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Multiple positive solutions for nonlocal boundary value problems of singular fractional differential equations

Xingqiu Zhang^{1,2} and Qiuyan Zhong^{1*}

*Correspondence:
zhqy197308@163.com
¹Department of Information Engineering, Jining Medical College, Jining, Shandong 272067, P.R. China
Full list of author information is available at the end of the article

Abstract

We consider the existence of multiple positive solutions for the following nonlinear fractional differential equations of nonlocal boundary value problems:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, & D_{0+}^{\beta} u(0) = 0, & D_{0+}^{\beta} u(1) = \sum_{i=1}^{\infty} \xi_i D_{0+}^{\beta} u(\eta_i), \end{cases}$$

where $2 < \alpha \leq 3$, $1 \leq \beta \leq 2$, $\alpha - \beta \geq 1$, $0 < \xi_i, \eta_i < 1$ with $\sum_{i=1}^{\infty} \xi_i \eta_i^{\alpha-\beta-1} < 1$. Existence result of at least two positive solutions is given via fixed point theorem on cones. The nonlinearity f may be singular both on the time and the space variables.

MSC: 26A33; 34B15; 34B18

Keywords: fractional differential equations; nonlocal boundary value problem; singularity; multiple positive solutions

1 Introduction

The purpose of this paper is to investigate the multiplicity of positive solutions for the following nonlocal boundary value problems of singular fractional differential equations:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, & D_{0+}^{\beta} u(0) = 0, & D_{0+}^{\beta} u(1) = \sum_{i=1}^{\infty} \xi_i D_{0+}^{\beta} u(\eta_i), \end{cases} \quad (1)$$

where $2 < \alpha \leq 3$, $1 \leq \beta \leq 2$, $\alpha - \beta \geq 1$, $0 < \xi_i, \eta_i < 1$ with $\sum_{i=1}^{\infty} \xi_i \eta_i^{\alpha-\beta-1} < 1$, $f \in C(J \times \mathbb{R}_{++}, \mathbb{R}_+)$, $J = (0, 1)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_{++} = (0, +\infty)$, D_{0+}^{α} is the standard Riemann-Liouville's fractional derivative of order α . The nonlinearity f permits singularities at $t = 0, 1$ and $u = 0$. A function $u \in C[0, 1]$ is said to be a positive solution of BVP (1) if $u(t) > 0$ on $(0, 1)$ and u satisfies (1) on $[0, 1]$.

Recently, much attention has been paid on the study of nonlocal boundary value problems of fractional differential equations; see [1–16] and [17–25]. By virtue of the contraction map principle and the fixed point index theory, Bai [1] investigated the existence and uniqueness of positive solutions for the following fractional differential equation:

$$D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, 1 < \alpha \leq 2, \quad (A)$$

subject to three point boundary value conditions

$$u(0) = 0, u(1) = \mu u(\xi),$$

where $0 < \mu \xi^{\alpha-1} < 1$, $0 \leq \mu \leq 1$, $0 < \xi < 1$, f is continuous on $[0, 1] \times [0, +\infty)$. When $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ satisfies Carathéodory type conditions, by using some fixed point theorems, like Leray-Schauder nonlinear alternative and a mixed monotone method, Li *et al.* [2], Xu *et al.* [3, 4] obtained the existence and multiplicity results of positive solutions for the fractional differential equation (A) with fractional derivative in boundary conditions

$$u(0) = 0, \quad D_{0+}^{\beta} u(1) = \mu D_{0+}^{\beta} u(\xi), \quad 0 \leq \beta \leq 1.$$

In 2011, Lv [5] and Yang *et al.* [6] discussed the existence of minimal and maximal and the uniqueness of positive solutions for fractional differential equation (A) under multi-point boundary value conditions,

$$u(0) = 0, \quad D_{0+}^{\beta} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\beta} u(\eta_i), \quad 0 \leq \beta \leq 1.$$

Motivated by the above papers, when $2 < \alpha \leq 3$ and f is continuous, Li *et al.* [7] obtained the existence results of at least one and unique solutions for fractional differential equation (A) subject to more general multi-point boundary value conditions

$$u(0) = 0, \quad D_{0+}^{\beta} u(0) = 0, \quad D_{0+}^{\beta} u(1) = \sum_{i=1}^{m-2} \xi_i D_{0+}^{\beta} u(\eta_i).$$

The tools to obtain the main results are the nonlinear alternative of the Leray-Schauder and the Banach contraction mapping principle.

Compared with the existing literature, this paper has the following two new features. First, different from [7], infinite-point boundary value conditions are considered in this paper. At the same time, the nonlinearity f in this paper permits singularities with respect to both the time and the space variables which is seldom considered at present. Second, the purpose of this paper is to investigate the existence of multiple positive solutions for BVP (1). As to multiple positive solutions, it is worth pointing out that conditions imposed on f are different from that in [4]. To achieve this goal, first we convert the expression of the unique solution into an integral form and then get the Green function BVP (1). After further discussion of the properties of the Green function, a suitable cone is constructed to obtain the main result in this paper by means of the Guo-Krasnoselskii fixed point theorem.

2 Preliminaries and several lemmas

Definitions and useful lemmas from fractional calculus theory can be found in the recent literature [26–28], we omit them here.

Lemma 1 ([1]) *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_N t^{\alpha-N}$$

for some $C_i \in R, i = 1, 2, \dots, N$, where N is the smallest integer greater than or equal to α .

Lemma 2 *Let $y \in L^1[0,1]$ and $2 < \alpha \leq 3$, the unique solution of*

$$\begin{cases} D_{0+}^\alpha u(t) + y(t) = 0, \\ u(0) = 0, \quad D_{0+}^\beta u(0) = 0, \quad D_{0+}^\beta u(1) = \sum_{i=1}^\infty \xi_i D_{0+}^\beta u(\eta_i), \end{cases} \tag{2}$$

is

$$u(t) = \int_0^1 G(t,s)y(s) ds,$$

where

$$G(t,s) = \frac{1}{p(0)\Gamma(\alpha)} \begin{cases} p(s)(1-s)^{\alpha-\beta-1}t^{\alpha-1} - p(0)(t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ p(s)(1-s)^{\alpha-\beta-1}t^{\alpha-1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{3}$$

here $p(s) = 1 - \sum_{s \leq \eta_i} \xi_i (\frac{\eta_i - s}{1-s})^{\alpha-\beta-1}$.

Proof By Lemma 1, we get

$$u(t) = -I_{0+}^\alpha y(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + C_3 t^{\alpha-3}. \tag{4}$$

It follows from the condition $u(0) = 0$ that $C_3 = 0$. Considering the relation $D_{0+}^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}$, we have

$$\begin{aligned} D_{0+}^\beta u(t) &= D_{0+}^\beta \left(- \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} y(s) ds \right) \\ &\quad + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} + C_2 \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)} t^{\alpha-\beta-2}. \end{aligned}$$

Since $2 < \alpha \leq 3, 1 \leq \beta \leq 2$, and $\alpha \geq \beta + 1$, we have $-1 \leq \alpha - \beta - 2 \leq 0$. Thus, $C_2 = 0$. As deduced by the boundary value condition $D_{0+}^\beta u(1) = \sum_{i=1}^\infty \xi_i D_{0+}^\beta u(\eta_i)$, we have

$$\begin{aligned} & - \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s) ds + C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \\ &= - \sum_{i=1}^\infty \frac{1}{\Gamma(\alpha-\beta)} \xi_i \int_0^{\eta_i} (\eta_i - s)^{\alpha-\beta-1} y(s) ds + C_1 \sum_{i=1}^\infty \xi_i \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \eta_i^{\alpha-\beta-1}, \end{aligned}$$

which implies that

$$\begin{aligned} C_1 &= \int_0^1 \frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1 - \sum_{i=1}^\infty \xi_i \eta_i^{\alpha-\beta-1})} y(s) ds - \sum_{i=1}^\infty \xi_i \int_0^{\eta_i} \frac{(\eta_i - s)^{\alpha-\beta-1}}{\Gamma(\alpha)(1 - \sum_{i=1}^\infty \xi_i \eta_i^{\alpha-\beta-1})} y(s) ds \\ &= \int_0^1 \frac{(1-s)^{\alpha-\beta-1} p(s)}{\Gamma(\alpha) p(0)} y(s) ds, \end{aligned}$$

where $p(s) = 1 - \sum_{s \leq \eta_i} \xi_i \left(\frac{\eta_i - s}{1-s}\right)^{\alpha - \beta - 1}$. Thus, we have

$$\begin{aligned} u(t) &= - \int_0^t \frac{1}{\Gamma(\alpha)} (t-s)^{\alpha-1} y(s) \, ds + \int_0^1 \frac{(1-s)^{\alpha-\beta-1} p(s)}{\Gamma(\alpha)p(0)} y(s) \, ds \cdot t^{\alpha-1} \\ &= \int_0^t \frac{p(s)(1-s)^{\alpha-\beta-1} t^{\alpha-1} - p(0)(t-s)^{\alpha-1}}{\Gamma(\alpha)p(0)} y(s) \, ds \\ &\quad + \int_t^1 \frac{p(s)(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{\Gamma(\alpha)p(0)} y(s) \, ds \\ &= \int_0^1 G(t,s) \, ds. \end{aligned}$$

□

Lemma 3 *The function $p(s) > 0, s \in [0, 1]$, and p is nondecreasing on $[0, 1]$.*

Proof By direct computation, we have

$$\begin{aligned} p'(s) &= \sum_{s \leq \eta_i} \xi_i (\alpha - \beta - 1) (\eta_i - s)^{\alpha - \beta - 2} (1 - s)^{\beta + 1 - \alpha} \\ &\quad + \xi_i (\beta + 1 - \alpha) (\eta_i - s)^{\alpha - \beta - 1} (1 - s)^{\beta - \alpha} \\ &= \sum_{s \leq \eta_i} \xi_i (\eta_i - s)^{\alpha - \beta - 2} (1 - s)^{\beta - \alpha} [(\alpha - \beta - 1)(1 - s) + (\beta + 1 - \alpha)(\eta_i - s)] \\ &= \sum_{s \leq \eta_i} \xi_i (\eta_i - s)^{\alpha - \beta - 2} (1 - s)^{\beta - \alpha} (\alpha - \beta - 1)(1 - \eta_i) \geq 0. \end{aligned}$$

Then we have p is a nondecreasing function on $[0, 1]$, which implies that $p(s) \geq p(0) = \sum_{i=1}^{\infty} \xi_i \eta_i^{\alpha - \beta - 1} > 0, s \in [0, 1]$. □

Lemma 4 *The function $G(t, s)$ defined by (3) admits the following properties:*

- (i) $G(t, s) > 0, \frac{\partial}{\partial t} G(t, s) > 0, 0 < t, s < 1;$
- (ii) $\max_{t \in [0, 1]} G(t, s) = G(1, s) = \frac{1}{p(0)\Gamma(\alpha)} [p(s)(1-s)^{\alpha-\beta-1} - p(0)(1-s)^{\alpha-1}], 0 \leq s \leq 1;$
- (iii) $G(t, s) \geq t^{\alpha-1} G(1, s), 0 \leq t, s \leq 1.$

Proof (i) For $0 < s \leq t < 1$, noticing that $2 < \alpha \leq 3$, by Lemma 3, is easy to see that

$$\begin{aligned} G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} [p(s)(1-s)^{\alpha-\beta-1} t^{\alpha-1} - p(0)(t-s)^{\alpha-1}] \\ &\geq \frac{1}{p(0)\Gamma(\alpha)} [p(0)(1-s)^{\alpha-1} t^{\alpha-1} - p(0)(t-s)^{\alpha-1}] \\ &= \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \left[(1-s)^{\alpha-1} - \left(1 - \frac{s}{t}\right)^{\alpha-1} \right] \geq 0. \end{aligned}$$

It is clear that for $0 < t \leq s < 1, G(t, s) > 0$.

By direct computation, we have

$$\frac{\partial}{\partial t} G(t, s) = \frac{1}{p(0)\Gamma(\alpha)} \begin{cases} (\alpha - 1)p(s)(1-s)^{\alpha-\beta-1} t^{\alpha-2} \\ \quad - (\alpha - 1)p(0)(t-s)^{\alpha-2}, & 0 \leq s \leq t \leq 1, \\ (\alpha - 1)p(s)(1-s)^{\alpha-\beta-1} t^{\alpha-2}, & 0 \leq t \leq s \leq 1. \end{cases}$$

It is clear that $\frac{\partial}{\partial t} G(t, s)$ is continuous on $[0, 1] \times [0, 1]$. For $0 < s \leq t < 1$, noticing that $1 \leq \beta \leq 2$, we get

$$\begin{aligned} \frac{\partial}{\partial t} G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} [(\alpha - 1)p(s)(1 - s)^{\alpha-\beta-1}t^{\alpha-2} - (\alpha - 1)p(0)(t - s)^{\alpha-2}] \\ &= \frac{(\alpha - 1)}{p(0)\Gamma(\alpha)} t^{\alpha-2} \left[p(s)(1 - s)^{\alpha-\beta-1} - p(0) \left(1 - \frac{s}{t}\right)^{\alpha-2} \right] \\ &\geq \frac{(\alpha - 1)p(s)}{p(0)\Gamma(\alpha)} t^{\alpha-2} \left[(1 - s)^{\alpha-2} - \left(1 - \frac{s}{t}\right)^{\alpha-2} \right] \geq 0. \end{aligned}$$

It is clear that for $0 < t \leq s < 1$, $\frac{\partial}{\partial t} G(t, s) > 0$.

(ii) By (i), we know that $G(t, s)$ is increasing with respect to t , thus we have $\max_{t \in [0, 1]} G(t, s) = G(1, s) = \frac{1}{p(0)\Gamma(\alpha)} [p(s)(1 - s)^{\alpha-\beta-1} - p(0)(1 - s)^{\alpha-1}]$, $0 \leq s \leq 1$.

(iii) For $0 \leq s \leq t \leq 1$, we get

$$\begin{aligned} G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} [t^{\alpha-1}p(s)(1 - s)^{\alpha-\beta-1} - p(0)(t - s)^{\alpha-1}] \\ &= \frac{1}{p(0)\Gamma(\alpha)} t^{\alpha-1} \left[p(s)(1 - s)^{\alpha-\beta-1} - p(0) \left(1 - \frac{s}{t}\right)^{\alpha-1} \right] \\ &\geq \frac{1}{p(0)\Gamma(\alpha)} t^{\alpha-1} [p(s)(1 - s)^{\alpha-\beta-1} - p(0)(1 - s)^{\alpha-1}] \\ &= t^{\alpha-1} G(1, s). \end{aligned}$$

For $0 \leq t \leq s \leq 1$, we have

$$\begin{aligned} G(t, s) &= \frac{1}{p(0)\Gamma(\alpha)} t^{\alpha-1} p(s)(1 - s)^{\alpha-\beta-1} \\ &= t^{\alpha-1} \cdot \frac{1}{p(0)\Gamma(\alpha)} p(s)(1 - s)^{\alpha-\beta-1} \\ &\geq t^{\alpha-1} G(1, s). \end{aligned}$$

□

We make the following assumptions:

(H₁) $f : (J \times \mathbb{R}_{++}, \mathbb{R}_+)$ is continuous.

(H₂) There exist $a, b \in C(J, \mathbb{R}_+)$, $g \in C(\mathbb{R}_{++}, \mathbb{R}_+)$ such that

$$f(t, u) \leq a(t)g(u) + b(t), \quad \forall t \in J, u \in \mathbb{R}_{++}$$

and

$$a_r^* = \int_0^1 a(t)g_r(t) dt < +\infty$$

for any $r > 0$, where

$$g_r(t) = \max \{g(u) : t^{\alpha-1}r \leq u \leq r\}$$

and

$$b^* = \int_0^1 b(t) dt < +\infty.$$

(H₃) There exists $c \in C(J, \mathbb{R}_+)$ such that

$$\frac{f(t, u)}{c(t)u} \rightarrow +\infty \text{ as } u \rightarrow +\infty$$

uniformly for $t \in J$, and

$$c^* = \int_0^1 c(t) dt < +\infty.$$

(H₄) There exists $d \in C(J, \mathbb{R}_+)$ such that

$$\frac{f(t, u)}{d(t)} \rightarrow +\infty \text{ as } u \rightarrow 0^+$$

uniformly for $t \in J$, and

$$d^* = \int_0^1 d(t) dt < +\infty.$$

Let $E = C[0,1]$ be the Banach space equipped with the maximum norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$ and let P be the cone of nonnegative functions in $C[0,1]$ with the following form:

$$P = \{u \in E | u(t) \geq t^{\alpha-1} \|u\|, t \in [0,1]\}.$$

Denote $P_+ = \{u \in P : \|u\| > 0\}$ and $P_{mn} = \{u \in P, m \leq \|u\| \leq n\}$ for any $n > m > 0$.

Define the operator T as follows:

$$(Tu)(t) = \int_0^1 G(t, s) f(s, u(s)) ds, \quad 0 \leq t \leq 1. \tag{5}$$

Clearly, $T : P \setminus \{0\} \rightarrow C[0,1]$.

Lemma 5 *Suppose that (H₁) and (H₂) hold, then for any $j > i > 0$, $T : P_{ij} \rightarrow P$ is completely continuous.*

Proof For any $u \in P_{ij}$, then $i \leq \|u\| \leq j$. By the definition of cone P , we have

$$it^{\alpha-1} \leq u(t) \leq j, \quad \forall t \in [0,1]. \tag{6}$$

It is not difficult to see that condition (H₂) implies that

$$a_{ij}^* = \int_0^1 a(t) g_{ij}(t) dt < +\infty \tag{7}$$

for any $j > i > 0$, where

$$g_{ij}(t) = \max\{g(u) : it^{\alpha-1} \leq u \leq j\}. \tag{8}$$

By (H_1) , (H_2) , (3), and Lemma 4(ii), we get

$$f(t, u(t)) \leq a(t)g_{ij}(t) + b(t), \quad \forall t \in J \tag{9}$$

and

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t, s)f(s, u(s)) \, ds \\ &\leq \frac{1}{p(0)\Gamma(\alpha)} \int_0^1 [p(s)(1-s)^{\alpha-\beta-1} - p(0)(1-s)^{\alpha-1}]f(s, u(s)) \, ds \\ &\leq \frac{1}{p(0)\Gamma(\alpha)} \int_0^1 p(s)(1-s)^{\alpha-\beta-1}f(s, u(s)) \, ds \\ &\leq \frac{1}{p(0)\Gamma(\alpha)} \int_0^1 [a(s)g_{ij}(s) + b(s)] \, ds \\ &\leq \frac{1}{p(0)\Gamma(\alpha)}(a_{ij}^* + b^*), \quad \forall t \in [0, 1], \end{aligned} \tag{10}$$

which implies that T is well defined. By Lemma 4(iii), we have

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t, s)f(s, u(s)) \, ds \\ &\leq \int_0^1 G(1, s)f(s, u(s)) \, ds, \quad \forall t \in [0, 1] \end{aligned} \tag{11}$$

and

$$\begin{aligned} (Tu)(t) &= \int_0^1 G(t, s)f(s, u(s)) \, ds \\ &\geq t^{\alpha-1} \int_0^1 G(1, s)f(s, u(s)) \, ds, \quad \forall t \in [0, 1], \end{aligned} \tag{12}$$

which means that T maps P_{ij} into P .

Next, we are in a position to show that T is completely continuous. Let $u_n, \bar{u} \in P_{ij}$, $\|u_n - \bar{u}\| \rightarrow 0$ ($n \rightarrow \infty$), then $\lim_{n \rightarrow \infty} u_n(t) = \bar{u}(t)$, $t \in [0, 1]$. Let

$$\begin{aligned} (T_1u)(t) &= f(t, u(t)), \quad 0 < t < 1, u \in P_{ij}, \\ (T_2u)(t) &= \int_0^1 G(t, s)u(s) \, ds, \quad 0 < t < 1, u \in L^1[0, 1]. \end{aligned}$$

By (H_1) ,

$$\lim_{n \rightarrow \infty} f(t, u_n(t)) = f(t, \bar{u}(t)), \quad 0 < t < 1. \tag{13}$$

Similar to (6) and (9), for $u_n, \bar{u} \in P_{ij}$, we have

$$f(t, u_n(t)) \leq a(t)g_{ij}(t) + b(t), \quad f(t, \bar{u}(t)) \leq a(t)g_{ij}(t) + b(t), \quad \forall t \in J.$$

Thus, we have

$$|f(t, u_n(t)) - f(t, \bar{u}(t))| \leq 2[a(t)g_{ij}(t) + b(t)] = \sigma(t) \in L^1[0, 1]. \tag{14}$$

It follows from (13), (14), and the Lebesgue dominated convergence theorem that $\lim_{n \rightarrow \infty} \int_0^1 |(T_1 u_n)(t) - (T_1 \bar{u})(t)| dt = 0$, which implies that $T_1 : P_{ij} \rightarrow L^1[0, 1]$ is continuous. By the Arzela-Ascoli theorem, we know that $T_2 : L^1[0, 1] \rightarrow C[0, 1]$ is completely continuous. As a consequence, $T = T_2 \circ T_1 : P_{ij} \rightarrow C[0, 1]$ is completely continuous. \square

In order to prove the main theorem, we state the following Guo-Krasnoselskii fixed point theorem.

Lemma 6 ([29]) *Let Ω_1 and Ω_2 be two bounded open sets in Banach space E such that $\theta \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$, $A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow P$ a completely continuous operator, where θ denotes the zero element of E and P a cone of E . Suppose that one of the two conditions holds:*

- (i) $\|Au\| \leq \|u\|, \forall u \in P \cap \partial\Omega_1; \|Au\| \geq \|u\|, \forall u \in P \cap \partial\Omega_2;$
- (ii) $\|Au\| \geq \|u\|, \forall u \in P \cap \partial\Omega_1; \|Au\| \leq \|u\|, \forall u \in P \cap \partial\Omega_2.$

Then A has a fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Main result

Theorem 1 *Let conditions (H₁)-(H₄) be satisfied. Assume in addition that there exists $r > 0$ such that*

$$\frac{1}{p(0)\Gamma(\alpha)}(a_r^* + b^*) < r, \tag{15}$$

where a_r^ and b^* are defined in condition (H₂). Then the boundary value problem (1) has at least two positive solutions u^* and u^{**} with $0 < \|u^*\| < r < \|u^{**}\|$.*

Proof By Lemma 5, the operator T defined by (5) is completely continuous from P_{mn} into P for any $n > m > 0$. We need only to prove that T has two fixed points u^* and $u^{**} \in P_+$ with $0 < \|u^*\| < r < \|u^{**}\|$.

By condition (H₃), there exists $r_1 > 0$ such that

$$f(t, u) \geq 2^{\alpha-1} \left(\int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) c(s) ds \right)^{-1} c(t)u, \quad \forall t \in J, u \geq r_1. \tag{16}$$

Choose

$$r_2 > \max\{2^{\alpha-1}r_1, r\}. \tag{17}$$

For $u \in P, \|u\| = r_2$, we have, by the construction of cone P ,

$$u(t) \geq t^{\alpha-1}r_2 \geq \left(\frac{1}{2}\right)^{\alpha-1} r_2 > r_1, \quad \forall t \in \left[\frac{1}{2}, 1\right]. \tag{18}$$

So, we get from (16) and (18)

$$\begin{aligned}
 (Tu)\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) f(s, u(s)) \, ds \\
 &\geq 2^{\alpha-1} \left(\int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) c(s) \, ds\right)^{-1} \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) c(s) u(s) \, ds \\
 &\geq 2^{\alpha-1} \left(\int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) c(s) \, ds\right)^{-1} \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) c(s) \, ds \cdot \left(\frac{1}{2}\right)^{\alpha-1} r_2 = r_2.
 \end{aligned} \tag{19}$$

Therefore,

$$\|Tu\| = \max_{t \in [0,1]} \|(Tu)(t)\| \geq \left\| (Tu)\left(\frac{1}{2}\right) \right\| \geq r_2 = \|u\|, \quad \forall u \in P, \|u\| = r_2. \tag{20}$$

By condition (H₄), there exists $r_3 > 0$ such that

$$f(t, u) \geq \left(\int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) d(s) \, ds\right)^{-1} d(t)r, \quad \forall t \in J, 0 < u < r_3. \tag{21}$$

Choose

$$0 < r_4 < \min\{r_3, r\}. \tag{22}$$

For $u \in P$, $\|u\| = r_4$, we have

$$r_3 > r_4 = \|u\| \geq r_4 t^{\alpha-1} > 0, \quad \forall t \in J.$$

So, we get

$$\begin{aligned}
 (Tu)\left(\frac{1}{2}\right) &= \int_0^1 G\left(\frac{1}{2}, s\right) f(s, u(s)) \, ds \\
 &\geq \left(\int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) d(s) \, ds\right)^{-1} \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) d(s) r \, ds \\
 &\geq \left(\int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) d(s) \, ds\right)^{-1} \int_{\frac{1}{2}}^1 G\left(\frac{1}{2}, s\right) d(s) \, ds \cdot r = r > r_4.
 \end{aligned} \tag{23}$$

Therefore,

$$\|Tu\| = \max_{t \in [0,1]} \|(Tu)(t)\| \geq \left\| (Tu)\left(\frac{1}{2}\right) \right\| > r_4 = \|u\|, \quad \forall u \in P, \|u\| = r_4. \tag{24}$$

On the other hand, for $u \in P$, $\|u\| = r$, similar to (10), by (H₂), Lemma 3(iii) and (15), we get

$$\begin{aligned}
 (Tu)(t) &= \int_0^1 G(t, s) f(s, u(s)) \, ds \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \int_0^1 [p(s)(1-s)^{\alpha-\beta-1} - p(0)(1-s)^{\alpha-1}] f(s, u(s)) \, ds
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \int_0^1 p(s)(1-s)^{\alpha-\beta-1} f(s, u(s)) \, ds \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} f(s, u(s)) \, ds \\
 &\leq \frac{1}{p(0)\Gamma(\alpha)} (a_r^* + b^*) < r, \quad \forall t \in [0, 1].
 \end{aligned}
 \tag{25}$$

Thus, from (25), we get

$$\|Tu\| < \|u\|, \quad \forall u \in P, \|u\| = r.
 \tag{26}$$

We know from (20), (24), (26), and Lemma 6 that T has two fixed points $u^*, u^{**} \in P_{r_4 r_2}$ such that $0 < r_4 < \|u^*\| < r < \|u^{**}\| \leq r_2$. □

4 An example

Example 1 Consider the following infinite-point boundary value problems:

$$\begin{cases}
 D_{0+}^{\frac{5}{2}} u(t) + \frac{1}{15\sqrt[3]{1-t}}(u^2 + \frac{1}{6\sqrt{u}}) + \frac{1}{20\sqrt[3]{t(1-t)}} = 0, & 0 < t < 1, \\
 u(0) = 0, \quad D_{0+}^{\frac{5}{4}} u(0) = 0, \quad D_{0+}^{\frac{5}{4}} u(1) = \sum_{i=1}^{\infty} \frac{3}{7} i^{-2} D_{0+}^{\frac{5}{4}} u(\frac{1}{i}).
 \end{cases}
 \tag{27}$$

Conclusion BVP (27) has at least two positive solutions u^* and u^{**} with $0 < \|u^*\| < r < \|u^{**}\|$.

Proof In this problem, $\alpha = \frac{5}{2}, \beta = \frac{5}{4}, \xi_i = \frac{3}{7}i^{-2}, \eta_i = \frac{1}{i}, f(t, u) = \frac{1}{20\sqrt[3]{t(1-t)}}(u^2 + \frac{1}{3\sqrt{u}}) + \frac{1}{15\sqrt[3]{t(1-t)}}$. By simple computation, we have $\Gamma(\frac{5}{2}) = 1.3293, \sum_{i=1}^{\infty} \xi_i \eta_i^{\alpha-\beta-1} = 0.6258 < 1, p(0) = 0.3742$. For any $r > 0$, it is easy to see that (H₂) holds for $a(t) = \frac{1}{15\sqrt[3]{(1-t)}}$, $g(u) = u^2 + \frac{1}{6\sqrt{u}}, b(t) = \frac{1}{20\sqrt[3]{t(1-t)}}$, and

$$a_r^* = \int_0^1 a(t)g_r(t) \, dt < \int_0^1 \frac{1}{15\sqrt[3]{(1-t)}} \left(r^2 + \frac{1}{6t^{\frac{3}{4}}\sqrt{r}} \right) dt < +\infty,
 \tag{28}$$

$b^* = \int_0^1 \frac{1}{20\sqrt[3]{t(1-t)}} \, dt = 0.1027$. Obviously, (H₃) and (H₄) hold for $c(t) = d(t) = \frac{1}{15\sqrt[3]{(1-t)}}$, and $c^* = d^* = 0.0600$. Take $r = 1$, we have, by (27),

$$\begin{aligned}
 \frac{1}{p(0)\Gamma(\alpha)} (a_1^* + b^*) &< \frac{1}{0.3742 \times 1.3293} \times (0.0600 + 0.0463 + 0.1027) \\
 &= 0.4202 < 1 = r.
 \end{aligned}$$

Consequently, (15) holds, and our conclusion follows from Theorem 1. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors, XZ and QZ, contributed to each part of this work equally and read and approved the final version of the manuscript.

Author details

¹Department of Information Engineering, Jining Medical College, Jining, Shandong 272067, P.R. China. ²School of Mathematics, Liaocheng University, Liaocheng, Shandong 252059, P.R. China.

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