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The existence of a ground state solution for a class of fractional differential equation with p-Laplacian operator

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Abstract

In this paper, we apply the Nehari manifold method to study the fractional p-Laplacian differential equation

$$\begin{cases} {}_tD_T^{\alpha}\phi_p({}_0D_t^{\alpha}u(t)) = f(t,u(t)), & t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where ${}_0D_t^{\alpha}$, ${}_tD_T^{\alpha}$ are the left and right Riemann-Liouville fractional derivatives of order $0 \le \alpha < 1$, respectively. We prove the existence of a ground state solution for the boundary value problem.

MSC: 34A08; 35A15

Keywords: fractional differential equations; boundary value problem; Nehari manifold; ground state solution

1 Introduction

Fractional differential equations have played an important role in many fields such as engineering, science, electrical circuits, diffusion and applied mathematics. In the recent years, some authors have studied the fractional differential equation by using different methods, such as fixed point theorem, coincidence degree theory, critical point theory, *etc.* (see [1–12]).

By using the mountain pass theorem, Jiao and Zhou [13] studied the existence of solutions for the following boundary value problem:

$$\begin{cases} \frac{d}{dt}\left(\frac{1}{2}{}_0D_t^{-\beta}(u'(t)) + \frac{1}{2}{}_tD_T^{-\beta}(u'(t))\right) + \nabla F(t,u(t)) = 0, & t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $0 < \beta < 1$, ${}_0D_t^{-\beta}$, and ${}_tD_T^{-\beta}$ are the left and right fractional integrals of order β, respectively, $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$, and $\nabla F(t,x)$ is the gradient of F with respect to x.

The authors in [14–19] studied the existence and multiplicity of solutions for the related problems with the help of critical point theory. Furthermore, the author in [20] studied the Boundary value problem with fractional p-Laplacian operator by using the mountain pass theorem.

Motivated by the above results, we would like to investigate the ground state solution for the following fractional boundary value problem:

$$\begin{cases} {}_tD_T^\alpha \phi_p({}_0D_t^\alpha u(t)) = f(t,u(t)), \quad t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.1}$$

where ${}_0D_t^\alpha$, ${}_tD_T^\alpha$ are the left and right Riemann-Liouville fractional derivatives of order $0 \le \alpha < 1$ and $\phi_p(s) = |s|^{p-2}s$, $p > 1$. The technical tool is the method of Nehari manifold, see [21, 22]. It is worth mentioning that there are real applications of such equations when $p = 2$ in [23, 24].

This article is organized as follows. In Section 2, some preliminaries on the fractional calculus are presented. In Section 3, we set up the variational framework of problem (1.1) and give some necessary lemmas. Finally, Section 4 presents the main result and its proof.

2 Preliminaries on the fractional calculus

In this section, we will introduce some notations, definitions and preliminary facts on fractional calculus which are used throughout this paper.

Definition 2.1 (Left and right Riemann-Liouville fractional integrals) Let f be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order α for function f denoted by ${}_aD_t^{-\alpha}f(t)$ and ${}_tD_b^{-\alpha}f(t)$ function, respectively, are defined by

$$\begin{aligned} {}_aD_t^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds, \quad t \in [a,b], \alpha > 0, \\ {}_tD_T^{-\alpha}f(t) &= \frac{1}{\Gamma(\alpha)}\int_t^T (t-s)^{\alpha-1}f(s)\,ds, \quad t \in [a,b], \alpha > 0, \end{aligned}$$

provided that the right-hand side integral is pointwise defined on $[a,b]$.

Definition 2.2 (Left and right Riemann-Liouville fractional derivatives) Let f be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order α for a function f denoted by ${}_aD_t^\alpha f(t)$ and ${}_tD_b^\alpha f(t)$, respectively, are defined by

$$\begin{aligned} {}_aD_t^\alpha f(t) &= \frac{d^n}{dt^n}\,{}_aD_t^{\alpha-n}f(t) \\ &= \frac{1}{\Gamma(\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}f(s)\,ds, \quad t \in [a,b], \alpha > 0, \\ {}_tD_b^\alpha f(t) &= (-1)^n\frac{d^n}{dt^n}\,{}_tD_b^{\alpha-n}f(t) \\ &= \frac{(-1)^n}{\Gamma(\alpha)}\frac{d^n}{dt^n}\int_t^b (s-t)^{n-\alpha-1}f(s)\,ds, \quad t \in [a,b], \alpha > 0, \end{aligned}$$

provided that the right-hand side integral is pointwise defined on $[a,b]$.

Definition 2.3 (Left and right Caputo fractional derivatives) If $\alpha \in (n-1,n)$ and $f \in AC^n([a,b],\mathbb{R})$, then the left and right Caputo fractional derivatives of order α for function f denoted by ${}_a^C D_t^\alpha f(t)$ and ${}_t^C D_b^\alpha f(t)$ function, respectively, are defined by

$$\begin{aligned}
{}_a^C D_t^\alpha f(t) &= {}_a D_t^{\alpha-n} \frac{d^n}{dt^n} f(t) \\
&= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds, \quad t \in [a,b], \alpha > 0, \\
{}_t^C D_b^\alpha f(t) &= (-1)^n {}_t D_b^{\alpha-n} \frac{d^n}{dt^n} f(t) \\
&= \frac{(-1)^n}{\Gamma(\alpha)} \int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s)\,ds, \quad t \in [a,b], \alpha > 0,
\end{aligned}$$

respectively, where $t \in [a,b]$.

Lemma 2.1 ([25]) *The left and right Riemann-Liouville fractional integral operators have the property of a semigroup, i.e.*

$$\int_a^b \big[{}_a D_t^{-\alpha} f(t)\big] g(t)\,dt = \int_a^b \big[{}_t D_b^{-\alpha} g(t)\big] f(t)\,dt, \quad \alpha > 0,$$

provided that $f \in L^p([a,b],\mathbb{R})$, $g \in L^q([a,b],\mathbb{R})$, *and* $p \geq q$, $q \geq 1$, $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ *or* $p \neq 1$, $q \neq 1$, $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$.

Lemma 2.2 ([25]) *Assume that* $n-1 < \alpha < n$ *and* $f \in C^n[a,b]$. *Then*

$$\begin{aligned}
{}_a D_t^{-\alpha} \big({}_a^C D_t^\alpha f(t)\big) &= f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{j!} (t-a)^j, \\
{}_t D_b^{-\alpha} \big({}_t^C D_b^\alpha f(t)\big) &= f(t) - \sum_{j=0}^{n-1} \frac{(-1)^j f^{(j)}(b)}{j!} (b-t)^j
\end{aligned}$$

for $t \in [a,b]$.

Lemma 2.3 ([25]) *Assume that* $n-1 < \alpha < n$, *then*

$$\begin{aligned}
{}_a^C D_t^\alpha f(t) &= {}_a D_t^\alpha f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(a)}{\Gamma(j-\alpha+1)} (t-a)^{j-\alpha}, \quad t \in [a,b], \\
{}_t^C D_b^\alpha f(t) &= {}_t D_b^\alpha f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(b)}{\Gamma(j-\alpha+1)} (b-t)^{j-\alpha}, \quad t \in [a,b].
\end{aligned}$$

3 A variational setting

To apply critical point theory of the existence of solutions for boundary value problem (1.1), we shall state some basic notation and results, which will be used in the proof of our main results.

Throughout this paper, we assume that the following conditions are satisfied.

(H_1) $f \in C^1(\mathbb{R} \times \mathbb{R})$;
(H_2) $f(t,0) = 0 = \frac{\partial f}{\partial s}(t,0)$ for every $t \in \mathbb{R}$;

(H_3) for $t \in [0,T]$, $x \in \mathbb{R}$, one has

$$\limsup_{|x|\to 0} \frac{F(t,x)}{|x|^p} < \frac{(\Gamma(\alpha+1))^p}{pT^{\alpha p}};$$

here

$$F(t,s) = \int_0^s f(t,x)\,dx;$$

(H_4) there exist constants $\mu \in (0, 1/p)$, $M > 0$ such that

$$0 < F(t,x) \le \mu x f(t,x), \quad \forall t \in [0,T], x \in \mathbb{R} \text{ with } |x| \ge M;$$

(H_5) the map $t \to t^{-(p-1)} s f(x,ts)$ is increasing on $(0,+\infty)$, for every $x \in \mathbb{R}$ and $s \in \mathbb{R}$.

In order to establish a variational structure which enables us to transform the existence of solutions for boundary value problem (1.1) into the existence of critical points for the corresponding functional, it is necessary to construct an appropriate function space. In the following, we introduce some results from [13, 14].

Definition 3.1 (see [13, 14]) Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by

$$E_0^{\alpha,p} = \bigl\{u \in L^p\bigl([0,T],\mathbb{R}\bigr) \big| {}_0^cD_t^\alpha u \in L^p\bigl([0,T],\mathbb{R}\bigr), u(0) = u(T) = 0\bigr\}$$

with the norm

$$\|u\|_{\alpha,p} = \bigl(\|u\|_{L^p}^p + \bigl\|{}_0^cD_t^\alpha u\bigr\|_{L^p}^p\bigr)^{\frac{1}{p}}, \quad \forall u \in E_0^{\alpha,p}, \tag{3.1}$$

where $\|u\|_{L^p} = (\int_0^T |u(t)|^p\,dt)^{1/p}$ is the norm of $L^p([0,T],\mathbb{R})$.

Remark 3.1 For any $u \in E_0^{\alpha,p}$, noting the fact that $u(0) = u(T) = 0$, we have ${}_0^cD_t^\alpha u(t) = {}_0D_t^\alpha u(t)$, ${}_t^cD_T^\alpha u(t) = {}_tD_T^\alpha u(t)$, $t \in [0,T]$.

Lemma 3.1 (see [13, 14]) *Let $0 < \alpha \le 1$ and $1 < p < \infty$. The fractional derivative space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Lemma 3.2 (see [13, 14]) *Let $0 < \alpha \le 1$ and $1 < p < \infty$. For $u \in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p} \le \frac{T^\alpha}{\Gamma(\alpha+1)} \bigl\|{}_0^cD_t^\alpha u\bigr\|_{L^p}. \tag{3.2}$$

Moreover, if $\alpha > 1/p$ and $1/p + 1/q = 1$, then

$$\|u\|_\infty \le \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \bigl\|{}_0^cD_t^\alpha u\bigr\|_{L^p}, \tag{3.3}$$

where $\|u\|_\infty = \max_{t\in[0,T]} |u(t)|$ is the norm of $C([0,T],\mathbb{R})$.

Remark 3.2 According to (3.2), we know that the norm (3.1) is equivalent to the norm of the form

$$\|u\|_{\alpha,p} = \left\| {}_0^cD_t^\alpha u \right\|_{L^p}. \tag{3.4}$$

Hence, we can consider $E_0^{\alpha,p}$ with the norm (3.4) in the following analysis.

Lemma 3.3 (see [13, 14]) *Let* $0 < \alpha \le 1$ *and* $1 < p < \infty$. *Assume that* $\alpha > 1/p$ *and the sequence* $\{u_k\}$ *converges weakly to* u *in* $E_0^{\alpha,p}$, *i.e.* $u_k \rightharpoonup u$. *Then* $u_k \to u$ *in* $C([0,T],\mathbb{R})$, *i.e.*

$$\|u_k - u\|_\infty \to 0, \quad k \to \infty.$$

Now we give the definition of weak solutions of boundary value problem (1.1).

Definition 3.2 By a weak solution of boundary value problem (1.1), we mean that the function $u \in E_0^{\alpha,p}$ such that $f(\cdot,u(\cdot)) \in L^1([0,T],\mathbb{R})$ satisfies the following equation:

$$\int_0^T \left[{}_tD_T^\alpha \phi_p\left({}_0D_t^\alpha u(t)\right)\right]v(t)\,dt = \int_0^T f\left(t,u(t)\right)v(t)\,dt, \quad \forall v \in E_0^{\alpha,p}.$$

Then define the functional $I : E_0^{\alpha,p} \to \mathbb{R}$ by

$$I(u) = \frac{1}{p}\int_0^T \left|{}_0^cD_t^\alpha u(t)\right|^p dt - \int_0^T F\left(t,u(t)\right)dt, \tag{3.5}$$

where $F(t,s) = \int_0^s f(t,\tau)\,d\tau$.

Next we will establish a variational structure of boundary value problem (1.1) on $E_0^{\alpha,p}$. Also, we will show that the critical points of I are weak solutions of boundary value problem (1.1).

Remark 3.3 From Lemma 3.3, we can see that the functional $u \to \int_0^T F(t,u(t))\,dt$ is weakly continuous on $E_0^{\alpha,p}$. As the above functional is convex continuous and weakly continuous, we know that I is a weakly lower semi-continuous functional on $E_0^{\alpha,p}$ with $\alpha > 1/p$. Then following [26], we can see that $I \in C^1(E_0^{\alpha,p},\mathbb{R})$ and we have

$$I'(u)v = \int_0^T \phi_p\left({}_0^cD_t^\alpha u(t)\right){}_0^cD_t^\alpha v(t)\,dt - \int_0^T f\left(t,u(t)\right)v(t)\,dt, \quad \forall v \in E_0^{\alpha,p}. \tag{3.6}$$

Lemma 3.4 *Let* $0 < \alpha \le 1$ *and* I *be defined by* (3.5). *If* (H_1) *is satisfied and* $u \in E_0^{\alpha,p}$ *is a solution of the corresponding Euler equation* $I'(u) = 0$, *then* u *is a weak solution of boundary value problem* (1.1).

Proof For $u, v \in E_0^{\alpha,p}$, by Remark 3.1 and Definition 2.2, one has

$$\begin{aligned}
\int_0^T \left[{}_tD_T^\alpha \phi_p\left({}_0D_t^\alpha u(t)\right)\right]v(t)\,dt &= \int_0^T \left[{}_tD_T^\alpha \phi_p\left({}_0^cD_t^\alpha u(t)\right)\right]v(t)\,dt \\
&= -\int_0^T v(t)d\left[{}_tD_T^{\alpha-1}\phi_p\left({}_0^cD_t^\alpha u(t)\right)\right] \\
&= \int_0^T \left[{}_tD_T^{\alpha-1}\phi_p\left({}_0^cD_t^\alpha u(t)\right)\right]v'(t)\,dt.
\end{aligned}$$

Thus, from Lemma 2.1, we have

$$\begin{aligned}\int_0^T \left[{}_tD_T^\alpha \phi_p\left({}_0D_t^\alpha u(t)\right)\right]v(t)\,dt &= \int_0^T \phi_p\left({}_0^cD_t^\alpha u(t)\right){}_0D_t^{\alpha-1}v'(t)\,dt \\ &= \int_0^T \phi_p\left({}_0^cD_t^\alpha u(t)\right){}_0^cD_t^\alpha v(t)\,dt,\end{aligned}$$

which together with (3.6) and $I'(u) = 0$ shows that

$$0 = I'(u)v = \int_0^T \left[{}_tD_T^\alpha \phi_p\left({}_0D_t^\alpha u(t)\right)\right]v(t)\,dt - \int_0^T f\left(t,u(t)\right)v(t)\,dt$$

for any $v \in E_0^{\alpha,p}$. Hence, according to Definition 3.2, u is weak solution of boundary value problem (1.1). The proof is complete. □

4 Main result

In this section, we shall investigate the solvability of boundary value problem (1.1) with the aid of the Nehari manifold method.

There is one-to-one correspondence between the critical points of I and weak solutions of boundary value problem (1.1). Now, we define

$$\mathcal{N} = \left\{u \in E_0^{\alpha,p} \setminus \{0\} | I'(u)u = 0\right\}. \tag{4.1}$$

Then we know any non-zero critical point of I must be on $\mathcal{N}$. Define

$$G(u) = I'(u)u = \int_0^T \phi_p\left({}_0^cD_t^\alpha u(t)\right){}_0^cD_t^\alpha u(t)\,dt - \int_0^T f\left(t,u(t)\right)u(t)\,dt. \tag{4.2}$$

Lemma 4.1 *Assume the hypotheses* (H_1)-(H_5) *hold. If* $u \in \mathcal{N}$ *is a critical point of* $I|_{\mathcal{N}}$, *then* $I'(u) = 0$.

Proof For $u \in \mathcal{N}$, together with (H_5),

$$\begin{aligned}G'(u)u &= \int_0^T p\phi_p\left({}_0^cD_t^\alpha u(t)\right){}_0^cD_t^\alpha u(t) - \frac{\partial}{\partial u}f\left(t,u(t)\right)\cdot u^2(t) - f\left(t,u(t)\right)u(t)\,dt \\ &= \int_0^T pf\left(t,u(t)\right)u(t) - \frac{\partial}{\partial u}f\left(t,u(t)\right)\cdot u^2(t) - f\left(t,u(t)\right)u(t)\,dt \\ &= \int_0^T (p-1)f\left(t,u(t)\right)u(t) - \frac{\partial}{\partial u}f\left(t,u(t)\right)\cdot u^2(t)\,dt < 0.\end{aligned} \tag{4.3}$$

If $u \in \mathcal{N}$ is a critical point of $I|_{\mathcal{N}}$, then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$, such that $I'(u) = \lambda G'(u)$. Then we have

$$I'(u)u = \lambda G'(u)u = 0.$$

By (4.3), $G'(u)u \neq 0$, we have $\lambda = 0$. So we can see that $I'(u) = 0$. The proof is complete. □

Lemma 4.2 *Assume the hypotheses* (H_1)-(H_5) *hold. For any* $u \in E_0^{\alpha,p} \setminus \{0\}$, *there is a unique* $y = y(u)$ *such that* $y(u)u \in \mathcal{N}$ *and we have* $I(yu) = \max_{y\geq 0} I(yu) > 0$.

Proof First, we claim that there exist constants $\sigma > 0$, $\rho > 0$ such that $I(u) > 0$ for all $u \in B_\rho(0) \setminus \{0\}$ and $I(u) \geq \sigma$ for all $u \in \partial B_\rho(0)$. That is, 0 is a strict local minimizer of I. In fact, by (H_3) we can see that there exist $\varepsilon \in (0,1)$, $\delta > 0$ such that

$$F(t,x) \leq \frac{(1-\varepsilon)(\Gamma(\alpha+1))^p}{pT^{\alpha p}}|x|^p, \quad t \in [0,T], x \in \mathbb{R} \text{ with } |x| \leq \delta. \tag{4.4}$$

Let $\rho = \frac{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}}{T^{\alpha-1/p}}\delta > 0$ and $\sigma = \varepsilon\rho^p/p > 0$. Then, by Lemma 3.2, we have

$$\|u\|_\infty \leq \frac{T^{\alpha-\frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}}\|u\|_{\alpha,p} = \delta, \quad u \in E_0^{\alpha,p} \text{ with } \|u\|_{\alpha,p} = \rho,$$

which together with Lemma 3.2 and (4.4) implies that

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|_{\alpha,p}^p - \int_0^T F\bigl(t,u(t)\bigr)\,dt \\ &\geq \frac{1}{p}\|u\|_{\alpha,p}^p - \frac{(1-\varepsilon)(\Gamma(\alpha+1))^p}{pT^{\alpha p}}\int_0^T \bigl|u(t)\bigr|^p\,dt \\ &\geq \frac{1}{p}\|u\|_{\alpha,p}^p - \frac{1-\varepsilon}{p}\|u\|_{\alpha,p}^p \\ &= \frac{\varepsilon}{p}\|u\|_{\alpha,p}^p = \sigma, \quad \forall u \in E_0^{\alpha,p} \text{ with } \|u\|_{\alpha,p} = \rho. \end{aligned}$$

Next, we claim that $I(yu) \to -\infty$, as $y \to \infty$. In fact, by (H_4), there exists a constant $A > 0$ such that $F(t,u) \geq A|u|^{\frac{1}{\mu}}$ for $|u| \geq M$. On the other hand, we can see that there exists a constant B such that $F(t,u) \geq B$ for $|u| \leq M$. For any $u \in E_0^{\alpha,p} \setminus \{0\}$, $y \in \mathbb{R}^+$, noting that $\mu \in (0,1/p)$, we get

$$\begin{aligned} I(yu) &= \frac{1}{p}\|yu\|_{\alpha,p}^p - \int_0^T F\bigl(t,yu(t)\bigr)\,dt \\ &\leq \frac{y^p}{p}\|u\|_{\alpha,p}^p - A\int_0^T \bigl|yu(t)\bigr|^{\frac{1}{\mu}}\,dt - BT \\ &= \frac{y^p}{p}\|u\|_{\alpha,p}^p - Ay^{\frac{1}{\mu}}\|u\|_{L^{\frac{1}{\mu}}}^{\frac{1}{\mu}} - BT \\ &\to -\infty, \quad y \to \infty. \end{aligned}$$

Let $g(y) := I(yu)$ for $y > 0$. From what we have proved, there has to be at least one $y_u = y(u) > 0$ such that

$$g(y_u) = \max_{y\geq 0} g(y) = \max_{y \geq 0} I(yu) = I(y_u u).$$

We will prove $g(y)$ has a unique critical point for $y > 0$. Consider a critical point

$$\begin{aligned} g'(y) &= I'(yu)u \\ &= \frac{1}{y}\|yu\|_{\alpha,p}^p - \int_0^T f(t,yu)u\,dt \\ &= 0. \end{aligned}$$

Then together with (H_5), we have

$$\begin{aligned} g''(y) &= \frac{p-1}{y^2}\|yu\|_{\alpha,p}^p - \int_0^T \frac{\partial f(t,yu)}{\partial(yu)}u^2\,dt \\ &= (p-1)\int_0^T \frac{f(t,yu)u}{y}\,dt - \int_0^T \frac{\partial f(t,yu)}{\partial(yu)}u^2\,dt \\ &= \frac{1}{y^2}\left[(p-1)\int_0^T f(t,yu)yu\,dt - \int_0^T \frac{\partial f(t,yu)}{\partial(yu)}(yu)^2\,dt\right] \\ &< 0. \end{aligned}$$

So we know that if y is a critical point of g, then it should be a strict local maximum. This implies the uniqueness of the critical point. The proof is complete. □

Remark 4.1 From

$$g'(y) = I'(yu)u = \frac{1}{y}I'(yu)yu,$$

we see y is a critical point if $yu \in \mathcal{N}$. Define $m = \inf_{\mathcal{N}} I$. Then we can see that $m \geq \inf_{\partial B_\rho(0)} I \geq \sigma > 0$.

Lemma 4.3 *Assume the hypotheses* (H_1)-(H_5) *hold and* $m = \inf_{\mathcal{N}} I$. *Then there exists* $u \in \mathcal{N}$ *such that* $I(u) = m$.

Proof From Remark 3.3, we can see that the functional $u \to \int_0^T F(t,u(t))\,dt$ and I is a weakly lower semi-continuous functional on $E_0^{\alpha,p}$ with $\alpha > 1/p$. Similarly we can see that G is a weakly lower semi-continuous functional on $E_0^{\alpha,p}$ with $\alpha > 1/p$.

Since $F(t,x) - \mu x f(t,x)$ is continuous, there exists $c \in \mathbb{R}^+$ such that

$$F(t,x) \leq \mu x f(t,x) + c, \quad t \in [0,T], |x| \leq M.$$

Thus, together with (H_4), we get

$$F(t,x) \leq \mu x f(t,x) + c, \quad t \in [0,T], x \in \mathbb{R}. \tag{4.5}$$

Let $\{u_k\} \subset E_0^{\alpha,p}$ where

$$\left|I(u_k)\right| \leq K, \qquad I'(u_k) \to 0 \quad \text{as } k \to \infty.$$

According to (3.6), one has

$$I'(u_k)u_k = \|u_k\|_{\alpha,p}^p - \int_0^T f\bigl(t,u_k(t)\bigr)u_k(t)\,dt,$$

which together with (4.5) shows that

$$\begin{aligned} K &\geq I(u_k) \\ &= \frac{1}{p}\|u_k\|_{\alpha,p}^p - \int_0^T F\bigl(t,u_k(t)\bigr)\,dt \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{p}\|u_k\|_{\alpha,p}^p - \mu\int_0^T f\big(t,u_k(t)\big)u_k(t)\,dt - cT \\
&= \left(\frac{1}{p}-\mu\right)\|u_k\|_{\alpha,p}^p + \mu I'(u_k)u_k - cT \\
&\geq \left(\frac{1}{p}-\mu\right)\|u_k\|_{\alpha,p}^p - \mu\big\|I'(u_k)\big\|_{\alpha,q}\|u_k\|_{\alpha,p} - cT,
\end{aligned}$$

where q is a constant such that $1/p + 1/q = 1$. Since $I'(u_k) \to 0$, there exists $N_0 \in \mathbb{N}$ such that

$$K \geq \left(\frac{1}{p}-\mu\right)\|u_k\|_{\alpha,p}^p - \|u_k\|_{\alpha,p} - cT, \quad k > N_0.$$

It follows from $\mu \in (0, 1/p)$ that $\{u_k\}$ is bounded in $E_0^{\alpha,p}$. Since $E_0^{\alpha,p}$ is a reflexive space, going to a subsequence if necessary, we can assume that $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$. Hence we have

$$\begin{aligned}
\big(I'(u_k) - I'(u)\big)(u_k - u) &= I'(u_k)(u_k-u) - I'(u)(u_k-u) \\
&\leq \big\|I'(u_k)\big\|_{\alpha,q}\|u_k - u\|_{\alpha,p} - I'(u)(u_k-u) \\
&\to 0, \quad k \to \infty.
\end{aligned} \tag{4.6}$$

Moreover, by Lemma 3.3, we see that u_k is bounded in $C([0,T],\mathbb{R})$ and $\|u_k - u\|_\infty \to 0$ as $k \to \infty$. Then we get

$$\int_0^T \big(f\big(t,u_k(t)\big) - f\big(t,u(t)\big)\big)\big(u_k(t) - u(t)\big)\,dt \to 0, \quad k \to \infty. \tag{4.7}$$

Noting that

$$\begin{aligned}
&\big(I'(u_k) - I'(u)\big)(u_k - u) \\
&\quad= \int_0^T \big(\phi_p\big({}_0^cD_t^\alpha u_k(t)\big) - \phi_p\big({}_0^cD_t^\alpha u(t)\big)\big)\big({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\big)\,dt \\
&\qquad - \int_0^T \big(f\big(t,u_k(t)\big) - f\big(t,u(t)\big)\big)\big(u_k(t) - u(t)\big)\,dt,
\end{aligned}$$

then from (4.6) and (4.7), we have

$$\int_0^T \big(\phi_p\big({}_0^cD_t^\alpha u_k(t)\big) - \phi_p\big({}_0^cD_t^\alpha u(t)\big)\big)\big({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\big)\,dt \to 0 \tag{4.8}$$

as $k \to \infty$.

Following [27], we can see that there exist $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\int_0^T \big(\phi_p\big({}_0^cD_t^\alpha u_k(t)\big) - \phi_p\big({}_0^cD_t^\alpha u(t)\big)\big)\big({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\big)\,dt \\
&\quad\geq \begin{cases} c_1\int_0^T |{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)|^p\,dt, & p \geq 2, \\ c_2\int_0^T \frac{|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)|^2}{(|{}_0^cD_t^\alpha u_k(t)| + |{}_0^cD_t^\alpha u(t)|)^{2-p}}\,dt, & 1 < p < 2. \end{cases}
\end{aligned} \tag{4.9}$$

When $1 < p < 2$, one has

$$\begin{aligned}&\int_0^T \left|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\right|^p dt\\&\quad\le \left(\int_0^T \frac{|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)|^2}{(|{}_0^cD_t^\alpha u_k(t)| + |{}_0^cD_t^\alpha u(t)|)^{2-p}}\,dt\right)^{\frac{p}{2}}\\&\qquad\cdot\left(\int_0^T \left(\left|{}_0^cD_t^\alpha u_k(t)\right| + \left|{}_0^cD_t^\alpha u(t)\right|\right)^p dt\right)^{\frac{2-p}{2}}.\end{aligned}$$

Noting that $(s_1 + s_2)^\gamma \le 2^{\gamma-1}(s_1^\gamma + s_2^\gamma)$, where $s_1, s_2 \ge 0$, $\gamma \ge 1$ (see [28]), we have

$$\begin{aligned}&\int_0^T \left|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\right|^p dt\\&\quad\le c_3\left(\|u_k\|_{\alpha,p}^p + \|u\|_{\alpha,p}^p\right)^{\frac{2-p}{2}}\left(\int_0^T \frac{|{}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)|^2}{(|{}_0^cD_t^\alpha u_k(t)| + |{}_0^cD_t^\alpha u(t)|)^{2-p}}\,dt\right)^{\frac{p}{2}}\end{aligned}$$

with $c_3 = 2^{(p-1)(2-p)/2}$, which together with (4.9) implies that

$$\begin{aligned}&\int_0^T \left(\phi_p\left({}_0^cD_t^\alpha u_k(t)\right) - \phi_p\left({}_0^cD_t^\alpha u(t)\right)\right)\left({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\right) dt\\&\quad\ge c_2 c_3^{-\frac{2}{p}}\left(\|u_k\|_{\alpha,p}^p + \|u\|_{\alpha,p}^p\right)^{\frac{p-2}{p}}\|u_k - u\|_{\alpha,p}^p,\quad 1 < p < 2.\end{aligned}\tag{4.10}$$

When $p \ge 2$, by (4.9), we get

$$\begin{aligned}&\int_0^T \left(\phi_p\left({}_0^cD_t^\alpha u_k(t)\right) - \phi_p\left({}_0^cD_t^\alpha u(t)\right)\right)\left({}_0^cD_t^\alpha u_k(t) - {}_0^cD_t^\alpha u(t)\right) dt\\&\quad\ge c_1\|u_k - u\|_{\alpha,p}^p,\quad p \ge 2.\end{aligned}\tag{4.11}$$

It follows from (4.8), (4.10), and (4.11) that

$$\|u_k - u\|_{\alpha,p} \to 0,\quad k \to \infty.$$

Namely $\{u_k\}$ converges strongly to u in $E_0^{\alpha,p}$. Since G is weakly lower semi-continuous and $\{u_k\} \in \mathcal{N}$, we have

$$G(u) \le \varliminf_{k\to\infty} G(u_k) = 0$$

and $u \ne 0$. In fact, if $u = 0$, then $u_k \to 0$ in $E_0^{\alpha,p}$. From $G(u_k) = 0$, we get $\|u_k\|_{\alpha,p} \to 0$. This is a contradiction with $\{u_k\} \in \mathcal{N}$.

Then from Lemma 4.2, there exists a unique $y > 0$ such that $yu \in \mathcal{N}$. Together with the fact that I is weakly lower semi-continuous, we have

$$m \le I(yu) \le \varliminf_{k\to\infty} I(yu_k) \le \lim_{k\to\infty} I(yu_k) \le \lim_{k\to\infty} I(u_k) = m.$$

Then we see that m is obtained at $yu \in \mathcal{N}$. The proof is complete. □

Theorem 4.1 *Assume the hypotheses* (H_1)-(H_5) *hold, boundary value problem* (1.1) *has a weak solution such that* $I(u) = m$, *i.e. there exists a ground state solution of boundary value problem* (1.1).

Proof By using Lemmas 4.2 and 4.3, we can see that there exists $u \in \mathcal{N}$ such that $I(u) = m = \inf_{\mathcal{N}} I$. Then u is a critical point of $I|_{\mathcal{N}}$. From Lemma 4.1 we have $I'(u) = 0$. So boundary value problem (1.1) has a weak solution such that $I(u) = m$. The proof is complete. □

5 An example

In this section, we will give an example to illustrate our main result.

Example 5.1 Consider the following boundary value problem for a fractional p-Laplacian equation:

$$\begin{cases} {}_tD_T^{\frac{1}{2}}\phi_3({}_0D_t^{\frac{1}{2}}u(t)) = \cos^2 t \cdot u(t)|u(t)|^3, & t \in [0, T], \\ u(0) = u(T) = 0. \end{cases} \tag{5.1}$$

Corresponding to boundary value problem (1.1), we see that $p = 3$, $\alpha = \frac{1}{2}$, and

$$f(t,u) = \cos^2 t \cdot u|u|^3.$$

Then we have

$$F(t,u) = \int_0^u f(t,s)\,ds = \int_0^u \cos^2 t \cdot s|s|^3\,ds.$$

Obviously, (H_1) and (H_2) hold. Furthermore,

$$\limsup_{|u|\to 0} \frac{F(t,u)}{|u|^3} = 0 < \frac{(\Gamma(\frac{1}{2}+1))^3}{3T^{\frac{1}{2}3}}.$$

Then we see that (H_3) holds. Choose $\mu = \frac{1}{4}$ and $M = 1$. By a simple calculation, we can see that

$$0 < F(t,u) \le \frac{1}{4} u f(t,u), \quad \forall t \in [0,T], u \in \mathbb{R} \text{ with } |u| \ge 1.$$

So, (H_4) holds. On the other hand, $t^{-(p-1)} s f(x, ts) = \cos^2 x \cdot t^2 s^2 |s|^3$ is increasing on $t \in (0, +\infty)$, for every $x \in \mathbb{R}$ and $s \in \mathbb{R}$. So, we see that (H_5) holds.

Then, boundary value problem (5.1) satisfies all assumptions of Theorem 4.1. Hence, there exists a ground state solution of boundary value problem (5.1).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in this article. All authors read and approved the final manuscript.

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