Pan et al. *Boundary Value Problems* (2015) 2015:141
DOI 10.1186/s13661-015-0404-7

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A high-accuracy compact conservative scheme for generalized regularized long-wave equation

Xintian Pan[1,2*], Haitao Che[2] and Yiju Wang[1]

*Correspondence: panxintian@126.com
[1]School of Management, Qufu Normal University, Rizhao Shandong, 276800, China
[2]School of Mathematics and Information Science, Weifang University, Weifang, 261061, China

Abstract

In this article, we develop a high-order compact conservative numerical scheme to solve the initial-boundary problem of GRLW equation. The proposed scheme is three-level and linear-implicit based on a finite difference method. A detailed numerical analysis of the scheme is presented including a convergence analysis result. Some numerical examples are provided to show the present scheme is efficient, reliable, and of high accuracy.

Keywords: GRLW equation; compact conservative scheme; solvability; convergence; stability

1 Introduction

The Cauchy problem of the generalized regularized long-wave (GRLW) equation reads

$$u_t - \beta u_{xxt} + u_x + \alpha\left(u^p\right)_x = 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

where α, β are positive constants and p is a positive integer [1]. The GRLW equation was first put forward by Peregrine [2] and Benjamin *et al.* [3] as a model for small-amplitude long waves on the surface of water in a channel. Many authors [4–8] have recently studied models for long waves in nonlinear dispersive systems. When $p = 2$, (1.1) is usually called the RLW equation. When $p = 3$, (1.1) is called a modified regularized long-wave (MRLW) equation. Various numerical techniques have been developed to solve the equation. These partly include the finite difference method, finite element methods, the least squares method, and a collocation method with quadratic B-splines, cubic B-splines and septic splines; we refer to [9–20], and references therein.

In general, the solutions of the system (1.1)-(1.2) decays rapidly to zero for $|x| \gg 0$. Therefore, numerically we can solve the system (1.1)-(1.2) in a compact domain $\Omega = (x_l, x_r)$ with $-x_l \gg 0$ and $x_r \gg 0$. We can add the boundary conditions to the Cauchy problem (1.1)-(1.2),

$$u(x_l,t) = u(x_r,t) = 0. \tag{1.3}$$

It is well known that the system (1.1)-(1.2) possesses the following conservative law:

$$E(t) = \|u\|_{L_2}^2 + \|u_x\|_{L_2}^2 = E(0). \tag{1.4}$$

In [1], Zhang considered a linear conservative scheme for GRLW equation, however, the accuracy of the scheme is only second-order. Recently, there has been growing interest in high-order compact methods to solve the partial differential equations [21–31], where fourth-order compact finite difference approximation solutions for the transient wave equations, a N-carrier system, the Klein-Gordon equation, the Sine-Gordon equation, the one-dimensional heat and advection-diffusion equations, the Schrödinger equation, the Klein-Gordon-Schrödinger equation and the RLW equation were shown, respectively. These numerical methods may give us many enlightenments to design a new numerical scheme for the GRLW equation. For a wide and most complete vision concerning the importance, the breadth, and the interest of the topics covered, we should also recall the study done on the long waves in [32–36].

The main purpose of this paper is to construct a new numerical scheme which has the following advantages:

1. Coupling with the Richardson extrapolation, the new scheme is high-accuracy and without refined mesh; it has an accuracy of $O(\tau^2 + h^4)$.
2. The new scheme is linearized and preserves the original conservative property.
3. The coefficient matrices of the scheme is symmetric and pentadiagonal, and the Thomas algorithm can be employed to solve it effectively.
4. Useful numerical examples are given to show the efficiency of the scheme.

The rest of this paper is organized as follows. In Section 2, a high-accuracy linear-compact difference scheme for the GRLW equation is described. In Section 3, we discuss the solvability of the scheme and the estimate of the difference solution. In Section 4, the fourth-order convergence and stability of the scheme are proved by the discrete energy method. Numerical results are reported in Section 5.

2 High-accuracy compact scheme and its discrete conservative law

In this section, we describe a high-order linear-compact difference scheme for (1.1)-(1.2).

Let $h = \frac{x_r - x_l}{J}$ and $\tau = \frac{T}{N}$ be the uniform step size in the spatial and the temporal direction, respectively. Denote $x_j = jh$ $(0 \leq j \leq J)$, $t_n = n\tau$ $(0 \leq n \leq N)$, $u_j^n \approx u(x_j, t_n)$ and $Z_h^0 = \{u = (u_j) | u_{-1} = u_0 = u_J = u_{J+1} = 0, j = 0, 1, \ldots, J\}$. For simplicity, we introduce the following notations of the difference operators:

$$\delta_x u_j^n = \frac{u_{j+1}^n - u_j^n}{h}, \qquad \delta_{\bar{x}} u_j^n = \frac{u_j^n - u_{j-1}^n}{h}, \qquad \delta_{\hat{x}} u_j^n = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \qquad \delta_{\ddot{x}} u_j^n = \frac{u_{j+2}^n - u_{j-2}^n}{4h},$$

$$\delta_{\hat{t}} u_j^n = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau}, \qquad \delta_t u_j^n = \frac{u_j^{n+1} - u_j^n}{\tau}, \qquad \bar{u}_j^n = \frac{u_j^{n+1} + u_j^{n-1}}{2}.$$

In the paper, C denotes a general positive constant which may have different values in different occurrences.

Based on the notations above, we consider the following high-accuracy linear-compact scheme for the initial-boundary problem (1.1)-(1.3),

$$
\begin{aligned}
&\delta_{\hat{t}}u_j^n-\frac{4}{3}\beta\delta_x\delta_{\bar{x}}\delta_{\hat{t}}u_j^n+\frac{1}{3}\beta\delta_{\hat{x}}\delta_{\hat{x}}\delta_{\hat{t}}u_j^n+\frac{4}{3}\delta_{\hat{x}}\left(u_j^n\right)-\frac{1}{3}\delta_{\ddot{x}}\left(u_j^n\right)\\
&\quad+\frac{p}{p+1}\alpha\left[\frac{4}{3}\left\{\left(\delta_{\hat{x}}\bar{u}_j^n\right)\left(u_j^n\right)^{p-1}+\delta_{\hat{x}}\left[\left(u_j^n\right)^{p-1}\left(\bar{u}_j^n\right)\right]\right\}\right.\\
&\quad\left.-\frac{1}{3}\left\{\left(\delta_{\ddot{x}}\bar{u}_j^n\right)\left(u_j^n\right)^{p-1}+\delta_{\ddot{x}}\left[\left(u_j^n\right)^{p-1}\left(\bar{u}_j^n\right)\right]\right\}\right]=0,\\
&\quad 1\le j\le J-1, 1\le n\le N-1,
\end{aligned}
\tag{2.1}
$$

$$
\begin{aligned}
&u_j^1-\frac{4}{3}\beta\delta_x\delta_{\bar{x}}u_j^1+\frac{1}{3}\beta\delta_{\hat{x}}\delta_{\hat{x}}u_j^1\\
&\quad=u_0(x_j)-\frac{d^2u_0}{dx^2}(x_j)-\tau\frac{du_0}{dx}(x_j)-\tau\alpha pu_0^{p-1}(x_j)\frac{du_0}{dx}(x_j),
\end{aligned}
\tag{2.2}
$$

$$
u_j^0=u_0(x_j),\quad 1\le j\le J,
\tag{2.3}
$$

$$
u_0^n=u_J^n=0,\quad 0\le n\le N.
\tag{2.4}
$$

For convenience, the last term of (2.1) is defined by

$$
\kappa\left(u^n,\bar{u}^n\right)=\kappa_1\left(u^n,\bar{u}^n\right)+\kappa_2\left(u^n,\bar{u}^n\right),
\tag{2.5}
$$

where

$$
\begin{aligned}
\kappa_1\left(u^n,\bar{u}^n\right)&=\frac{4p}{3(p+1)}\alpha\left\{\left(\delta_{\hat{x}}\bar{u}^n\right)\left(u^n\right)^{p-1}+\delta_{\hat{x}}\left[\left(u^n\right)^{p-1}\left(\bar{u}^n\right)\right]\right\},\\
\kappa_2\left(u^n,\bar{u}^n\right)&=-\frac{p}{3(p+1)}\alpha\left\{\left(\delta_{\ddot{x}}\bar{u}^n\right)\left(u^n\right)^{p-1}+\delta_{\ddot{x}}\left[\left(u^n\right)^{p-1}\left(\bar{u}^n\right)\right]\right\}.
\end{aligned}
$$

Theorem 2.1 *Suppose* $u_0\in H_0^1[x_l,x_r]$, *then the scheme* (2.1) *admits the following invariant*:

$$
\begin{aligned}
E^n&=\frac{1}{2}\left(\left\|u^{n+1}\right\|^2+\left\|u^n\right\|^2\right)+\frac{2}{3}\beta\left(\left\|\delta_xu^{n+1}\right\|^2+\left\|\delta_xu^n\right\|^2\right)-\frac{1}{6}\beta\left(\left\|\delta_{\hat{x}}u^{n+1}\right\|^2+\left\|\delta_{\hat{x}}u^n\right\|^2\right)\\
&\quad+\frac{4}{3}h\tau\sum_{j=1}^{J-1}\delta_xu_j^nu_j^{n+1}-\frac{1}{3}h\tau\sum_{j=1}^{J-1}\delta_{\ddot{x}}u_j^nu_j^{n+1}=E^{n-1}=\cdots=E^0.
\end{aligned}
\tag{2.6}
$$

Proof Taking in (2.1) the inner product with $2\bar{u}^n$ and using the boundary condition (2.4) yield

$$
\begin{aligned}
&\frac{1}{2\tau}\left(\left\|u^{n+1}\right\|^2-\left\|u^{n-1}\right\|^2\right)+\frac{2}{3\tau}\beta\left(\left\|\delta_xu^{n+1}\right\|^2-\left\|\delta_xu_x^{n-1}\right\|^2\right)\\
&\quad-\frac{1}{6\tau}\beta\left(\left\|\delta_{\hat{x}}u^{n+1}\right\|^2-\left\|\delta_{\hat{x}}u_x^{n-1}\right\|^2\right)\\
&\quad+\frac{4}{3}\left(\delta_xu^n,2\bar{u}^n\right)-\frac{1}{3}\left(\delta_{\ddot{x}}u^n,2\bar{u}^n\right)+\left(\kappa_1\left(u^n,\bar{u}^n\right)+\kappa_2\left(u^n,\bar{u}^n\right),2\bar{u}^n\right)=0.
\end{aligned}
\tag{2.7}
$$

Notice that

$$\frac{4}{3}\left(\delta_x u^n, 2\bar{u}^n\right) = \frac{4}{3}h\sum_{j=1}^{J-1}\left(\delta_x u_j^n u_j^{n+1} - u_j^n \delta_x u_j^{n-1}\right) \tag{2.8}$$

and

$$\frac{1}{3}\left(\delta_{\ddot{x}} u^n, 2\bar{u}^n\right) = \frac{1}{3}h\sum_{j=1}^{J-1}\left(\delta_{\ddot{x}} u_j^n u_j^{n+1} - u_j^n \delta_{\ddot{x}} u_j^{n-1}\right). \tag{2.9}$$

Now, computing the last term of the left-hand side in (2.7), we have

$$\begin{aligned}
\left(\kappa_1\left(u^n, \bar{u}^n\right), 2\bar{u}^n\right) &= \frac{8p}{3(p+1)}\alpha h\sum_{j=1}^{J-1}\left\{\left(u_j^n\right)^{p-1}\delta_{\hat{x}}\left(\bar{u}_j^n\right) + \delta_{\hat{x}}\left[\left(u_j^n\right)^{p-1}\bar{u}_j^n\right]\right\}\bar{u}_j^n \\
&= \frac{4p}{3(p+1)}\alpha\sum_{j=1}^{J-1}\left[\left(u_j^n\right)^{p-1}\bar{u}_{j+1}^n\bar{u}_j^n - \left(u_{j+1}^n\right)^{p-1}\bar{u}_{j+1}^n\bar{u}_j^n\right] \\
&\quad - \frac{4p}{3(p+1)}\alpha\sum_{j=1}^{J-1}\left[\left(u_{j-1}^n\right)^{p-1}\bar{u}_j^n\bar{u}_{j-1}^n - \left(u_j^n\right)^{p-1}\bar{u}_j^n\bar{u}_{j-1}^n\right] \\
&= 0.
\end{aligned} \tag{2.10}$$

Similarly to the proof of (2.10), we get

$$\left(\kappa_2\left(u^n, \bar{u}^n\right), 2\bar{u}^n\right) = 0. \tag{2.11}$$

Substituting (2.8)-(2.11) into (2.7). Let

$$\begin{aligned}
E^n = {}& \frac{1}{2}\left(\left\|u^{n+1}\right\|^2 + \left\|u^n\right\|^2\right) + \frac{2}{3}\beta\left(\left\|\delta_x u^{n+1}\right\|^2 + \left\|\delta_x u^n\right\|^2\right) - \frac{1}{6}\beta\left(\left\|\delta_{\hat{x}} u^{n+1}\right\|^2 + \left\|\delta_{\hat{x}} u^n\right\|^2\right) \\
& + \frac{4}{3}h\tau\sum_{j=1}^{J-1}\delta_x u_j^n u_j^{n+1} - \frac{1}{3}h\tau\sum_{j=1}^{J-1}\delta_{\ddot{x}} u_j^n u_j^{n+1}.
\end{aligned}$$

By the definition of E^n, (2.6) follows. □

3 Solvability and estimate for the difference solution

In this section, we shall discuss the estimate for the difference solution and the solvability of the difference scheme (2.1). For $\forall v^n$, $w^n \in Z_h^0$, we define the discrete inner products and norms on Z_h^0 via

$$\left(v^n, w^n\right) = h\sum_{j=1}^{J-1}v_j^n w_j^n, \qquad \left(\delta_x v^n, \delta_x w^n\right)_l = h\sum_{j=0}^{J-1}\delta_x v_j^n \delta_x w_j^n, \qquad \left\|v^n\right\|^2 = \left(v^n, v^n\right),$$

$$\left\|\delta_x v^n\right\| = \sqrt{\left(\delta_x v^n, \delta_x v^n\right)_l}, \qquad \left\|\delta_{\ddot{x}} v^n\right\| = \sqrt{\left(\delta_{\ddot{x}} v^n, \delta_{\ddot{x}} v^n\right)_l}, \qquad \left\|v^n\right\|_\infty = \max_{1\le j\le J-1}\left|v_j^n\right|.$$

To analyze the estimates of difference solution for the scheme (2.1)-(2.4), the following lemmas should be introduced.

Lemma 3.1 ([31]) *For a mesh function $u \in Z_h^0$, by the Cauchy-Schwarz inequality, we have*

$$\|\delta_{\ddot{x}} u\|^2 \le \|\delta_{\hat{x}} u\|^2 \le \|\delta_x u\|^2.$$

Lemma 3.2 (Discrete Sobolev's inequality [37]) *There exist two constants C_1 and C_2 such that*

$$\|u^n\|_\infty \le C_1\|u^n\| + C_2\|\delta_x u^n\|^2.$$

Theorem 3.1 *Suppose that $u_0 \in H^1$, then there is the estimation for the solution u^n of the scheme* (2.1): $\|u^n\| \le C$, $\|\delta_x u^n\| \le C$, *which yields* $\|u^n\|_\infty \le C$.

Proof It follows from (2.6) and the Cauchy-Schwartz inequality that

$$\begin{aligned}
&\frac{1}{2}\big(\|u^{n+1}\|^2 + \|u^n\|^2\big) + \frac{2}{3}\beta\big(\|\delta_x u^{n+1}\|^2 + \|\delta_x u^n\|^2\big) - \frac{1}{6}\beta\big(\|\delta_{\hat{x}} u^{n+1}\|^2 + \|\delta_{\hat{x}} u^n\|^2\big)\\
&\quad\le C + \frac{4}{3}h\tau\sum_{j=1}^{J-1}\big|\delta_x u_j^n u_j^{n+1}\big| + \frac{1}{3}h\tau\sum_{j=1}^{J-1}\big|\delta_{\ddot{x}} u_j^n u_j^{n+1}\big|\\
&\quad\le C + \frac{2}{3}\tau\big(\|\delta_x u^n\|^2 + \|u^{n+1}\|^2\big) + \frac{1}{6}\tau\big(\|\delta_{\ddot{x}} u^n\|^2 + \|u^{n+1}\|^2\big).
\end{aligned} \tag{3.1}$$

According to Lemma 3.1, we obtain from (3.1)

$$\frac{1}{2}\left[\left(1 - \frac{5}{3}\tau\right)\|u^{n+1}\|^2 + \|u^n\|^2\right] + \frac{1}{2}\left[\beta\|\delta_x u^{n+1}\|^2 + \left(\beta - \frac{5}{3}\tau\right)\|\delta_x u^n\|^2\right] \le C. \tag{3.2}$$

This implies for small τ which satisfies $\beta - \frac{5}{3}\tau > 0$ that we have

$$\|u^n\| \le C, \qquad \|\delta_x u^n\| \le C. \tag{3.3}$$

Using Lemma 3.2, we obtain

$$\|u^n\|_\infty \le C. \tag{3.4}$$

□

Remark 3.1 Theorem 3.1 implies that the scheme (2.1) is unconditionally stable.

Theorem 3.2 *The difference scheme* (2.1) *is uniquely solvable.*

Proof Let us prove the unique solvability by induction. It is obvious that u^0 and u^1 are uniquely determined by (2.3) and (2.2), respectively. Suppose that $u^0, u^1, \ldots, u^n$ be solved uniquely. Consider u^{n+1} in (2.1) which satisfies

$$\begin{aligned}
&\frac{1}{2\tau}u_j^{n+1} - \frac{2}{3\tau}\beta\delta_x\delta_{\bar{x}}u_j^{n+1} + \frac{1}{6\tau}\delta_{\hat{x}}\delta_{\hat{x}}\big(u_j^{n+1}\big) + \frac{2p}{3(p+1)}\alpha\big\{\big(u_j^n\big)^{p-1}\delta_{\hat{x}}u_j^{n+1} + \delta_{\hat{x}}\big[\big(u_j^n\big)^{p-1}u_j^{n+1}\big]\big\}\\
&\quad - \frac{p}{6(p+1)}\big\{\big(u_j^n\big)^{p-1}\delta_{\ddot{x}}u_j^{n+1} + \delta_{\ddot{x}}\big[\big(u_j^n\big)^{p-1}u_j^{n+1}\big]\big\} = 0.
\end{aligned} \tag{3.5}$$

Taking the inner product of (3.5) with u^{n+1}, we obtain

$$\frac{1}{2\tau}\|u^{n+1}\|^2+\frac{2}{3\tau}\beta\|\delta_x u^{n+1}\|^2-\frac{1}{6\tau}\beta\|\delta_{\hat{x}} u^{n+1}\|^2+(I-II,u^{n+1})=0, \tag{3.6}$$

where

$$I=\frac{2p}{3(p+1)}\alpha\{(u_j^n)^{p-1}\delta_{\hat{x}}u_j^{n+1}+\delta_{\hat{x}}[(u_j^n)^{p-1}u_j^{n+1}]\},$$
$$II=\frac{p}{6(p+1)}\{(u_j^n)^{p-1}\delta_{\ddot{x}}u_j^{n+1}+\delta_{\ddot{x}}[(u_j^n)^{p-1}u_j^{n+1}]\}.$$

Similarly to the proof of (2.10), we get

$$(I,u^{n+1})=0,\qquad (II,u^{n+1})=0. \tag{3.7}$$

It follows from (3.6)-(3.7) and Lemma 3.1 that

$$\frac{1}{2\tau}\|u^{n+1}\|^2+\frac{1}{2\tau}\beta\|\delta_x u^{n+1}\|^2\le 0. \tag{3.8}$$

That is, (3.5) has only a trivial solution. Hence, (2.1) determines u_j^{n+1} uniquely. This completes the proof of Theorem 3.2. □

4 Convergence and stability of the difference scheme

First, we shall consider the truncation error of the difference scheme (2.8)-(2.10). Let $v_j^n = u(x_j,t_n)$. We define the truncation error as follows:

$$\begin{aligned}Er_j^n={}&\delta_{\hat{t}}v_j^n-\frac{4}{3}\beta\delta_x\delta_{\bar{x}}\delta_{\hat{t}}v_j^n+\frac{1}{3}\beta\delta_{\hat{x}}\delta_{\hat{x}}\delta_{\hat{t}}v_j^n+\frac{4}{3}\delta_{\hat{x}}(v_j^n)-\frac{1}{3}\delta_{\ddot{x}}(v_j^n)\\&+\frac{p}{p+1}\alpha\bigg[\frac{4}{3}\{(\delta_{\hat{x}}\bar{v}_j^n)(v_j^n)^{p-1}+\delta_{\hat{x}}[(v_j^n)^{p-1}(\bar{v}_j^n)]\}\\&-\frac{1}{3}\{(\delta_{\ddot{x}}\bar{v}_j^n)(v_j^n)^{p-1}+\delta_{\ddot{x}}[(v_j^n)^{p-1}(\bar{v}_j^n)]\}\bigg],\\&1\le j\le J-1, 1\le n\le N-1,\end{aligned} \tag{4.1}$$

$$\begin{aligned}s_j^0={}&v_j^1-\frac{4}{3}\beta\delta_x\delta_{\bar{x}}v_j^1+\frac{1}{3}\beta\delta_{\hat{x}}\delta_{\hat{x}}v_j^1-u_0(x_j)\\&+\frac{d^2u_0}{dx^2}(x_j)+\tau\frac{du_0}{dx}(x_j)+\tau\alpha p u_0^{p-1}(x_j)\frac{du_0}{dx}(x_j),\end{aligned} \tag{4.2}$$

$$v_j^0=u_0(x_j),\quad 1\le j\le J, \tag{4.3}$$

$$v_0^n=v_J^n=0,\quad 0\le n\le N. \tag{4.4}$$

Using a Taylor expansion, we obtain $|Er^n|+|s^0|=O(\tau^2+h^4)$ holds if $\tau,h\to 0$.

Next, we shall discuss the convergence and stability of the scheme (2.1)-(2.4).

Lemma 4.1 (Discrete Gronwall inequality [37]) *Suppose that the discrete mesh function* $\{w^n|n=1,2,\ldots,N;N\tau=T\}$ *satisfies the recurrence formula*

$$w^n-w^{n-1}\le A\tau w^n+B\tau w^{n-1}+C_n\tau,$$

where A, B, and C_n ($n = 1,\dots,N$) are nonnegative constants. Then

$$\|w^n\|_\infty \le \left(w^0 + \tau\sum_{k=1}^{N} C_k\right)e^{2(A+B)T},$$

where τ is small, such that $(A+B)\tau \le \frac{N-1}{2N}$ ($N > 1$).

Theorem 4.1 *Assume that $u_0 \in H^1$, then the solution u^n of the scheme* (2.1)-(2.4) *converges to the solution of the initial-boundary problem* (1.1)-(1.3) *and the rate of convergence is $O(\tau^2 + h^4)$ by the $\|\cdot\|_\infty$ norm.*

Proof Let $e_j^n = v_j^n - u_j^n$. From (4.1)-(4.4) and (2.1)-(2.4), we have

$$\begin{aligned} Er_j^n ={}& \delta_{\hat t} e_j^n - \frac{4}{3}\beta\delta_x\delta_{\bar x}\delta_{\hat t} e_j^n + \frac{1}{3}\beta\delta_{\hat x}\delta_{\hat x}\delta_{\hat t} e_j^n + \frac{4}{3}\delta_{\hat x}\left(e_j^n\right) - \frac{1}{3}\delta_{\ddot x}\left(e_j^n\right) \\ &+ \frac{p}{p+1}\alpha\left[\frac{4}{3}\left\{\left(\delta_{\hat x}\bar v_j^n\right)\left(v_j^n\right)^{p-1} + \delta_{\hat x}\left[\left(v_j^n\right)^{p-1}\left(\bar v_j^n\right)\right]\right\}\right. \\ &\left. - \frac{4}{3}\left\{\left(\delta_{\hat x}\bar u_j^n\right)\left(u_j^n\right)^{p-1} + \delta_{\hat x}\left[\left(u_j^n\right)^{p-1}\left(\bar u_j^n\right)\right]\right\}\right] \\ &- \frac{p}{p+1}\alpha\left[\frac{1}{3}\left\{\left(\delta_{\ddot x}\bar v_j^n\right)\left(v_j^n\right)^{p-1} + \delta_{\ddot x}\left[\left(v_j^n\right)^{p-1}\left(\bar v_j^n\right)\right]\right\}\right. \\ &\left. - \frac{1}{3}\left\{\left(\delta_{\ddot x}\bar u_j^n\right)\left(u_j^n\right)^{p-1} + \delta_{\ddot x}\left[\left(u_j^n\right)^{p-1}\left(\bar u_j^n\right)\right]\right\}\right], \quad 1 \le j \le J-1, 1 \le n \le N-1, \end{aligned} \tag{4.5}$$

$$s_j^0 = e_j^1 - \frac{4}{3}\beta\delta_x\delta_{\bar x} e_j^1 + \frac{1}{3}\beta\delta_{\hat x}\delta_{\hat x} e_j^1, \tag{4.6}$$

$$e_j^0 = 0, \quad 1 \le j \le J, \tag{4.7}$$

$$e_0^n = e_J^n = 0, \quad 0 \le n \le N. \tag{4.8}$$

Taking in (4.5) the inner product with $2\bar e^n$ (*i.e.* $e^{n+1} + e^{n-1}$), we obtain

$$\begin{aligned} \left(Er^n, 2\bar e^n\right) ={}& \delta_{\hat t}\left\|e^n\right\|^2 + \frac{4}{3}\beta\delta_{\hat t}\left\|\delta_x e^n\right\|^2 - \frac{1}{3}\beta\delta_{\hat t}\left\|\delta_{\hat x} e^n\right\|^2 + \frac{4}{3}h\sum_{j=1}^{J-1}\left(\delta_x e_j^n e_j^{n+1} - e_j^n\delta_x e_j^{n-1}\right) \\ &- \frac{1}{3}h\sum_{j=1}^{J-1}\left(\delta_{\ddot x} e_j^n e_j^{n+1} - e_j^n\delta_{\ddot x} e_j^{n-1}\right) + \left(P_1 + P_2 + Q_1 + Q_2, 2\bar e^n\right), \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} P_1 &= \frac{4p\alpha}{3(p+1)}\left[\left(\delta_{\hat x}\bar v^n\right)\left(v^n\right)^{p-1} - \left(\delta_{\hat x}\bar u^n\right)\left(u^n\right)^{p-1}\right], \\ P_2 &= \frac{4p\alpha}{3(p+1)}\left\{\delta_{\hat x}\left[\left(v^n\right)^{p-1}\bar v^n\right] - \delta_{\hat x}\left[\left(u^n\right)^{p-1}\bar u^n\right]\right\}, \\ Q_1 &= -\frac{p\alpha}{3(p+1)}\left[\left(\delta_{\ddot x}\bar v^n\right)\left(v^n\right)^{p-1} - \left(\delta_{\ddot x}\bar u^n\right)\left(u^n\right)^{p-1}\right], \\ Q_2 &= -\frac{p\alpha}{3(p+1)}\left\{\delta_{\ddot x}\left[\left(v^n\right)^{p-1}\bar v^n\right] - \delta_{\ddot x}\left[\left(u^n\right)^{p-1}\bar u^n\right]\right\}. \end{aligned}$$

Computing the sixth term on the right-hand side of (4.9) and using Lemma 3.1 and Theorem 3.1 yield

$$\begin{aligned}
(P_1, 2\bar{e}^n) &= \frac{4p\alpha}{3(p+1)}\big(\big[(\delta_{\hat{x}}\bar{v}^n)(v^n)^{p-1} - (\delta_{\hat{x}}\bar{u}^n)(u^n)^{p-1}\big], 2\bar{e}^n\big) \\
&= \frac{8p\alpha}{3(p+1)}\big(\big[(\delta_{\hat{x}}\bar{e}^n)(v^n)^{p-1} + (\delta_{\hat{x}}\bar{u}^n)\big((v^n)^{p-1} - (u^n)^{p-1}\big)\big], \bar{e}^n\big) \\
&= \frac{8p\alpha}{3(p+1)}h\left\{\sum_{j=1}^{J-1}(\delta_{\hat{x}}\bar{e}_j^n)(v_j^n)^{p-1}\bar{e}_j^n + \sum_{j=1}^{J-1}(\delta_{\hat{x}}\bar{u}_j^n)\big[(v_j^n)^{p-1} - (u_j^n)^{p-1}\big]\bar{e}_j^n\right\} \\
&= \frac{8p\alpha}{3(p+1)}h\left(\sum_{j=1}^{J-1}(\delta_{\hat{x}}\bar{e}_j^n)(v_j^n)^{p-1}\bar{e}_j^n + \sum_{j=1}^{J-1}(\delta_{\hat{x}}\bar{u}_j^n)\left[e_j^n\sum_{k=0}^{p-2}(v_j^n)^{p-2-k}(u_j^n)^k\right]\bar{e}_j^n\right) \\
&\le C\big(\|\delta_{\hat{x}}\bar{e}^n\|^2 + \|e^n\|^2 + \|\bar{e}^n\|^2\big) \\
&\le C\big(\|\delta_x e^{n+1}\|^2 + \|\delta_x e^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2\big), \qquad (4.10)
\end{aligned}$$

where the Cauchy-Schwartz inequality and Lemma 3.1 are used.

Similarly, we can also obtain

$$(P_2, 2\bar{e}^n) \le C\big(\|\delta_x e^{n+1}\|^2 + \|\delta_x e^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2\big), \qquad (4.11)$$

$$(Q_1, 2\bar{e}^n) \le C\big(\|\delta_x e^{n+1}\|^2 + \|\delta_x e^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2\big), \qquad (4.12)$$

$$(Q_2, 2\bar{e}^n) \le C\big(\|\delta_x e^{n+1}\|^2 + \|\delta_x e^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2\big). \qquad (4.13)$$

In addition, it is obvious that

$$(Er^n, 2\bar{e}^n) \le \|Er^n\|^2 + \frac{1}{2}\big(\|e^{n+1}\|^2 + \|e^{n-1}\|^2\big), \qquad (4.14)$$

$$\frac{4}{3}h\sum_{j=1}^{J-1}\big(\delta_x e_j^n e_j^{n+1} - e_j^n\delta_x e_j^{n-1}\big) \le C\big(\|\delta_x e^n\|^2 + \|\delta_x e^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2\big), \qquad (4.15)$$

$$-\frac{1}{3}h\sum_{j=1}^{J-1}\big(\delta_{\ddot{x}} e_j^n e_j^{n+1} - e_j^n\delta_{\ddot{x}} e_j^{n-1}\big) \le C\big(\|\delta_x e^n\|^2 + \|\delta_x e^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2\big). \qquad (4.16)$$

It follows from (4.9)-(4.16) that

$$\begin{aligned}
&\delta_{\hat{t}}\|e^n\|^2 + \frac{4}{3}\beta\delta_{\hat{t}}\|\delta_x e^n\|^2 - \frac{1}{3}\beta\delta_{\hat{t}}\|\delta_{\hat{x}} e^n\|^2 \\
&\quad\le \|Er^n\|^2 + C\big(\|\delta_x e^{n+1}\|^2 + \|\delta_x e^n\|^2 \\
&\qquad + \|\delta_x e^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2\big). \qquad (4.17)
\end{aligned}$$

Let $B^n = \frac{1}{2}(\|e^{n+1}\|^2 + \|e^n\|^2) + \frac{\beta}{2}(\|\delta_x e^{n+1}\|^2 + \|\delta_x e^n\|^2)$. Using Lemma 3.1, (4.17) can be written as follows:

$$B^n - B^{n-1} \le \tau\|Er^n\|^2 + C\tau\big(B^n + B^{n-1}\big). \qquad (4.18)$$

According to Lemma 4.1, we can immediately obtain

$$B^n \le \Big(B^0 + T \sup_{l\le n\le N} \big\|Er^n\big\|^2\Big) e^{CT}. \tag{4.19}$$

Taking the inner product of (4.6) with e^1 yields

$$\big(s^0, e^1\big) = \big\|e^1\big\|^2 + \frac{4}{3}\beta\big\|\delta_x e^1\big\|^2 - \frac{1}{3}\beta\big\|\delta_{\hat{x}} e^1\big\|^2. \tag{4.20}$$

This, together with $(s^0, e^1) \le \frac{1}{2}(\|s^0\|^2 + \|e^1\|^2)$, $|s^0| = O(\tau^2 + h^4)$, and Lemma 3.1, gives

$$\big\|e^1\big\| \le O\big(\tau^2 + h^4\big), \qquad \big\|\delta_x e^1\big\| \le O\big(\tau^2 + h^4\big). \tag{4.21}$$

From the discrete initial condition (4.7), we know that $B^0 = [O(\tau^2 + h^4)]^2$.

It follows from (4.19) that

$$B^n \le \big[O\big(\tau^2 + h^4\big)\big]^2. \tag{4.22}$$

Then we have

$$\big\|e^n\big\| \le O\big(\tau^2 + h^4\big), \qquad \big\|\delta_x e^n\big\| \le O\big(\tau^2 + h^4\big). \tag{4.23}$$

This, together with Lemmas 3.2, gives

$$\big\|e^n\big\|_\infty \le O\big(\tau^2 + h^4\big). \tag{4.24}$$

This completes the proof of Theorem 4.1. □

Similarly, we can prove stability of the difference solution.

Theorem 4.2 *Under the conditions of Theorem* 4.1, *the solution of the scheme* (2.1)-(2.4) *is unconditionally stable by the* $\|\cdot\|_\infty$ *norm.*

5 Numerical experiments

In this section, we give some numerical experiments to demonstrate our theoretical results obtained in the previous sections. We will measure the accuracy of the proposed scheme using the absolute error defined by $e^n = \|v^n - u^n\|_\infty$.

Consider the initial-boundary value problem (1.1)-(1.3). In the numerical experiments, we take $x_l = -40$, $x_r = 60$, $T = 10$, and choose three cases $p = 2, 3, 4$, respectively. In order to verify the accuracy $O(\tau^2 + h^4)$, we take τ and h small enough to verify the fourth-order accuracy and second-order accuracy in the spatial and temporal directions, respectively. The convergence order figures of $\log(e^n)$- $\log(h)$ with h and the ones of $\log(e^n)$- $\log(\tau)$ with τ small enough are given in Figures 1-6 for various mesh steps h and τ at $t = 10$. From Figures 1-6, it is obvious that the scheme (2.1)-(2.4) is convergent in the maximum norm, and the convergence order is $O(\tau^2 + h^4)$.

We show in Theorem 2.1 that the numerical solution u^n of the scheme (2.1) satisfies the conservation of discrete energy. In Tables 1-3, the values of E^n for the scheme (2.1) are

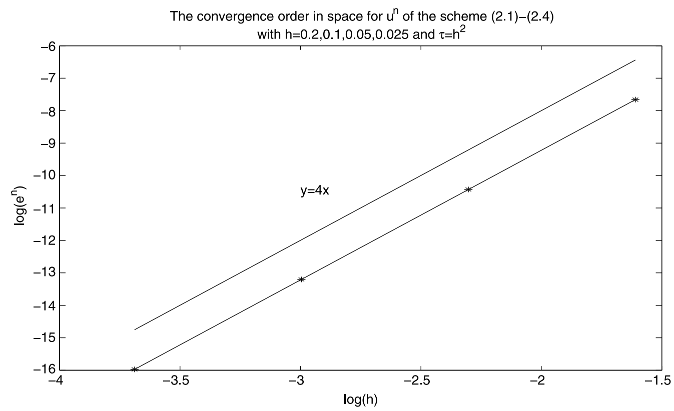

Figure 1 The spatial convergence order in maximal norm for u^n at $t = 10$ with different h and τ computed by the scheme (2.1)-(2.4).

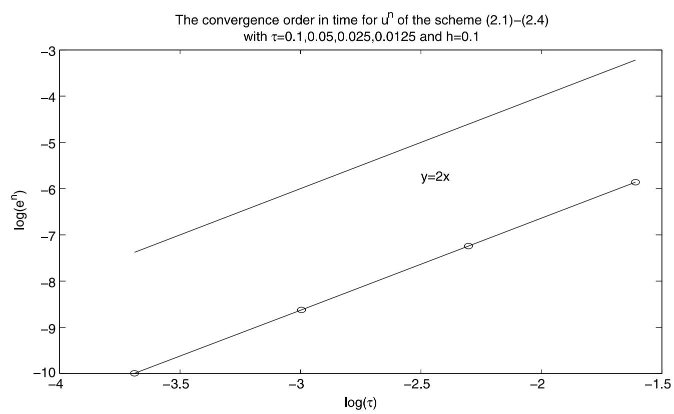

Figure 2 The temporal convergence order in maximal norm for u^n at $t = 10$ with different h and τ computed by the scheme (2.1)-(2.4).

Table 1 Discrete energy E^n of the scheme (2.1) at different time t when $h = 0.1, 0.05$ and $p = 2$

t	E^n	
	$h = 0.1$, $\tau = 0.1$	$h = 0.05$, $\tau = 0.025$
2	5.599999981471710	5.599999994989064
4	5.599999981452243	5.599999990049693
6	5.599999981432834	5.599999985109076
8	5.599999981413214	5.599999980169102
10	5.599999981393715	5.599999975228256

presented for three cases $p = 2, 3, 4$ under various mesh steps h and τ, respectively. It is easy to see from Tables 1-3 that the scheme (2.1) preserves the discrete energy very well, which also shows the accuracy and efficiency of the scheme in this paper.

Case 5.1 Take $p = 2$. Consider the following initial-boundary problem of RLW equation:

$$u_t - u_{xxt} + u_x + uu_x = 0, \tag{5.1}$$

$$u(x, 0) = u_0(x), \tag{5.2}$$

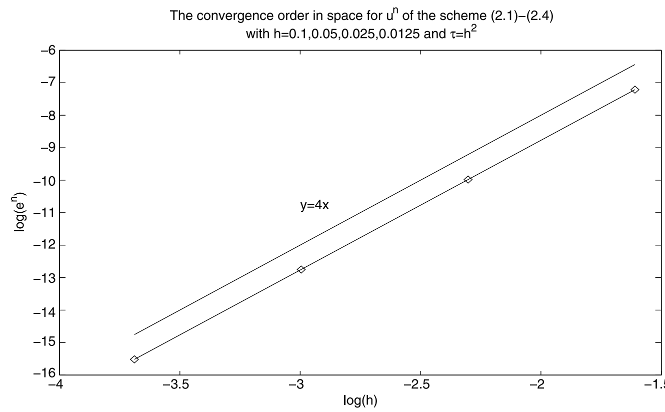

Figure 3 The spatial convergence order in maximal norm for u^n at $t = 10$ with different h and τ computed by the scheme (2.1)-(2.4).

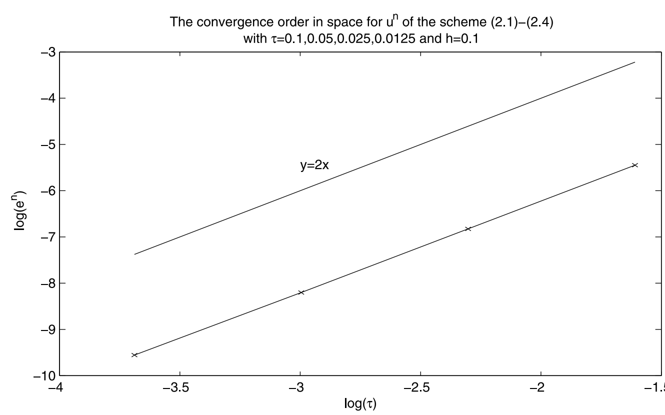

Figure 4 The temporal convergence order in maximal norm for u^n at $t = 10$ with different h and τ computed by the scheme (2.1)-(2.4).

Table 2 Discrete energy E^n of the scheme (2.1) at different times t when $h = 0.1, 0.05$ and $p = 3$

t	E^n	
	$h = 0.1$, $\tau = 0.1$	$h = 0.05$, $\tau = 0.025$
2	2.888888820624460	2.888888884464111
4	2.888888820614826	2.888888884309116
6	2.888888820605145	2.888888884153947
8	2.888888820595414	2.888888883998621
10	2.888888820585634	2.888888883843508

$$u(x_l, t) = u(x_r, t) = 0. \tag{5.3}$$

In computations, we choose the initial condition $u(x,0) = \operatorname{sech}^2(\frac{1}{4}x)$ [10]. The convergence order figures and the values of E^n are shown in Figures 1-2 and Table 1, respectively.

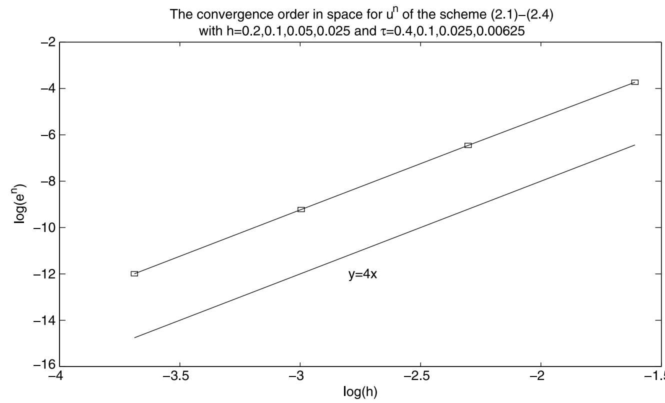

Figure 5 The spatial convergence order in maximal norm for u^n at $t = 10$ with different h and τ computed by the scheme (2.1)-(2.4).

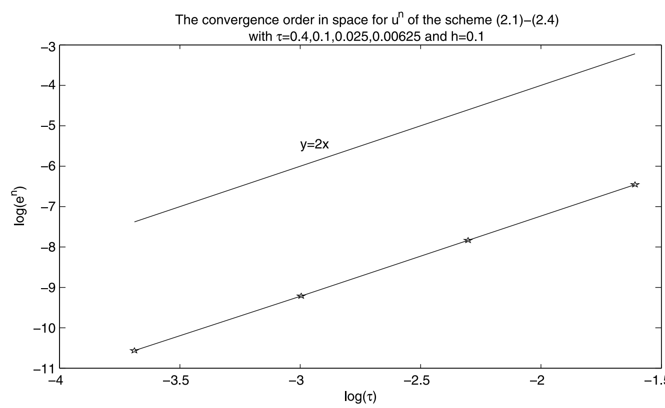

Figure 6 The temporal convergence order in maximal norm for u^n at $t = 10$ with different h and τ computed by the scheme (2.1)-(2.4).

Table 3 Discrete energy E^n of the scheme (2.1) at different time t when $(h, \tau) = (0.1, 0.1)$ and (0.05,0.025) for $p = 4$

t	E^n	
	$h = 0.1$, $\tau = 0.1$	$h = 0.05$, $\tau = 0.025$
2	5.943183838327444	5.943183859464757
4	5.943183838326568	5.943183859449224
6	5.943183838325569	5.943183859434159
8	5.943183838324471	5.943183859418023
10	5.943183838323502	5.943183859402211

Case 5.2 Take $p = 3$. We consider the following initial-boundary problem of MRLW equation:

$$u_t - u_{xxt} + u_x + \left(u^3\right)_x = 0, \tag{5.4}$$

$$u(x, 0) = u_0(x), \tag{5.5}$$

$$u(x_l, t) = u(x_r, t) = 0. \tag{5.6}$$

In experiments, we choose the initial condition $u(x,0) = \frac{\sqrt{6}}{3}\operatorname{sech}(\frac{1}{2}x)$ [1]. The convergence order figures and the values of E^n are shown in Figures 3-4 and Table 2, respectively.

Case 5.3 Take $p = 4$. We consider the initial-boundary problem (1.1)-(1.3) of GRLW equation:

$$u_t - u_{xxt} + u_x + u^3 u_x = 0, \tag{5.7}$$

$$u(x,0) = u_0(x), \tag{5.8}$$

$$u(x_l,t) = u(x_r,t) = 0. \tag{5.9}$$

In the following experiments, we choose the initial condition $u(x,0) = \operatorname{sech}^{\frac{2}{3}}(\frac{3\sqrt{11}}{22}x)$ [1]. The convergence order figures and the values of E^n are shown in Figures 5-6 and Table 3, respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The article was carried out in collaboration between all authors. The three authors have contributed to, read, and approved the manuscript.

Acknowledgements
This work is supported by the Natural Science Foundation of China (No. 11201343, 11401438), Natural Science Foundation of Shandong Province (ZR2012AM017, ZR2013FL032), a Project of Shandong Province Higher Educational Science and Technology Program (No. J14LI52, J15LI56), the Youth Research Foundation of WFU (No. 2013Z11) and the Project of Science and Technology Program of Weifang (Grant no. 201301006).

Received: 25 May 2015 Accepted: 27 July 2015 Published online: 16 August 2015

References
1. Zhang, L: A finite difference scheme for the generalized regularized long-wave equation. Appl. Math. Comput. **168**, 962-972 (2005)
2. Peregrine, DH: Calculations of the development of an undular bore. J. Fluid Mech. **25**, 321-330 (1966)
3. Benjamin, TB, Bona, JL, Mahony, JJ: Model equations for long waves in nonlinear dispersive systems. Philos. Trans. R. Soc. Lond. A **272**, 47-78 (1972)
4. Albert, J: Dispersion of low-energy waves for the generalized Benjamin-Bona-Mahony equation. J. Differ. Equ. **63**(1), 117-134 (1986)
5. Ruggieri, M, Speciale, MP: Similarity reduction and closed form solutions for a model derived from two-layer fluids. Adv. Differ. Equ. **2013**, 355 (2013)
6. Lmaco, J, Clark, HR, Medeiros, LA: Remarks on equations of Benjamin-Bona-Mahony type. J. Math. Anal. Appl. **328**(2), 1117-1140 (2007)
7. Ruggieri, M, Speciale, MP: New exact solutions for a coupled KdV-like model. J. Phys. Conf. Ser. **482**, 012036 (2014)
8. Gear, JA, Grimshaw, R: Weak and strong interactions between internal solitary waves. Stud. Appl. Math. **70**, 235-258 (1984)
9. Kutluay, S, Esen, A: A finite difference solution of the regularized long wave equation. Math. Probl. Eng. **2006**, 1-14 (2006)
10. Zhang, L, Chang, Q: A new finite difference method for regularized long-wave equation. Chinese J. Numer. Methods Comput. Appl. **23**, 58-66 (2001)
11. Avilez-Valente, P, Seabra-Santos, FJ: A Petrov-Galerkin finite element scheme for the regularized long wave equation. Comput. Mech. **34**, 256-270 (2004)
12. Esen, A, Kutluay, S: Application of a lumped Galerkin method to the regularized long wave equation. Appl. Math. Comput. **174**(2), 833-845 (2006)
13. Guo, L, Chen, H: H^1-Galerkin mixed finite element method for the regularized long wave equation. Computing **77**, 205-221 (2006)
14. Gu, H, Chen, N: Least-squares mixed finite element methods for the RLW equations. Numer. Methods Partial Differ. Equ. **24**, 749-758 (2008)
15. Saka, B, Dag, I: A numerical solution of the RLW equation by Galerkin method using quartic B-splines. Commun. Numer. Methods Eng. **24**, 1339-1361 (2008)
16. Dag, I: Least squares quadratic B-spline finite element method for the regularized long wave equation. Comput. Methods Appl. Mech. Eng. **182**, 205-215 (2000)

17. Dag, I, Ozer, MN: Approximation of the RLW equation by the least square cubic B-spline finite element method. Appl. Math. Model. **25**, 221-231 (2001)
18. Dag, I, Saka, B, Irk, D: Application of cubic B-splines for numerical solution of the RLW equation. Appl. Math. Comput. **159**, 373-389 (2004)
19. Soliman, AA, Raslan, KR: Collocation method using quadratic B-spline for the RLW equation. Int. J. Comput. Math. **78**, 399-412 (2001)
20. Soliman, AA, Hussien, MH: Collocation solution for RLW equation with septic spline. Appl. Math. Comput. **161**, 623-636 (2005)
21. Cohen, G: High-Order Numerical Methods for Transient Wave Equations. Springer, New York (2002)
22. Dai, W: An improved compact finite difference scheme for solving a N-carrier system with Neumann boundary conditions. Numer. Methods Partial Differ. Equ. **27**, 436-446 (2011)
23. Dai, W, Tzou, DY: A fourth-order compact finite difference scheme for solving a N-carrier system with Neumann boundary conditions. Numer. Methods Partial Differ. Equ. **25**, 274-289 (2010)
24. Dehghan, M, Mohebbi, A, Asgari, Z: Fourth-order compact solution of the nonlinear Klein-Gordon equation. Numer. Algorithms **52**, 523-540 (2009)
25. Mohebbia, A, Dehghan, M: High-order solution of one-dimensional Sine-Gordon equation using compact finite difference and DIRKN methods. Math. Comput. Model. **51**, 537-549 (2010)
26. Mohebbia, A, Dehghan, M: High-order compact solution of the one-dimensional heat and advection-diffusion equations. Appl. Math. Model. **34**, 3071-3084 (2010)
27. Dehghan, M, Taleei, A: A compact split-step finite difference method for solving the nonlinear Schrodinger equations with constant and variable coefficients. Comput. Phys. Commun. **181**, 43-51 (2010)
28. Xie, S, Li, G, Yi, S: Compact finite difference schemes with high accuracy for one-dimensional nonlinear Schrödinger equation. Comput. Methods Appl. Mech. Eng. **198**, 1052-1060 (2009)
29. Wang, T, Guo, B: Unconditional convergence of two conservative compact difference schemes for non-linear Schrödinger equation in one dimension. Sci. Sin., Math. **41**(3), 207-233 (2011) (in Chinese)
30. Pan, X, Zhang, L: High-order linear compact conservative method for the nonlinear Schrodinger equation coupled with the nonlinear Klein-Gordon equation. Nonlinear Anal. **92**, 108-118 (2013)
31. Zheng, K, Hu, J: High-order conservative Crank-Nicolson scheme for regularized long wave equation. Adv. Differ. Equ. **2013**, 287 (2013)
32. Alias, A, Grimshaw, RHJ, Khusnutdinova, KR: On strongly interacting internal waves in a rotating ocean and coupled Ostrovsky equations. Chaos **23**(2), 023121 (2013)
33. Ruggieri, M, Speciale, MP: KdV-like equations for fluid dynamics. AIP Conf. Proc. **1637**, 918-924 (2014)
34. Alias, A, Grimshaw, RHJ, Khusnutdinova, KR: Coupled Ostrovsky equations for internal waves in a shear flow. Phys. Fluids **26**, 126603 (2014)
35. Ruggieri, M, Speciale, MP: Quasi self-adjoint coupled KdV-like equations. In: 11th International Conference of Numerical Analysis and Applied Mathematics, vol. 1558, pp. 1220-1223. AIP, New York (2013)
36. Ruggieri, M, Speciale, MP: On a hierarchy of traveling wave solutions in a shallow stratified fluid. In: 11th International Conference of Numerical Analysis and Applied Mathematics, vol. 1558, pp. 1793-1796. AIP, New York (2013)
37. Zhou, Y: Application of Discrete Functional Analysis to the Finite Difference Method. Inter. Acad. Publishers, Beijing (1990)