -[u, u] \quad \text{in } \Omega \times (0,\infty), \tag{1.2}$$

$$v = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0,\infty), \tag{1.3}$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_0 \times (0,\infty), \tag{1.4}$$

$$\mathcal{B}_1 u - \mathcal{B}_1\left(\int_0^t g(t-s)u(s)\,ds\right) = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.5}$$

$$\mathcal{B}_2 u - \alpha\frac{\partial u''}{\partial \nu} - \mathcal{B}_2\left(\int_0^t g(t-s)u(s)\,ds\right) = -y' \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.6}$$

$$u' + p(x)y' + q(x)y = 0 \quad \text{on } \Gamma_1 \times (0,\infty), \tag{1.7}$$

$$u(0) = u_0, \qquad u'(0) = u_1 \quad \text{in } \Omega, \qquad y(0) = y_0 \quad \text{on } \Gamma_1, \tag{1.8}$$

where $\alpha > 0$, $\rho > 0$. The functions g, p and q satisfy some conditions to be specified later, the von Karman bracket $[\cdot,\cdot]$ is given by

$$[u,\phi] \equiv u_{x_1x_1}\phi_{x_2x_2} + u_{x_2x_2}\phi_{x_1x_1} - 2u_{x_1x_2}\phi_{x_1x_2}$$

and

$$\begin{aligned} \mathcal{B}_1 u &= \Delta u + (1-\mu)B_1 u, \\ \mathcal{B}_2 u &= \frac{\partial}{\partial \nu}\Delta u + (1-\mu)B_2 u, \end{aligned}$$

here $\mu \in (0, \frac{1}{2})$ is Poisson's ratio,

$$\begin{aligned} B_1 u &= 2\nu_1\nu_2 u_{x_1} u_{x_2} - \nu_1^2 u_{x_2x_2} - \nu_2^2 u_{x_1x_1}, \\ B_2 u &= \frac{\partial}{\partial \tau}\bigl[\bigl(\nu_1^2 - \nu_2^2\bigr)u_{x_1x_2} + \nu_1\nu_2(u_{x_2x_2} - u_{x_1x_1})\bigr]. \end{aligned}$$

The physical applications of the above system are related to the problem of noise control and suppression in practical applications. In this model, problem (1.1)-(1.8) describes small vibrations of a thin homogeneous isotropic plate of uniform thickness of α with acoustic boundary conditions on a portion of the boundary and the Dirichlet boundary condition on the rest, $u(x,t)$ denotes the transversal displacement of the plate, the Airy stress function, $v(x,t)$ a vibrating plate and $y(x,t)$ the normal displacement to the boundary; and Eq. (1.1) was interpreted by the stresses at any instant dependent on the complete history of strains. In this model, the portion of the boundary denoted by Γ_1 is a locally reacting plate, with each point on the plate acting like a damped harmonic oscillator in the response to excess stress from the fluid in the interior Eq. (1.7). The coupling between the acoustic stress and the displacement of the boundary is given by Eqs. (1.5)-(1.6). The noise sound propagates through some acoustic medium, for example, through air, in a room which is characterized by a bounded domain Ω and whose walls, ceiling and floor are described by the boundary conditions. This is the description of Wu in [1]. For more physical explanation of wave equations with acoustic boundary, we refer the reader to [2–10]. The acoustic boundary conditions were introduced by Beale and Rosencrans in [3, 11], where the authors proved the global existence and regularity of solutions in a Hilbert space of the linear problem

$$\begin{aligned} &u_{tt} = \Delta u \quad \text{in } \Omega \times (0,\infty), \\ &\frac{\partial u}{\partial \nu} = z_t \quad \text{on } \partial\Omega \times (0,\infty), \\ &u_t + m(x)z_{tt} + p(x)z_t + q(x)z = 0 \quad \text{on } \partial\Omega \times (0,\infty), \end{aligned}$$

where m, p and q are nonnegative functions on the boundary with m and q being strictly positive.

Frota and Larkin [12] eliminated the term z_{tt} and established global solvability and decay estimates for a linear wave equation with boundary conditions

$$u' + p(x)z' + q(x)z = 0 \quad \text{in } \Gamma_1 \times (0, \infty),$$
$$\frac{\partial u}{\partial \nu} = h(x)z' \quad \text{on } \Gamma_1 \times (0, \infty),$$
$$u = 0 \quad \text{on } \Gamma_0 \times (0, \infty).$$

The decay rate estimated for wave equations of memory type with acoustic boundary conditions was studied by Park and Park [9], and Park *et al.* [13] investigated the general decay for a von Karman equation of memory type with acoustic boundary conditions. Park and Ha [14] considered the Klein-Gordon equation with damping $|u_t|^\rho u_t$ and acoustic boundary conditions. Wu [1] also proved the well-posedness for variable-coefficient wave equation with nonlinear damped acoustic boundary conditions. Recently Boukhatem and Benabderrahmane [15] proved the existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions. The semilinear wave equation with porous acoustic boundary conditions was studied by Graber and Said-Houari [16], and Graber [17] investigated the strong stability and uniform decay of solutions to a wave equation with semilinear porous acoustic boundary conditions. The uniform decay for a von Karman plate equation with a boundary memory condition was studied by Park and Park [18]. Park and Kang [19] considered the uniform decay of solutions for von Karman equations of dynamic viscoelasticity with memory. The asymptotic behavior and energy decay of the solutions for a quasilinear viscoelastic problems were studied by many authors [20–22], and Kang [23] proved the exponential decay for quasilinear von Karman equation with memory.

Motivated by [13] and [23], in this paper we prove the exponential decay of a quasilinear von Karman equation of memory type with acoustic boundary conditions for problem (1.1)-(1.8) satisfying

$$\frac{d}{dt}\mathcal{L}(t) \le -C\xi(t)\mathcal{L}(t) \quad \text{for some } C > 0 \text{ and for all } t \ge t_0.$$

This is done by applying the idea presented in [4, 5] with some necessary modification due to the nature of the problem treated here. To the best of our knowledge, there are no results for a quasilinear von Karman equation of memory type with acoustic boundary conditions. Thus this work is meaningful. In particular, the nonlinear term $|u'|^\rho u''$ is difficult to analyze, and the result of the energy decay is dependent on the kernel g. So we overcome the issue using the change of Lyapunov functional. The structure of this paper is as follows. In Section 2, we give some notation and material needed for our work. In Section 3, we prove the main results.

2 Preliminaries

In this section, we present some material needed in the proof of our result. Throughout this paper, we define

$$V = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_0\},$$
$$W = \left\{u \in H^2(\Omega) : u = \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_0\right\},$$

$$(u,v) = \int_\Omega u(x)v(x)\,dx,$$
$$(u,v)_{\Gamma_1} = \int_{\Gamma_1} u(x)v(x)\,d\Gamma.$$

For a Banach space X, $\|\cdot\|_X$ denotes the norm of X. For simplicity, we denote $\|\cdot\|_{L^2(\Omega)}$ by norm $\|\cdot\|$ and $\|\cdot\|_{L^1(\Gamma_1)}$ by $\|\cdot\|_{\Gamma_1}$, respectively. We define, for all $1 \leq p < \infty$,

$$\|u\|_p^p = \int_\Omega |u(x)|^p\,dx.$$

For $0 < \mu < \frac{1}{2}$, the bilinear form $a(\cdot,\cdot)$ is defined by

$$\begin{aligned} a(u,v) = \int_\Omega \{ & u_{x_1x_1}v_{x_1x_1} + u_{x_2x_2}v_{x_2x_2} + \mu(u_{x_1x_1}v_{x_2x_2} \\ & + u_{x_2x_2}v_{x_1x_1}) + 2(1-\mu)u_{x_1x_2}v_{x_1x_2}\}\,dx. \end{aligned} \tag{2.1}$$

A simple calculation, based on the integration by part formula, yields

$$\int_\Omega \Delta^2 uv\,dx = a(u,v) - \left(\mathcal{B}_1 u, \frac{\partial v}{\partial \nu}\right)_\Gamma + (\mathcal{B}_2 u, v)_\Gamma.$$

Thus, for $(u,v) \in (H^4(\Omega)\cap W)\times W$, it holds

$$\int_\Omega \Delta^2 uv\,dx = a(u,v) - \left(\mathcal{B}_1 u, \frac{\partial v}{\partial \nu}\right)_{\Gamma_1} + (\mathcal{B}_2 u, v)_{\Gamma_1}.$$

Since $\Gamma_0 \neq \emptyset$, we have (see [6]) that $\sqrt{a(u,u)}$ is equivalent to the $H^2(\Omega)$ norm in W, that is,

$$C_1\|u\|^2_{H^2(\Omega)} \leq a(u,u) \leq C_2\|u\|^2_{H^2(\Omega)} \quad \text{for some } C_1, C_2 > 0. \tag{2.2}$$

This and the Sobolev imbedding theorem imply that for some positive constants C_p, $\tilde{C}_p$ and C_s,

$$\begin{aligned} &\|u\|^2 \leq C_p a(u,u), \qquad \|u\|^2_{\Gamma_1} \leq \tilde{C}_p a(u,u) \quad \text{and} \\ &\|\nabla u\|^2 \leq C_s a(u,u) \quad \text{for all } u \in W. \end{aligned} \tag{2.3}$$

And since $V \hookrightarrow L^{\rho+2}(\Omega)$, $\rho > 0$, there exists a positive constant K such that

$$\|u\|_{\rho+2} \leq K\|\nabla u\|_2, \quad u \in V. \tag{2.4}$$

By (2.1) and Young's inequality, we deduce that

$$a(u,v) \leq \delta\|u\|^2_{H^2(\Omega)} + \frac{5}{8\delta}\|v\|^2_{H^2(\Omega)} \quad \text{for all } \delta > 0.$$

From this and (2.2), it holds that

$$a(u,v) \leq \delta\|u\|^2_{H^2(\Omega)} + \frac{5}{8C_1\delta}a(v,v) \quad \text{for all } \delta > 0. \tag{2.5}$$

Now we introduce the relative results of the Airy stress function and von Karman bracket $[\cdot,\cdot]$.

Lemma 2.1 [24] *Let u, w be functions in $H^2(\Omega)$ and v in $H_0^2(\Omega)$, where Ω is an open bounded and connected set of R^2 with regular boundary. Then*

$$\int_\Omega w[u,v]\,dx = \int_\Omega v[w,u]\,dx.$$

Lemma 2.2 [25] *Let $u \in H^2(\Omega)$ and v be the Airy stress function satisfying* (1.2) *and* (1.3). *Then the following relations hold*:

$$[u,v] \in L^2(\Omega) \quad \textit{and} \quad \big\|[u,v]\big\| \le C\|u\|_{H^2(\Omega)}\|v\|_{W^{2,\infty}(\Omega)} \le C'\|u\|_{H^2(\Omega)}\|u\|_{H^2(\Omega)}^2, \tag{2.6}$$

where C and C' are some positive constants.

Now we state the assumptions for problem (1.1)-(1.8). For the relaxation function g, we assume that $g: R_+ \to R_+$ is continuously differentiable verifying that

$$g(0) > 0, \qquad l := \int_0^\infty g(s)\,ds < 1 \tag{2.7}$$

and

$$g'(t) \le -\xi(t)g(t) \quad \text{for } t \ge 0, \tag{2.8}$$

where $\xi: R_+ \to R_+$ is a nonincreasing differentiable function. Condition (2.8) was considered by Messaoudi and Mustafa [26] when studying the stability of a memory-type Timoshenko system. For the functions p and q, we assume that $p, q \in C(\Gamma_1)$ and $p(x), q(x) > 0$ on Γ_1. This assumption implies that there exist positive constants p_i, q_i ($i = 0,1$) such that

$$p_0 \le p(x) \le p_1, \qquad q_0 \le q(x) \le q_1 \quad \text{for all } x \in \Gamma_1. \tag{2.9}$$

The existence of solution can be proved by the Faedo-Galerkin method (see [14, 27, 28]).

Theorem 2.1 *Let the initial data $(u_0,u_1,y_0) \in (H^4(\Omega)\cap W) \times (H^3(\Omega)\cap W) \times L^2(\Gamma_1)$ and the conditions above on g, p and q hold. Then problem* (1.1)-(1.8) *admits a unique solution (u,y) in the class*

$$u \in C\big(0,T; W \cap H^4(\Omega)\big) \cap C^1\big(0,T; V\cap H^3(\Omega)\big), \qquad y, y' \in L^2\big(0,T; L^2(\Gamma_1)\big).$$

3 Main result

In this section, we shall prove the general decay rate of the solution for problem (1.1)-(1.8). For simplicity of notation, we define

$$(g * u)(t) = \int_0^t g(t-s)u(s)\,ds,$$
$$(g \,\Box\, u)(t) = \int_0^t g(t-s)\big\|u(t)-u(s)\big\|^2\,ds$$

and

$$\big(g \,\Box\, \partial^2 u\big)(t) = \int_0^t g(t-s)a\big(u(t)-u(s), u(t)-u(s)\big)\,ds.$$

From (2.3), we deduce

$$(g \,\Box\, u)(t) \le C_p\big(g \,\Box\, \partial^2 u\big)(t) \quad \text{for some constant } C_p. \tag{3.1}$$

From (1.1)-(1.8), we have

$$E'(t) = a\big((g * u)(t), u'(t)\big) - \int_{\Gamma_1} p(x)\big(y'(x,t)\big)^2\,d\Gamma, \tag{3.2}$$

where

$$\begin{aligned} E(t) = {} & \frac{1}{\rho+2}\big\|u'(t)\big\|_{\rho+2}^{\rho+2} + \frac{1}{2}a\big(u(t),u(t)\big) + \frac{\alpha}{2}\big\|\nabla u'(t)\big\|^2 \\ & + \frac{1}{4}\big\|\Delta v(t)\big\|^2 + \frac{1}{2}\int_{\Gamma_1} q(x)\big(y(x,t)\big)^2\,d\Gamma. \end{aligned} \tag{3.3}$$

A direct calculation gives

$$\begin{aligned} a\big((g * u)(t), u'(t)\big) = {} & -\frac{1}{2}g(t)a\big(u(t),u(t)\big) + \frac{1}{2}\big(g' \,\Box\, \partial^2 u\big)(t) \\ & - \frac{1}{2}\frac{d}{dt}\left[\big(g \,\Box\, \partial^2 u\big)(t) - \left(\int_0^t g(s)\,ds\right)a\big(u(t),u(t)\big)\right]. \end{aligned} \tag{3.4}$$

Moreover, (2.5) gives

$$\begin{aligned} a\big((g * u)(t), u(t)\big) = {} & \int_0^t g(t-s)a\big(u(s)-u(t), u(t)\big)\,ds + \left(\int_0^t g(s)\,ds\right)a\big(u(t),u(t)\big) \\ \le {} & 2la\big(u(t),u(t)\big) + \frac{5}{8C_1}\big(g \,\Box\, \partial^2 u\big)(t). \end{aligned} \tag{3.5}$$

Define a modified energy by

$$\begin{aligned} \mathcal{E}(t) = {} & \frac{1}{\rho+2}\big\|u'(t)\big\|_{\rho+2}^{\rho+2} + \frac{\alpha}{2}\big\|\nabla u'(t)\big\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)a\big(u(t),u(t)\big) \\ & + \big(g \,\Box\, \partial^2 u\big)(t) + \frac{1}{4}\big\|\Delta v(t)\big\|^2 + \frac{1}{2}\int_{\Gamma_1} q(x)\big|y(x,t)\big|^2\,d\Gamma. \end{aligned} \tag{3.6}$$

Then applying (3.4) to (3.2), we derive

$$\mathcal{E}'(t) = -\frac{1}{2}g(t)a\big(u(t),u(t)\big) + \frac{1}{2}\big(g' \,\Box\, \partial^2 u\big)(t) - \int_{\Gamma_1} p(x)\big(y'(x,t)\big)^2\,d\Gamma < 0. \tag{3.7}$$

This and the assumptions g and p imply that $\mathcal{E}(t)$ is nonincreasing and one easily sees that

$$E(t) \le C\mathcal{E}(t) \quad \text{for any } t > 0 \text{ and for some positive constant } C. \tag{3.8}$$

Therefore, it is enough to obtain the desired energy decay for the modified energy $\mathcal{E}(t)$, which will be done in what follows. For this object, let us define the functional

$$L(t) = N\mathcal{E}(t) + \varepsilon\Phi(t) + \Psi(t), \tag{3.9}$$

where

$$\begin{aligned}\Phi(t) = {} & \left(\frac{1}{\rho+1}\left|u'(t)\right|^{\rho}u'(t), u(t)\right) + \alpha\big(\nabla u'(t), \nabla u(t)\big)\\ & + \big(u(t), y(t)\big)_{\Gamma_1} + \frac{1}{2}\int_{\Gamma_1} p(x)y(x,t)^2\,d\Gamma\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\Psi(t) = {} & -\int_0^t g(t-s)\left(u(t)-u(s), \frac{1}{\rho+1}\left|u'(t)\right|^{\rho}u'(t)\right)ds\\ & -\alpha\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s), \nabla u'(t)\big)\,ds.\end{aligned} \tag{3.11}$$

Lemma 3.1 *For $N > 0$ large enough, there exist $\beta_1 > 0$ and $\beta_2 > 0$ such that*

$$\beta_1\mathcal{E}(t) \le L(t) \le \beta_2\mathcal{E}(t) \quad \textit{for any } t \ge 0. \tag{3.12}$$

Proof From (3.9), we have

$$\begin{aligned}L(t) - N\mathcal{E}(t) = {} & \frac{\varepsilon}{\rho+1}\int_\Omega \left|u'(t)\right|^{\rho}u'(t)u(t)\,dx + \varepsilon\alpha\int_\Omega \nabla u'(t)\nabla u(t)\,dx\\ & + \varepsilon\int_{\Gamma_1} u(t)y(t)\,d\Gamma + \varepsilon\int_{\Gamma_1} p(x)\big(y(x,t)\big)^2\,d\Gamma\\ & - \frac{1}{\rho+1}\int_\Omega \left|u'(t)\right|^{\rho}u'(t)\left(\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)dx\\ & - \alpha\int_\Omega \nabla u'(t)\left(\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right)dx\\ = {} & J_1 + J_2 + J_3 + J_4 + J_5 + J_6.\end{aligned}$$

Using Hölder's inequality and Young's inequality, (2.3), (2.4), (2.9), (3.6), (3.7) and after some calculation, we obtain

$$\begin{aligned}J_1 &= \frac{\varepsilon}{\rho+1}\int_\Omega \left|u'(t)\right|^{\rho}u'(t)u(t)\,dx\\ &\le \frac{\varepsilon}{\rho+1}\left(\int_\Omega \left|u'(t)\right|^{\rho+2}dx\right)^{\frac{\rho+1}{\rho+2}}\left(\int_\Omega \left|u(t)\right|^{\rho+2}dx\right)^{\frac{1}{\rho+2}}\\ &\le \frac{\varepsilon}{\rho+2}\left\|u'(t)\right\|_{\rho+2}^{\rho+2} + \frac{\varepsilon}{(\rho+1)(\rho+2)}\left\|u(t)\right\|_{\rho+2}^{\rho+2}\\ &\le \frac{\varepsilon}{\rho+2}\left\|u'(t)\right\|_{\rho+2}^{\rho+2} + \frac{\varepsilon}{(\rho+1)(\rho+2)}K^{\rho+2}C_s^{\frac{\rho+2}{2}}\big(2E(0)\big)^{\frac{\rho}{2}}a\big(u(t),u(t)\big),\\ J_2 &= \varepsilon\alpha\int_\Omega \nabla u'(t)\cdot\nabla u(t)\,dx \le \frac{\varepsilon\alpha}{2}\left\|\nabla u'(t)\right\|^2 + \frac{\varepsilon\alpha}{2}\left\|\nabla u(t)\right\|^2\end{aligned}$$

$$\le \frac{\varepsilon\alpha}{2}\|\nabla u'(t)\|^2 + \frac{\varepsilon\alpha C_s}{2}a\big(u(t),u(t)\big),$$

$$J_3 = \varepsilon\int_{\Gamma_1} u(t)y(t)\,d\Gamma \le \frac{\varepsilon}{2}\int_{\Gamma_1}\big|u(t)\big|^2\,d\Gamma + \frac{\varepsilon}{2}\int_{\Gamma_1} y(t)^2\,d\Gamma$$

$$\le \frac{\varepsilon\tilde{C}_p}{2}a\big(u(t),u(t)\big) + \frac{\varepsilon}{2q_0}\int_{\Gamma_1} q(x)\big(y(x,t)\big)^2\,d\Gamma$$

and

$$J_4 = \varepsilon\int_{\Gamma_1} p(x)\big(y(x,t)\big)^2\,d\Gamma \le \frac{\varepsilon p_1}{q_0}\int_{\Gamma_1} q(x)\big(y(x,t)\big)^2\,d\Gamma.$$

Using Fubini's theorem, Hölder's inequality and Young's inequality, (2.3), (2.4), (3.6), (3.7), and after some calculation, we obtain

$$\begin{aligned}
J_5 &= -\frac{1}{\rho+1}\int_\Omega \big|u'(t)\big|^\rho u'(t)\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\,dx \\
&= -\frac{1}{\rho+1}\int_0^t g(t-s)\int_\Omega \big|u'(t)\big|^\rho u'(t)\big(u(t)-u(s)\big)\,dx\,ds \\
&\le \frac{1}{\rho+1}\|u'(t)\|_{\rho+2}^{\rho+1}\int_0^t g(t-s)\|u(t)-u(s)\|_{\rho+2}\,ds \\
&\le \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} + \frac{1}{(\rho+1)(\rho+2)}\left(\int_0^t g(t-s)\|u(t)-u(s)\|_{\rho+2}\,ds\right)^{\rho+2} \\
&\le \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} \\
&\quad + \frac{1}{(\rho+1)(\rho+2)}\left[\left(\int_0^t g(t-s)\,ds\right)^{\frac{1}{2}}\left(\int_0^t g(t-s)\|u(t)-u(s)\|_{\rho+2}^2\,ds\right)^{\frac{1}{2}}\right]^{\rho+2} \\
&\le \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} \\
&\quad + \frac{1}{(\rho+1)(\rho+2)}l^{\frac{\rho+2}{2}}K^{\rho+2}\left(\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|^2\,ds\right)^{\frac{\rho+2}{2}} \\
&\le \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} \\
&\quad + \frac{1}{(\rho+1)(\rho+2)}l^{\frac{\rho+2}{2}}K^{\rho+2}C_s^{\frac{\rho+2}{2}}\left(\int_0^t g(t-s)a\big(u(t)-u(s)\big)\,ds\right)^{\frac{\rho+2}{2}} \\
&\le \frac{1}{\rho+2}\|u'(t)\|_{\rho+2}^{\rho+2} \\
&\quad + \frac{1}{(\rho+1)(\rho+2)}l^{\frac{\rho+2}{2}}K^{\rho+2}C_s^{\frac{\rho+2}{2}}\big(\mathcal{E}(0)\big)^{\frac{\rho}{2}}\big(g\,\Box\,\partial^2 u\big)(t)
\end{aligned}$$

and

$$\begin{aligned}
J_6 &= -\alpha\int_\Omega \nabla u'(t)\cdot\int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\,dx \\
&\le \frac{\alpha}{2}\|\nabla u'(t)\|^2 + \frac{\alpha}{2}l\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|^2\,ds \\
&\le \frac{\alpha}{2}\|\nabla u'(t)\|^2 + \frac{\alpha l C_s}{2}\big(g\,\Box\,\partial^2 u\big)(t).
\end{aligned}$$

From the results of $J_1, J_2, \ldots, J_6$, we get

$$\begin{aligned}\left|L(t) - N\mathcal{E}(t)\right| \leq{}& \frac{\varepsilon + 1}{\rho + 2}\left\|u'(t)\right\|_{\rho+2}^{\rho+2} + \frac{\alpha}{2}(\varepsilon + 1)\left\|\nabla u'(t)\right\|^2 \\ &+ \left[\frac{\varepsilon}{(\rho + 1)(\rho + 2)}K^{\rho+2}C_s^{\frac{\rho+2}{2}}\left(2E(0)\right)^{\frac{\rho}{2}} + \frac{\varepsilon\alpha C_s}{2} + \frac{\varepsilon\tilde{C}_p}{2}\right]a\left(u(t), u(t)\right) \\ &+ \left[\frac{1}{(\rho + 1)(\rho + 2)}l^{\frac{\rho+2}{2}}K^{\rho+2}C_s^{\frac{\rho+2}{2}}\left(\mathcal{E}(0)\right)^{\frac{\rho}{2}} + \frac{\alpha l C_s}{2}\right]\left(g \,\Box\, \partial^2 u\right)(t) \\ &+ \left(\frac{\varepsilon}{2q_0} + \frac{p_1\varepsilon}{q_0}\right)\int_{\Gamma_1} q(x)\left(y(x,t)\right)^2\,d\Gamma.\end{aligned}$$

Therefore, for N is sufficiently large,

$$\left|L(t) - N\mathcal{E}(t)\right| \leq C\mathcal{E}(t).$$

Here we can take

$$\begin{aligned}C :={}& \max\left(\varepsilon + 1, \frac{2}{1 - l}\left[\frac{\varepsilon}{(\rho + 1)(\rho + 2)}K^{\rho+2}C_s^{\frac{\rho+2}{2}}\left(2E(0)\right)^{\frac{\rho}{2}} + \frac{\varepsilon\alpha C_s}{2} + \frac{\varepsilon\tilde{C}_p}{2}\right],\right. \\ &\left.\left[\frac{1}{(\rho + 1)(\rho + 2)}l^{\frac{\rho+2}{2}}K^{\rho+2}C_s^{\frac{\rho+2}{2}}\left(\mathcal{E}(0)\right)^{\frac{\rho}{2}} + \frac{\alpha l C_s}{2}\right], \left[\frac{\varepsilon}{q_0}(1 + 2p_1)\right]\right) > 0.\end{aligned}$$

So that we have

$$\beta_1\mathcal{E}(t) \leq L(t) \leq \beta_2\mathcal{E}(t),$$

where $\beta_1 = N - C$, $\beta_2 = N + C$. We complete the proof of Lemma 3.1. □

Lemma 3.2 *For any $t_0 > 0$ and sufficiently large $N > 0$, there exist positive constants α_0 and α_1 such that*

$$L'(t) \leq -\alpha_0\mathcal{E}(t) + \alpha_1\left(g \,\Box\, \partial^2 u\right)(t) \quad \text{for all } t \geq t_0. \tag{3.13}$$

Proof Using (3.10), (1.1) and (2.3), we have

$$\begin{aligned}\Phi'(t) ={}& \int_\Omega \left(\left|u'(t)\right|^\rho u''(t) - \alpha\Delta u''(t)\right)u(t)\,dx + \frac{1}{\rho + 1}\left\|u'(t)\right\|_{\rho+2}^{\rho+2} + \alpha\left\|\nabla u'(t)\right\|^2 \\ &+ \alpha\left(\frac{\partial u''(t)}{\partial\nu}, u(t)\right)_{\Gamma_1} + \left(u'(t), y(t)\right)_{\Gamma_1} + \left(u(t), y'(t)\right)_{\Gamma_1} \\ &+ \int_{\Gamma_1} p(x)y(x,t)y'(x,t)\,d\Gamma \\ ={}& \frac{1}{\rho + 1}\left\|u'(t)\right\|_{\rho+2}^{\rho+2} + \alpha\left\|\nabla u'(t)\right\|^2 - a\left(u(t), u(t)\right) \\ &+ \left(\mathcal{B}_1 u(t), \frac{\partial u(t)}{\partial\nu}\right)_{\Gamma_1} - \left(\mathcal{B}_2 u(t), u(t)\right)_{\Gamma_1} + a\left((g * u)(t), u(t)\right) \\ &- \int_0^t g(t - s)\left[\left(\mathcal{B}_1 u(s), \frac{\partial u(t)}{\partial\nu}\right)_{\Gamma_1} - \left(\mathcal{B}_2 u(s), u(t)\right)_{\Gamma_1}\right]ds\end{aligned}$$

$$
\begin{aligned}
&+ \big([u(t),v(t)],v(t)\big) + \alpha\left(\frac{\partial u''(t)}{\partial\nu},u(t)\right)_{\Gamma_1} + \big(u'(t),y(t)\big)_{\Gamma_1} + \big(u(t),y'(t)\big)_{\Gamma_1} \\
&+ \int_{\Gamma_1} p(x)y(x,t)y'(x,t)\,d\Gamma \\
={}& \frac{1}{\rho+1}\|u'(t)\|_{\rho+2}^{\rho+2} + \alpha\|\nabla u'(t)\|^2 - a\big(u(t),u(t)\big) + a\big((g*u)(t),u(t)\big) \\
&+ 2\big(u(t),y'(t)\big)_{\Gamma_1} - \big(\Delta^2 v(t),v(t)\big) + \big(u'(t),y(t)\big)_{\Gamma_1} \\
&+ \int_{\Gamma_1} p(x)y(x,t)y'(x,t)\,d\Gamma \\
\le{}& \frac{1}{\rho+1}\|u'(t)\|_{\rho+2}^{\rho+2} + \alpha\|\nabla u'(t)\|^2 - (1-\delta\tilde{C}_p)a\big(u(t),u(t)\big) \\
&+ a\big((g*u)(t),u(t)\big) - \|\Delta v(t)\|^2 + \frac{1}{\delta}\|y'(t)\|_{\Gamma_1}^2 \\
&- \int_{\Gamma_1} q(x)\big|y(x,t)\big|^2\,d\Gamma.
\end{aligned}
\tag{3.14}
$$

Adapting (3.5) to (3.14), we get

$$
\begin{aligned}
\Phi'(t) \le{}& \frac{1}{\rho+1}\|u'(t)\|_{\rho+2}^{\rho+2} + \alpha\|\nabla u'(t)\|^2 - (1-2l-\delta\tilde{C}_p)a\big(u(t),u(t)\big) \\
&+ \frac{1}{\delta}\|y'(t)\|_{\Gamma_1}^2 - \int_{\Gamma_1} q(x)\big(y(x,t)\big)^2\,d\Gamma + \frac{5}{8C_1}\big(g\,\Box\,\partial^2 u\big)(t) - \|\Delta v(t)\|^2.
\end{aligned}
\tag{3.15}
$$

Moreover, from (3.11) and (1.1), we derive

$$
\begin{aligned}
\Psi'(t) ={}& -\left(\big|u'(t)\big|^\rho u''(t), \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right) \\
&- \alpha\left(\nabla u''(t), \int_0^t g(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right) \\
&- \left(\frac{1}{\rho+1}\big|u'(t)\big|^\rho u'(t), \int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\right) \\
&- \alpha\left(\nabla u'(t), \int_0^t g'(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right) \\
&- \alpha\left(\nabla u'(t), \int_0^t g(t-s)\nabla u'(t)\,ds\right) \\
&- \left(\frac{1}{\rho+1}\big|u'(t)\big|^\rho u'(t), \int_0^t g(t-s)u'(t)\,ds\right) \\
={}& \left(\Delta^2 u(t) - \int_0^t g(t-s)\Delta^2 u(s)\,ds - [u,v], \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right) \\
&- \alpha\left(\frac{\partial u''(t)}{\partial\nu}, \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right) \\
&- \left(\frac{1}{\rho+1}\big|u'(t)\big|^\rho u'(t), \int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\right) \\
&- \frac{1}{\rho+1}\left(\int_0^t g(s)\,ds\right)\|u'(t)\|_{\rho+2}^{\rho+2} - \alpha\left(\int_0^t g(s)\,ds\right)\|\nabla u'(t)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&- \alpha\left(\nabla u'(t), \int_0^t g'(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right) \\
&= \int_0^t g(t-s) a\big(u(t), u(t)-u(s)\big)\,ds \\
&\quad - \int_0^t g(t-s)\int_0^t g(t-\tau) a\big(u(\tau), u(t)-u(s)\big)\,d\tau\,ds \\
&\quad - \left(y', \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right)_{\Gamma_1} \\
&\quad - \alpha\left(\nabla u'(t), \int_0^t g'(t-s)\big(\nabla u(t)-\nabla u(s)\big)\,ds\right) \\
&\quad - \left(\frac{1}{\rho+1}\left|u'(t)\right|^{\rho} u'(t), \int_0^t g'(t-s)\big(u(t)-u(s)\big)\,ds\right) \\
&\quad - \left([u,v], \int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right) \\
&\quad - \alpha\left(\int_0^t g(s)\,ds\right)\left\|\nabla u'(t)\right\|^2 - \frac{1}{\rho+1}\left(\int_0^t g(s)\,ds\right)\left\|u'(t)\right\|_{\rho+2}^{\rho+2} \\
&= I_1 + I_2 + \cdots + I_6 \\
&\quad - \alpha\left(\int_0^t g(s)\,ds\right)\left\|\nabla u'(t)\right\|^2 - \frac{1}{\rho+1}\left(\int_0^t g(s)\,ds\right)\left\|u'(t)\right\|_{\rho+2}^{\rho+2}.
\end{aligned}
\tag{3.16}
$$

In what follows we will estimate the terms on the right-hand side of (3.16). From (2.5) and (3.5) we deduce that

$$
|I_1| \le \eta l a\big(u(t), u(t)\big) + \frac{5}{8C_1\eta}\big(g \,\square\, \partial^2 u\big)(t)
$$

and

$$
\begin{aligned}
|I_2| &\le \left|\int_0^t g(t-s)\int_0^t g(t-\tau) a\big(u(t)-u(\tau), u(t)-u(s)\big)\,d\tau\,ds\right| \\
&\quad + \left|\int_0^t g(t-s)\int_0^t g(t-\tau) a\big(u(t), u(t)-u(s)\big)\,d\tau\,ds\right| \\
&\le \int_0^t g(t-s)\int_0^t g(t-\tau)\bigg[a\big(u(t)-u(\tau), u(t)-u(\tau)\big) \\
&\quad + \frac{5}{8C_1} a\big(u(t)-u(s), u(t)-u(s)\big)\bigg]\,d\tau\,ds \\
&\quad + \left(\int_0^t g(\tau)\,d\tau\right)\left|\int_0^t g(t-s) a\big(u(t), u(t)-u(s)\big)\,ds\right| \\
&\le \eta l^2 a\big(u(t), u(t)\big) + \left(l + \frac{5l}{8C_1} + \frac{5l}{8C_1\eta}\right)\big(g \,\square\, \partial^2 u\big)(t).
\end{aligned}
$$

Also, using Young's inequality, (2.3) and (3.8), we obtain

$$
\begin{aligned}
|I_3| &\le \eta\left\|y'(t)\right\|_{\Gamma_1}^2 + \frac{C_p l}{4\eta}\big(g \,\square\, \partial^2 u\big)(t), \\
|I_4| &\le \eta\alpha\left\|\nabla u'(t)\right\|^2 + \frac{\alpha}{4\eta}\int_0^t -g'(s)\,ds \int_0^t -g'(t-s)\left\|\nabla u(t)-\nabla u(s)\right\|^2\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \eta\alpha \|\nabla u'(t)\|^2 - \frac{\alpha g(0)C_s}{4\eta}\big(g' \square \partial^2 u\big)(t),\\
|I_5| &\leq \frac{\eta}{\rho+1}\|u'(t)\|_{2(\rho+1)}^{2(\rho+1)} - \frac{g(0)C_s}{4\eta(\rho+1)}\big(g' \square \partial^2 u\big)(t)\\
&\leq \frac{\eta K^{2(\rho+1)}}{\rho+1}\big(2\alpha^{-1}\mathcal{E}(0)\big)^{\rho}\|\nabla u'(t)\|^2 - \frac{g(0)C_s}{4\eta(\rho+1)}\big(g' \square \partial^2 u\big)(t),\\
|I_6| &\leq \eta \big\|\big[u(t), v(t)\big]\big\|^2 + \frac{1}{4\eta}\left\|\int_0^t g(t-s)\big(u(t)-u(s)\big)\,ds\right\|^2\\
&\leq \eta C' \|u(t)\|_{H^2(\Omega)}^2 \|v(t)\|_{H^2(\Omega)}^2 + \frac{C_p l}{4\eta}\big(g \square \partial^2 u\big)(t)\\
&\leq C_1^{-1}\eta C' 4E(0) a\big(u(t),u(t)\big) + \frac{C_p l}{4\eta}\big(g \square \partial^2 u\big)(t).
\end{aligned}
$$

Combining these estimates I_i, $i = 1, \ldots, 6$ and (3.16), we get

$$
\begin{aligned}
\Psi'(t) \leq{}& -\left(\int_0^t g(s)\,ds\right)\frac{\|u'(t)\|_{\rho+2}^{\rho+2}}{\rho+1} + \eta\big(l + l^2 + C'C_1^{-1}4E(0)\big)a\big(u(t),u(t)\big)\\
&- \left[\alpha\left(\int_0^t g(s)\,ds\right) - \eta\left(\alpha + \frac{K^{2(\rho+1)}}{\rho+1}\big(2\alpha^{-1}\mathcal{E}(0)\big)^{\rho}\right)\right]\|\nabla u'(t)\|^2\\
&+ \left(\frac{5}{8C_1\eta} + l + \frac{5l}{8C_1} + \frac{5l}{8C_1\eta} + \frac{C_p l}{2\eta}\right)\big(g \square \partial^2 u\big)(t)\\
&- \frac{g(0)C_s}{4\eta}\left(\alpha + \frac{1}{\rho+1}\right)\big(g' \square \partial^2 u\big)(t) + \eta\|y'(t)\|_{\Gamma_1}^2. \qquad (3.17)
\end{aligned}
$$

Since g is continuous and positive for any $t \geq t_0 > 0$, we have

$$
\int_0^t g(s)\,ds \geq \int_0^{t_0} g(s)\,ds := g_0 > 0. \qquad (3.18)
$$

Thus, making use of (3.18) and combining (3.7), (3.9), (3.15) and (3.17), we arrive at

$$
\begin{aligned}
L'(t) \leq{}& -\left(g_0 - \frac{\varepsilon}{\rho+1}\right)\|u'(t)\|_{\rho+2}^{\rho+2}\\
&- \left[\frac{N}{2}g(t) + \varepsilon(1 - 2l - \delta\tilde{C}_p) - \eta\big(l + l^2 + C'C_1^{-1}4E(0)\big)\right]a\big(u(t),u(t)\big)\\
&- \left[\alpha g_0 - \eta\left(\alpha + \frac{K^{2(\rho+1)}}{\rho+1}\big(2\alpha^{-1}\mathcal{E}(0)\big)^{\rho}\right) - \varepsilon\alpha\right]\|\nabla u'(t)\|^2 - \varepsilon\|\Delta v(t)\|^2\\
&+ \left(\frac{5\varepsilon}{8C_1} + \frac{5}{8C_1\eta} + l + \frac{5l}{8C_1} + \frac{5l}{8C_1\eta} + \frac{C_p \cdot l}{2\eta}\right)\big(g \square \partial^2 u\big)(t)\\
&+ \left[\frac{N}{2} - \frac{g(0)C_s}{4\eta}\left(\alpha + \frac{1}{\rho+1}\right)\right]\big(g' \square \partial^2 u\big)(t) - \varepsilon\int_{\Gamma_1} q(x)\big(y(x,t)\big)^2\,d\Gamma\\
&- \left(N - \frac{\varepsilon}{\delta p_0} - \frac{\eta}{p_0}\right)\int_{\Gamma_1} p(x)\big(y'(x,t)\big)^2\,d\Gamma \quad \text{for any } t \geq t_0. \qquad (3.19)
\end{aligned}
$$

We first choose $\varepsilon > 0$ and $\delta > 0$ so small such that $g_0 - \varepsilon > 0$ and $1 - 2l - \delta\tilde{C}_p > 0$, respectively. We also take $\eta > 0$ sufficiently small and $N > 0$ large enough so that $\frac{N}{2}g(t) + \varepsilon(1 - 2l - \delta\tilde{C}_p) -$

$\eta(l + l^2 + C'C_1^{-1}4E(0)) > 0$, $\alpha g_0 - \eta(\alpha + \frac{K^{2(\rho+1)}}{\rho+1}(2\alpha^{-1}\mathcal{E}(0))^\rho) - \varepsilon\alpha > 0$, $\frac{N}{2} - \frac{g(0)C_s}{4\eta}(\alpha + \frac{1}{\rho+1}) > 0$ and $Np_0 - \varepsilon/\delta - \eta > 0$ for any $t \geq t_0$. Then we deduce that desired result. We complete the proof of Lemma 3.2. □

Our main result is the following.

Theorem 3.1 *Assume that $l < 1/2$. Then, for each $t_0 > 0$, there exist positive constants C_0 and C such that*

$$E(t) \leq C_0 e^{-C\int_{t_0}^t \xi(s)\,ds} \quad \text{for all } t \geq t_0.$$

Proof Multiplying (3.13) by $\xi(t)$, noting that ξ is nonincreasing, and using (2.8) and (3.7), we see that

$$\begin{aligned}\xi(t)L'(t) &\leq -\alpha_0\xi(t)\mathcal{E}(t) + \alpha_1\xi(t)\big(g\,\Box\,\partial^2 u\big)(t) \leq -\alpha_0\xi(t)\mathcal{E}(t) - \alpha_1\big(g'\,\Box\,\partial^2 u\big)(t)\\ &\leq -\alpha_0\xi(t)\mathcal{E}(t) - 2\alpha_1\mathcal{E}'(t) \quad \text{for all } t \geq t_0.\end{aligned}$$

This and the fact $\xi'(t) \leq 0$ yield

$$\frac{d}{dt}\big(\xi(t)L(t) + 2\alpha_1\mathcal{E}(t)\big) \leq -\alpha_0\xi(t)\mathcal{E}(t) \quad \text{for all } t \geq t_0. \tag{3.20}$$

Now, we define

$$\mathcal{L}(t) = \xi(t)L(t) + 2\alpha_1\mathcal{E}(t).$$

Since $\xi(t)$ is a nonincreasing positive function, we can easily observe that $\mathcal{L}(t)$ is equivalent to $\mathcal{E}(t)$. Thus (3.20) implies that

$$\frac{d}{dt}\mathcal{L}(t) \leq -C\xi(t)\mathcal{L}(t) \quad \text{for some } C > 0 \text{ and for all } t \geq t_0.$$

Integrating the above inequality, we get

$$\mathcal{L}(t) \leq \mathcal{L}(t_0)e^{-C\int_{t_0}^t \xi(s)\,ds} \quad \text{for all } t \geq t_0.$$

Consequently, from the equivalent relations of $\mathcal{L}$, L, $\mathcal{E}$ and (3.8), we obtain the result in Theorem 3.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Wu, J: Well-posedness for a variable-coefficient wave equation with nonlinear damped acoustic boundary conditions. Nonlinear Anal. **75**(18), 6562-6569 (2012)
2. Avalos, G, Lasiecka, I, Rebarber, R: Uniform decay properties of a model in structural acoustics. J. Math. Pures Appl. **79**(10), 1057-1072 (2000)
3. Beale, JT, Rosencrans, SI: Acoustic boundary conditions. Bull. Am. Math. Soc. **80**, 1276-1278 (1974)
4. Berrimi, S, Messaoudi, SA: Existence and decay of solutions of a viscoelastic equation with a nonlinear source. Nonlinear Anal. **64**, 2314-2331 (2006)
5. Guesmia, A, Messaoudi, SA: General energy decay estimates of Timoshenko systems with frictional versus viscoelastic damping. Math. Methods Appl. Sci. **32**, 2102-2122 (2009)
6. Bucci, F, Lasiecka, I: Exponential decay rates for structural acoustic model with an over damping on the interface and boundary layer dissipation. Appl. Anal. **81**(4), 977-999 (2002)
7. Lasiecka, I: Boundary stabilization of a 3-dimensional structural acoustic model. J. Math. Pures Appl. **78**(2), 203-232 (1999)
8. Mugnolo, D: Abstract wave equation with acoustic boundary conditions. Math. Nachr. **279**(3), 299-318 (2006)
9. Park, JY, Park, SH: Decay rate estimates for wave equation of memory type with acoustic boundary conditions. Nonlinear Anal. TMA **74**(3), 993-998 (2011)
10. Vicente, A: Wave equation with acoustic/memory boundary conditions. Bol. Soc. Parana. Mat. **27**(1), 29-39 (2009)
11. Beale, JT: Spectral properties of an acoustic boundary condition. Indiana Univ. Math. J. **25**(9), 895-917 (1976)
12. Frota, CL, Larkin, NA: Uniform stabilization for a hyperbolic equation with acoustic boundary conditions in simple connected domains. Prog. Nonlinear Differ. Equ. Appl. **66**, 297-312 (2005)
13. Park, SH, Park, JY, Kang, YH: General decay for a von Karman equation or memory type with acoustic boundary conditions. Z. Angew. Math. Phys. **63**, 813-823 (2012)
14. Park, JY, Ha, TG: Well-posedness and uniform decay rates for the Klein-Gordon equations with damping term and acoustic boundary conditions. J. Math. Phys. **50**, Article No. 013506 (2009)
15. Boukhatem, Y, Benabderrahmane, B: Existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions. Nonlinear Anal. **97**, 191-209 (2014)
16. Graber, PJ, Said-Houari, B: On the wave equation with semilinear porous acoustic boundary conditions. J. Differ. Equ. **252**, 4898-4941 (2012)
17. Graber, PJ: Strong stability and uniform decay of solutions to a wave equation with semilinear porous acoustic boundary conditions. Nonlinear Anal. **74**, 3137-3148 (2011)
18. Park, PJ, Park, SH: Uniform decay for a von Karman plate equation with a boundary memory condition. Math. Methods Appl. Sci. **28**, 2225-2240 (2005)
19. Park, JY, Kang, JR: Global existence and uniform decay for a nonlinear viscoelastic equation with damping. Acta Appl. Math. **110**, 1393-1406 (2010)
20. Liu, WJ: Uniform decay of solutions for quasilinear system of viscoelastic equations. Nonlinear Anal. **71**, 2257-2267 (2009)
21. Messaoudi, SA, Tatar, NE: Exponential and polynomial decay for a quasilinear viscoelastic equation. Nonlinear Anal. **68**, 785-793 (2008)
22. Messaoudi, SA, Tatar, NE: Exponential decay for a quasilinear viscoelastic equation. Math. Nachr. **282**, 1443-1450 (2009)
23. Kang, JR: Exponential decay for nonlinear von Karman equations with memory. Abstr. Appl. Anal. **2013**, Article ID 484596 (2013)
24. Lagnese, JE: Boundary Stabilization of Thin Plates. SIAM, Philadelphia (1999)
25. Chueshov, I, Lasiecka, I: Global attractors for von Karman evolutions with a nonlinear boundary dissipation. J. Differ. Equ. **198**, 196-231 (2004)
26. Messaoudi, SA, Mustafa, MI: A stability result in a memory-type Timoshenko system. Dyn. Syst. Appl. **18**, 457-468 (2009)
27. Raposo, CA, Santos, ML: General decay to a von Karman system with memory. Nonlinear Anal. **74**, 937-945 (2011)
28. Rivera, JE, Menzala, GP: Decay rates of solutions of a von Karman system for viscoelastic plates with memory. Q. Appl. Math. **LVII**, 181-200 (1999)