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Construction of lower and upper solutions for first-order periodic problem

Ruyun Ma* and Lu Zhang

*Correspondence:
mary@nwnu.edu.cn
Department of Mathematics,
Northwest Normal University,
Lanzhou, 730070, P.R. China

Abstract

In this paper, we construct nonconstant lower and upper solutions for the periodic boundary value problem $x' + f(t, x) = e(t)$, $x(0) = x(T)$ and find their estimates. We prove the existence of positive solutions for the singular problem $x' + g(x) = e(t)$, $x(0) = x(T)$ by using these results.

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1 Introduction

The method of lower and upper solutions is an elementary but powerful tool in the existence theory of solutions to initial value problems and for periodic boundary value problems, even in cases where no special structure is assumed on the nonlinearity. Starting with the pioneering work of Moretto [1] for locally Lipschitzian ordinary differential equations, the method of upper and lower solutions for the periodic boundary value problem

$$x' + f(t, x) = 0, \quad x(0) = x(T) \quad (1.1)$$

has been extended to the case of a continuous right-hand side $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ (see, e.g., [2–6]).

In Franco *et al.* [6], the method of lower and upper solutions was applied to the periodic problem

$$x' = f(t, x), \quad x(0) = x(T),$$

and they also obtained a similar result to Lemma 1.1 below.

In order to apply these results, finding upper and lower solutions is very important. But the problem of construction of lower and upper solutions has been solved very rarely (see, e.g., [7, 8]). In this paper we fill this gap and present conditions ensuring the existence of nonconstant lower and upper solutions to the first-order periodic boundary value problem (1.1) and find their estimates. This enables us to prove the existence result for the periodic problem with strong singularity.

In order to explain the main results of the paper, let us introduce some notation: $C[0, T]$ stands for the set of functions continuous on $[0, T]$. For $x \in C[0, T]$, we denote

$$\bar{x} = \frac{1}{T} \int_0^T x(s) ds, \quad \|x\|_1 = \int_0^T |x(s)| ds.$$

If $x \in C[0, T]$, then we write $x^+(t) = \max\{x(t), 0\}$ and $x^-(t) = \max\{-x(t), 0\}$. For $e \in C[0, T]$, we put

$$E = \int_0^T e(t) dt, \quad E_{\pm} = \int_0^T e^{\pm}(t) dt,$$

and note that $E = E_+ - E_-$.

Let us recall first some classical definitions and results. They are taken from [3].

Definition 1.1 A lower solution α (respectively, an upper solution β) of problem (1.1) is a function $\alpha \in C^1[0, T]$ such that

$$\alpha' + f(t, \alpha) \geq 0, \quad t \in [0, T], \quad \alpha(0) \geq \alpha(T),$$

(respectively, $\beta \in C^1[0, T]$ and

$$\beta'(t) + f(t, \beta) \leq 0, \quad t \in [0, T], \quad \beta(0) \leq \beta(T)).$$

The basic existence theorem of the method of upper and lower solutions for (1.1) can be stated as follows.

Lemma 1.1 *If problem (1.1) has a lower solution α and upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, T]$ (resp. $\beta(t) \leq \alpha(t)$ for all $t \in [0, T]$), then problem (1.1) has at least one solution x such that $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in [0, T]$ (resp. $\beta(t) \leq x(t) \leq \alpha(t)$ for all $t \in [0, T]$).*

The paper is organized as follows. In Section 2 we develop a method to construct lower and upper solutions for (1.1). As the application, in Section 3, we establish an existence result for a nonlinear first-order periodic problem with strong singularity.

2 Construction of lower and upper solutions

In this section we consider

$$x' + f(t, x) = e(t), \quad x(0) = x(T), \tag{2.1}$$

where $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ and $e : [0, T] \rightarrow \mathbb{R}$ are continuous.

Let us consider an auxiliary boundary value problem

$$x' + \delta(t) = 0, \quad x(0) = x(T) = \zeta, \tag{2.2}$$

where $\delta \in C[0, T]$ satisfies $\bar{\delta} = 0$ and $\zeta \in \mathbb{R}$.

Clearly, for all $\zeta \in \mathbb{R}$, problem (2.2) possesses a unique solution x_ζ as follows:

$$x_\zeta(t) = \zeta - \int_0^t \delta(s) ds.$$

Proposition 2.1 *Assume that there are $B_1 \in \mathbb{R}$, $c_1 \in C[0, T]$ such that*

$$f(t, x) \leq c_1(t) \quad \text{for all } t \in [0, T] \text{ and } x \in [B_1, B_1 + 2\|\eta_1\|_1], \quad (2.3)$$

where $\eta_1(t) = \phi^-(t) - \phi^+(t) \frac{\Phi_-}{\Phi_+}$ and $\phi = c_1 - e$. If

$$\bar{c}_1 - \bar{e} \leq 0, \quad (2.4)$$

then there exists an upper solution β of (2.1) such that

$$B_1 \leq \beta \leq B_1 + 2\|\eta_1\|_1 \quad \text{in } [0, T]. \quad (2.5)$$

Proof Consider the function $\phi = c_1 - e$. We have two cases.

Case 1. Assume that $\Phi_+ = \int_0^T \phi^+(t) dt = 0$. Taking $\beta = \xi \in [B_1, B_1 + 2\|\eta_1\|_1]$ and using $c_1 - e \leq 0$, it follows from (2.3) that β is an upper solution of (2.1).

Case 2. Assume that $\Phi_+ > 0$. Using

$$\int_0^T [\phi^+(t)\Phi_- - \phi^-(t)\Phi_+] dt = 0,$$

it follows that there exists ω such that

$$\omega'(t) + \phi^+(t)\Phi_- - \phi^-(t)\Phi_+ = 0, \quad \omega(0) = \omega(T) = \zeta.$$

Taking $x_0 = \frac{1}{\Phi_+}$ and

$$\beta(t) = \zeta - \int_0^t x_0 [\phi^+(s)\Phi_- - \phi^-(s)\Phi_+] ds,$$

it is easy to see that $\beta(0) = \beta(T)$. Since $|\int_0^t x_0 [\phi^+(s)\Phi_- - \phi^-(s)\Phi_+] ds| = |\int_0^t \eta_1(s) ds| \leq \|\eta_1\|_1$, choosing $\zeta = B_1 + \|\eta_1\|_1$, we obtain

$$B_1 \leq \beta(t) \leq B_1 + 2\|\eta_1\|_1,$$

which means that (2.5) holds.

Now, using (2.4), it follows that $\Phi_+ \leq \Phi_-$, implying that

$$\beta'(t) = x_0 \omega'(t) = -x_0 [\phi^+(s)\Phi_- - \phi^-(s)\Phi_+] \leq -\phi(t) = e(t) - c_1(t).$$

From (2.3) and (2.5) we deduce that

$$\beta'(t) + f(t, \beta) \leq \beta'(t) + c_1(t) \leq e(t) \quad \text{for all } t \in [0, T],$$

and the proof is completed. \square

Using similar arguments, one can prove the following proposition.

Proposition 2.2 *Assume that there are $B_2 \in \mathbb{R}$, $c_2 \in C[0, T]$ such that*

$$f(t, x) \geq c_2(t) \quad \text{for all } t \in [0, T] \text{ and } x \in [B_2, B_2 + 2\|\eta_2\|_1],$$

where $\eta_2(t) = \phi^-(t) - \phi^+(t) \frac{\Phi_-}{\Phi_+}$ and $\phi = c_2 - e$. If

$$\bar{c}_2 - \bar{e} \geq 0,$$

then there exists a lower solution α of (2.1) such that

$$B_2 \leq \alpha \leq B_2 + 2\|\eta_2\|_1 \quad \text{in } [0, T].$$

Theorem 2.1 and Theorem 2.2 are simple examples of existence results which follow immediately from Lemma 1.1, Propositions 2.1 and 2.2.

Theorem 2.1 *Assume that there are $B_i \in \mathbb{R}$, $c_i \in C[0, T]$ such that*

$$(-1)^{i+1}f(t, x) \leq (-1)^{i+1}c_i(t), \quad (-1)^{i+1}(\bar{c}_i - \bar{e}) \leq 0$$

for all $t \in [0, T]$, $x \in [B_i, B_i + 2\|\eta_i\|_1]$ and all $i \in \{1, 2\}$,

$$B_2 + 2\|\eta_2\|_1 \leq B_1. \quad (2.6)$$

Then problem (2.1) possesses a solution x such that

$$B_2 \leq x(t) \leq B_1 + 2\|\eta_1\|_1$$

(η_1, η_2 are given by Proposition 2.1 and Proposition 2.2, respectively).

Theorem 2.2 *Assume that there are $B_1, B_2 \in \mathbb{R}$, $c_1, c_2 \in C[0, T]$ such that the assumptions of Theorem 2.1 are satisfied with*

$$B_1 + 2\|\eta_1\|_1 \leq B_2$$

instead of (2.6). Then problem (2.1) possesses a solution x such that

$$B_1 \leq x(t) \leq B_2 + 2\|\eta_2\|_1.$$

Now, we consider problem (1.1), i.e., $e \equiv 0$.

Proposition 2.3 *Assume that there are $A_1 \in \mathbb{R}$, $b_1 \in C[0, T]$ such that $\bar{b}_1 = 0$ and*

$$f(t, x) \geq b_1(t) \quad \text{for all } t \in [0, T] \text{ and } x \in [A_1, A_1 + 2\|b\|_1]. \quad (2.7)$$

Then there exists a lower solution α of (1.1) such that

$$A_1 \leq \alpha \leq A_1 + 2\|b\|_1 \quad \text{in } [0, T]. \quad (2.8)$$

Proof Problem (2.2) with $\delta(t) = b_1(t)$ on $[0, T]$ has a unique solution

$$x_\zeta(t) = \zeta - \int_0^t b_1(s) ds.$$

Since $-\|b_1\|_1 \leq -\int_0^t b_1(s) ds \leq \|b_1\|_1$, choosing $\zeta = A_1 + \|b_1\|_1$, we obtain

$$A_1 \leq x_\zeta(t) \leq A_1 + 2\|b_1\|_1.$$

Let us take $\alpha(t) = x_\zeta(t)$, which means that (2.8) holds. According to (2.7) this yields

$$0 = \alpha'(t) + b_1(t) \leq \alpha'(t) + f(t, \alpha) \quad \text{on } [0, T].$$

Furthermore, we have $\alpha(0) = \alpha(T)$, by Definition 1.1, the function α is a lower solution of (1.1). \square

The following assertion is due to Proposition 2.1 and its proof can be omitted.

Proposition 2.4 *Assume that there are $A_2 \in \mathbb{R}$, $b_2 \in C[0, T]$ such that $\bar{b}_2 = 0$ and*

$$f(t, x) \leq b_2(t) \quad \text{for all } t \in [0, T] \text{ and } x \in [A_2, A_2 + 2\|b_2\|_1]. \quad (2.9)$$

Then there exists an upper solution β of (1.1) such that

$$A_2 \leq \beta \leq A_2 + 2\|b_2\|_1 \quad \text{in } [0, T].$$

It is easy to see that the following results can be shown in a similar manner as in Theorem 2.1.

Theorem 2.3 *Assume that there are $A_i \in \mathbb{R}$, $b_i \in C[0, T]$ such that $\bar{b}_i = 0$ and*

$$(-1)^i f(t, x) \leq (-1)^i b_i(t)$$

for all $t \in [0, T]$, $x \in [A_i, A_i + 2\|b_i\|_1]$ and all $i \in \{1, 2\}$,

$$A_2 + 2\|b_2\|_1 \leq A_1. \quad (2.10)$$

Then problem (1.1) possesses a solution x such that

$$A_2 \leq x(t) \leq A_1 + 2\|b_1\|_1.$$

Theorem 2.4 *Assume that there are $A_1, A_2 \in \mathbb{R}$, $b_1, b_2 \in C[0, T]$ such that the assumptions of Theorem 2.3 are satisfied with*

$$A_1 + 2\|b_1\|_1 \leq A_2$$

instead of (2.10). Then problem (1.1) possesses a solution x such that

$$A_1 \leq x(t) \leq A_2 + 2\|b_2\|_1.$$

Problem (1.1) has been considered by several authors (see, e.g., [6, 9]). In [9], Peng gave the existence result as follows.

Theorem A Assume that there exists a positive number M such that $Mx - f(t, x) \geq 0$ for $t \in [0, T]$ and $x \geq 0$. If

- (i) $f_0 = \liminf_{x \rightarrow 0^+} \min_{t \in [0, T]} \frac{f(t, x)}{x} > 0$ and $f^\infty = \limsup_{x \rightarrow \infty} \max_{t \in [0, T]} \frac{f(t, x)}{x} < 0$; or
- (ii) $f_\infty = \liminf_{x \rightarrow \infty} \min_{t \in [0, T]} \frac{f(t, x)}{x} > 0$ and $f^0 = \limsup_{x \rightarrow 0^+} \max_{t \in [0, T]} \frac{f(t, x)}{x} < 0$.

Then problem (1.1) has at least one positive solution.

In this paper, we do not require f to satisfy (i) or (ii). Now we give an example.

Example 2.1 Consider the periodic boundary value problem

$$x' + \arctan(x - 1) = 0, \quad x(0) = x(1). \quad (2.11)$$

It is easy to see that

$$f_0 = \lim_{x \rightarrow 0^+} \frac{f(x)}{x} = -\infty \quad \text{and} \quad f_\infty = \lim_{x \rightarrow \infty} \frac{f(x)}{x} = 0.$$

So we can not obtain existence result by Theorem A. But if we choose $b_1(t) = \frac{\pi}{3}t - \frac{\pi}{6}$, compute that $\bar{b}_1 = \int_0^1 \frac{\pi}{3}s - \frac{\pi}{6} ds = 0$, and (2.7) holds for $A_1 = \frac{\sqrt{3}}{3} + 1$ and

$$A_1 + 2\|b_1\|_1 = \frac{\sqrt{3}}{3} + 1 + 2 \int_0^1 \left| \frac{\pi}{3}s - \frac{\pi}{6} \right| ds = \frac{\sqrt{3}}{3} + 1 + \frac{\pi}{6}.$$

By Proposition 2.3, problem (2.11) has a lower solution. In fact,

$$\alpha(t) = A_1 + \|b_1\|_1 - \int_0^t b_1(s) ds = -\frac{\pi}{6}t^2 + \frac{\pi}{6}t + \frac{\pi}{12} + \frac{\sqrt{3}}{3} + 1$$

is a lower solution.

According to Proposition 2.4, we choose $b_2(t) = \frac{\pi}{4}t - \frac{\pi}{8}$, $A_2 = 1 - \frac{\sqrt{3}}{3} - \frac{\pi}{8}$, compute that $A_2 + 2\|b_2\|_1 = 1 - \frac{\sqrt{3}}{3}$ and (2.9) hold. Then problem (2.11) has an upper solution

$$\beta(t) = A_2 + \|b_2\|_1 - \int_0^t b_2(s) ds = 1 - \frac{\sqrt{3}}{3} - \frac{\pi}{16} - \frac{\pi}{8}(t^2 - t).$$

Obviously, $A_2 + 2\|b_2\|_1 < A_1$. Thus, from Theorem 2.2, problem (2.11) has a solution $x(t)$ satisfying

$$0 < 1 - \frac{\sqrt{3}}{3} - \frac{\pi}{8} \leq \beta(t) \leq x(t) \leq \alpha(t) \leq \frac{\sqrt{3}}{3} + 1 + \frac{\pi}{6}, \quad \forall t \in [0, 1],$$

i.e., x is a positive solution of (2.11).

3 Periodic problem with strong singularity

We will consider the following singular equation with periodic conditions:

$$x'(t) + g(x) = e(t), \quad x(0) = x(T), \quad (3.1)$$

where $g \in C(0, \infty)$, $e \in C[0, T]$ and g has strong singularity at 0, i.e.,

$$\lim_{x \rightarrow 0^+} \int_x^1 g(\xi) d\xi = \infty. \quad (3.2)$$

Theorem 3.1 Assume that (3.2) is fulfilled, there exists $\tilde{A}_1 \in (1, \infty)$ such that

$$g(x) \leq \bar{e} \quad \text{for all } x \in [\tilde{A}_1, \tilde{B}_1],$$

where

$$\tilde{B}_1 - \tilde{A}_1 = 2\|\eta\|_1, \quad \text{and} \quad \eta(t) = \phi^-(t) - \phi^+(t) \frac{\Phi_-}{\Phi_+}, \quad \phi = \bar{e} - e.$$

Then problem (3.1) has a positive solution.

Proof If $g \in C(0, \infty)$ satisfies (3.2), then $\limsup_{x \rightarrow 0^+} g(x) = \infty$, which implies the existence of a sequence $\{\varepsilon_n\}_{n=1}^\infty \subset (0, 1)$ such that

$$g(\varepsilon_n) > 0 \quad \text{for all } n \in \mathbb{N}, \quad \lim_{n \rightarrow \infty} \varepsilon_n = 0, \quad \lim_{n \rightarrow \infty} g(\varepsilon_n) = \infty. \quad (3.3)$$

Let $e_M = \max_{t \in [0, T]} e(t)$. By (3.3), there exists N such that

$$g(\varepsilon_n) > e_M \quad \text{for all } n > N.$$

We choose $\varepsilon = \varepsilon_{N+1}$ such that $g(\varepsilon) > e_M$. Let $R \geq \tilde{B}_1$ be a constant. For the proof of Theorem 3.1, we deal with the auxiliary family of problems

$$x' + \tilde{g}(x) = e(t), \quad x(0) = x(T), \quad (3.4)$$

where

$$\tilde{g}(x) = \begin{cases} 0 & \text{if } x < 0, \\ g(\varepsilon) \frac{x}{\varepsilon} & \text{if } x \in [0, \varepsilon], \\ g(x) & \text{if } x \in [\varepsilon, R], \\ g(R) & \text{if } x \geq R. \end{cases} \quad (3.5)$$

Then $\tilde{g}(x)$ fulfils the assumptions of Proposition 2.1 with $c_1 = \bar{e}$. Thus, by Proposition 2.1, problem (3.4) has an upper solution β such that

$$\tilde{A}_1 \leq \beta \leq \tilde{B}_1.$$

Since $\varepsilon < \tilde{A}_1 \leq \tilde{B}_1 \leq R$, then $\varepsilon < \beta \leq R$ is a lower solution of (3.1).

It is easy to see that $\alpha = \varepsilon$ is a lower solution of (3.1). Thus, by Lemma 1.1, problem (3.1) has a positive solution x such that $0 < \varepsilon < x \leq \tilde{B}_1$. \square

Example 3.1 Consider the periodic boundary value problem with strong singularity

$$x' + \frac{1}{x} = e(t), \quad x(0) = x(T). \quad (3.6)$$

Let $g(x) = \frac{1}{x}$, $e(t) = t$. We can prove that $g(x)$ satisfies condition (3.2). Since $\bar{e} = \int_0^1 t \, dt = \frac{1}{2}$, then $\hat{\phi} = \bar{e} - e = \frac{1}{2} - t$.

We have

$$\eta(t) = \hat{\phi}^-(t) - \hat{\phi}^+(t) \frac{\hat{\Phi}^-}{\hat{\Phi}^+} = -\hat{\phi} = t - \frac{1}{2}, \quad \|\eta\|_1 = \frac{1}{4}.$$

We can choose $A_1 = 2$, then $B_1 = A_1 + 2\|\eta\|_1 = \frac{5}{2}$ and g satisfies

$$g(x) = \frac{1}{x} \leq \frac{1}{2} \quad \forall x \in \left[2, \frac{5}{2}\right].$$

By Theorem 3.1, problem (3.6) has a positive solution.

In fact, $\alpha = \frac{1}{2}$ is a lower solution of (3.6).

$$\beta(t) = \frac{11}{4} + \frac{1}{2}t^2 - \frac{1}{2}t$$

is an upper solution of (3.6).

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' contributions

RM completed the main study, carried out the results of this article. LZ drafted the manuscript, checked the proofs and verified the calculation. All the authors read and approved the final manuscript.

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