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The global stabilization of the Camassa-Holm equation with a distributed feedback control

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Abstract

The global stabilization of the Camassa-Holm equation with a distributed feedback control of the form $-(\lambda u - \beta u_{xx} - \lambda[u])$ is investigated. The existence and uniqueness of global strong solutions and global weak solutions to the closed loop control system are obtained. The exponential asymptotical stabilization of weak solutions to the problem is established. Namely, the weak solutions to the problem exponentially uniformly decay to a constant. The main novelty in this paper is that the effects of the coefficients λ and β on the global existence and exponential asymptotical stabilization of solutions are given.

MSC: 35G25; 35L15; 35Q58

Keywords: Camassa-Holm equation; feedback control; strong solutions; weak solutions; exponential asymptotical stabilization

1 Introduction

This paper is concerned with the distributed feedback control problem for the Camassa-Holm equation

$$\begin{cases} u_t - u_{xxt} + 2ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx} - (\lambda u - \beta u_{xx} - \lambda[u]), & t > 0, x \in \mathbb{S}, \\ u(t,0) = u(t,1), \quad u_x(t,0) = u_x(t,1), \quad u_{xx}(t,0) = u_{xx}(t,1), & t \geq 0, \\ u(0,x) = u_0(x), \quad x \in \mathbb{S}, \end{cases} \tag{1.1}$$

where $k \in \mathbb{R}$, $\lambda, \beta \geq 0$ are constants, $\mathbb{S} = [0,1] \subset \mathbb{R}$, $[u] = \int_{\mathbb{S}} u(t,x)\,dx$ denotes the mean value of $u(t,x)$ on $\mathbb{S}$, $(t,x) \in [0,\infty) \times \mathbb{S}$ and $u_0 \in H^s(\mathbb{S})$ with $s \geq 1$. The action of the feedback $-(\lambda u - \beta u_{xx} - \lambda[u])$ consists in balancing the level of water.

Let us give a brief overview of several related works. For the classical Camassa-Holm equation [1]

$$u_t - u_{xxt} + 2ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.2}$$

where $u(t,x)$ is the height of water free surface above a flat bottom (or the fluid velocity in x direction) and k is a constant related to critical shallow water wave speed. The alternative derivation of (1.2) as a model for the unidirectional propagation of shallow water waves is found in [2]. Equation (1.2) has attracted attention of many researchers due to several remarkable features. The first one is the presence of solutions in the form of peaked solitary waves for $k = 0$. The solitary waves are the peakons, known to be stable [3]. The peakon $u(t,x) = ce^{-|x-ct|}$ with $c \neq 0$ is smooth except at its crest and the tallest among all waves of fixed energy. It is traveling waves of the largest amplitude, which is the Stokes waves of the greatest height (see the discussions in [4]). The stability here is in the sense of orbital stability. That is, the shape is stable under small perturbations. For $k > 0$, the solitary waves are smooth stable solitons [5]. Another feature is that the equation has breaking waves [6]. In other words, the solutions remain bounded while their slopes become unbounded in finite time. A further important property is that of integrability, in the sense of an infinite dimensional Hamiltonian system. That is, for a large class of initial data for which the solution is global in time. By means of an associated isospectrum problem, one can show that the flow is equivalent to a linear flow (see [7, 8]). Li and Olver [9] not only obtained the local well-posedness for the problem but also gave the conditions which lead to some solutions blowing up in finite time in the Sobolev space $H^s(\mathbb{R})$ ($s > \frac{3}{2}$). It is possible to continue the solutions after blow-up, either as conservative or as dissipative global weak solutions (see [10, 11]). Novruzova and Hagverdiyev [12] obtained the global existence and uniqueness of strong solutions to Cauchy problem of the Camassa-Holm equation in $H^2(\mathbb{R})$. For other methods to establish the local well-posedness for the Cauchy problem and global existence of solutions to the Camassa-Holm equation or other shallow water models, the reader is referred to [13–16] and the references therein. The global weak solutions to the Cauchy problem of equation (1.2) have been studied extensively [17–22]. For the case $k = 0$ in equation (1.2), Xin and Zhang [19] obtained the global existence of weak solutions in $H^1(\mathbb{R})$ by using the vanishing viscosity method. Xin and Zhang [20] proved the uniqueness of global weak solutions obtained in [19] with the condition that $m_0 = u_0 - u_{0,xx}$ is a positive Radon measure. Coclite *et al.* [17] investigated the global existence and uniqueness of weak solutions to the hyperelastic-rod wave equation in $H^1(\mathbb{R})$, which is similar to equation (1.2) in the structure of equation. Lai and Wu [22] studied the global existence of weak solutions to the generalized hyperelastic-rod wave equation in $H^1(\mathbb{R})$.

Recently, much literature was devoted to the study of control problem for the water wave equations. In [23, 24], the exact boundary control problems for the Korteweg-de Vries equation were considered. Komornik [25] studied the feedback control problem for the Korteweg-de Vries equation in $H^2(\mathbb{S})$ ($s \geq 2$), where the feedback control is $f(t,x) = -\lambda(u - [u])$. They obtained the existence of solutions to the problem by using the Galerkin method. The exponential asymptotical stabilization of strong solutions to the problem was investigated. In [26], the exact controllability and stabilization of solutions to the Korteweg-de Vries equation were established. Rosier and Zhang [27] proved the unique continuation property of solutions to the Benjamin-Bona-Mahony equation with small initial data in $H^1(\mathbb{S})$. Zong and Zhao [28] investigated the feedback control problem for the Degasperis-Procesi equation with feedback control $f(t,x) = -\lambda(u - u_{xx} - [u])$. They obtained the global existence of solutions to the control problem in $H^s(\mathbb{S})$ ($s \geq 2$) by using Kato's theory and energy estimates. It is worth pointing out that the obtained solutions in [25, 28] are strong solutions. Glass [29] investigated the exact controllability and global asymptotical stabilization of solutions to the Camassa-Holm equation on $\mathbb{S}$ by means of a distributed control. It was shown in [29] that the constant k in equation (1.2) was related to the equilibrium point of solutions. Perrollaz [30] studied the initial boundary value problem and asymptotical stabilization of solutions to equation (1.2) on $\mathbb{S}$. The local existence result and weak-strong uniqueness of solutions were obtained. The global asymptotical stabilization of solutions to the problem was established by means of a boundary feedback law.

Integrating (1.1) with respect to the time variable from 0 to t and using integration by parts, we get

$$\frac{d}{dt}\int_{\mathbb{S}} u\,dx = -\left(\lambda\int_{\mathbb{S}} u\,dx - \lambda[u]\right),$$

from which one derives $[u(t)] = [u_0]$, for all $t > 0$. Let $a = [u_0]$ and $v = u - [u] = u - a$, then $v_0 = u_0 - a$. We still denote v by u for convenient and rewrite the problem (1.1) as

$$\begin{cases} u_t - u_{xxt} + 2(k+a)u_x + a(u_x - u_{xxx}) + 3uu_x + \lambda u - \beta u_{xx} \\ \quad = 2u_x u_{xx} + uu_{xxx}, \quad t > 0, x \in \mathbb{S}, \\ u(t,0) = u(t,1), \qquad u_x(t,0) = u_x(t,1), \qquad u_{xx}(t,0) = u_{xx}(t,1), \quad t \geq 0, \\ u(0,x) = u_0(x), \quad x \in \mathbb{S} \end{cases} \tag{1.3}$$

or

$$\begin{cases} u_t + (u+a)u_x = -\partial_x P, \quad t > 0, x \in \mathbb{S}, \\ u(t,0) = u(t,1), \qquad u_x(t,0) = u_x(t,1), \qquad u_{xx}(t,0) = u_{xx}(t,1), \quad t \geq 0, \\ u(0,x) = u_0(x), \quad x \in \mathbb{S}, \end{cases} \tag{1.4}$$

where $\partial_x P = \partial_x(1-\partial_x^2)^{-1}[u^2 + (2k+2a)u + \frac{1}{2}u_x^2 - \beta u_x] + (1-\partial_x^2)^{-1}(\lambda u)$.

Motivated by the work in [17, 19, 20, 22, 29], we study the global stabilization of problem (1.3). Our main results are the existence and uniqueness of global strong solutions, global weak solutions and the exponential asymptotical stabilization of solutions to the problem. Due to the presence of feedback control term, the conserved law which plays an important role in studying the problem disappeared. This difficulty has been dealt with by establishing the energy inequality and using the estimates of solutions to the transport equation. For the low regularity of space in which we study the weak solutions and weakly compact priori estimates of the approximate solutions, we use the method in [17] to improve the weak convergence to strong convergence. The uniqueness of global weak solutions is established with certain assumptions.

We write the space

$$E^s_{p,r}(T) = \begin{cases} C([0,T]; B^s_{p,r}(\mathbb{S})) \cap C^1([0,T]; B^{s-1}_{p,r}(\mathbb{S})), & 1 \leq r < \infty, \\ L^\infty([0,T]; B^s_{p,\infty}(\mathbb{S})) \cap \mathrm{Lip}([0,T]; B^{s-1}_{p,\infty}(\mathbb{S})), & r = \infty, \end{cases}$$

where $T > 0$, $s \in \mathbb{R}$, $p \in [1,\infty]$, $r \in [1,\infty]$.

The main results of this paper are stated as follows.

Theorem 1.1 *Let $1 \leq p, r \leq \infty$, $s > \max(\frac{3}{2}, 1 + \frac{1}{p})$ and $u_0 \in B^s_{p,r}(\mathbb{S})$. Then there exists a time $T > 0$ such that the problem* (1.3) *admits a unique solution $u \in E^s_{p,r}(T)$. If $s' < s$, $r = \infty$ or $s' = s$, $r < \infty$, the map $u_0 \to u$ is continuous from a neighborhood of u_0 in $B^s_{p,r}(\mathbb{S})$ into $C([0,T]; B^{s'}_{p,r}(\mathbb{S})) \cap C^1([0,T]; B^{s'-1}_{p,r}(\mathbb{S}))$.*

Theorem 1.2 *Let $u_0 \in H^s(\mathbb{S})$ ($s > \frac{3}{2}$) and $T > 0$ be the maximal existence time of corresponding solution u to problem* (1.3). *Then the solution u blows up in finite time if and only if*

$$\lim_{t \to T^-} \inf_{x \in \mathbb{S}} u_x(t,x) = -\infty. \tag{1.5}$$

We obtain the global existence of strong solutions to problem (1.3).

Theorem 1.3 *Let $u_0 \in H^s(\mathbb{S})$ ($s \geq 2$), $(1 - \frac{2}{\coth \frac{1}{2}})\beta < \lambda < (1 + \frac{2}{\coth \frac{1}{2}})\beta$, $m_0 = u_0 - u_{0,xx}$, and $\|m_0\|_{L^2(\mathbb{S})} \leq \frac{2}{3}(2\beta - (\coth \frac{1}{2})|\beta - \lambda|)$. Then the corresponding strong solution u to problem* (1.3) *exists globally.*

We present the global existence of weak solutions to problem (1.3). First of all, we give the definition of weak solutions.

Definition 1.1 The function $u(t,x)$ is a weak solution to problem (1.3) if:

(i) $u(t,x) \in C([0,\infty); C(\mathbb{S})) \cap L^\infty([0,\infty); H^1(\mathbb{S}))$ and

$$\|u(t)\|_{H^1(\mathbb{S})} \leq \|u_0\|_{H^1(\mathbb{S})}, \quad \text{for all } t > 0. \tag{1.6}$$

(ii) $u(t,x)$ satisfies problem (1.3) in the sense of distributions.

Theorem 1.4 *Let $u_0 \in H^1(\mathbb{S})$. Then problem* (1.3) *admits a weak solution in the sense of Definition* 1.1. *Moreover, the weak solution $u(t,x)$ has the following properties.*

(i) *There exists a positive constant C_1 depending only on $\|u_0\|_{H^1(\mathbb{S})}$ and the coefficients in problem* (1.3) *such that*

$$\partial_x u(t,x) \leq \frac{2}{t} + C_1, \quad \textit{for all } t > 0. \tag{1.7}$$

(ii) *Let $\delta \in (0,1)$, $T > 0$, $[a_0, b_0] \subset \mathbb{S}$. There exists a positive constant C_2 such that*

$$\int_0^T \int_{a_0}^{b_0} \left|\partial_x u(t,x)\right|^{2+\delta} \, dx \, dt \leq C_2. \tag{1.8}$$

(iii) *The corresponding solution $u(t,x)$ to problem* (1.3) *is exponentially asymptotically stable. Namely, there exists a positive constant $C_3 = C_3(\|u_0\|_{H^1(\mathbb{S})})$ such that*

$$\left\|\left(u(t), u_t(t)\right)\right\|_{H^1(\mathbb{S}) \times L^2(\mathbb{S})} \leq C_3 e^{-\lambda_1 t}, \quad \textit{for all } t > 0, \tag{1.9}$$

where $\lambda_1 = \min\{\lambda, \beta\}$.

We present the uniqueness of the global weak solutions to problem (1.3).

Theorem 1.5 *Let $u_0 \in H^1(\mathbb{S})$ and $m_0 = u_0 - u_{0,xx}$ be a positive Radon measure. Then problem* (1.3) *has a unique global weak solution $u(t,x) \in C([0,\infty); H^1(\mathbb{S}))$.*

Remark 1.1 The existence and uniqueness of global strong solutions in $H^s(\mathbb{S})$ ($s \geq 2$) and global weak solutions in $H^1(\mathbb{S})$ to problem (1.3) are obtained. The exponential asymptotical stabilization of solutions is established. Similar to [16], we deduce that the coefficient β in (1.3) is related to the blow-up rate of solutions. Theorem 1.3 in this paper contains Theorem 1 in [12] as a special case. The problem (1.3) studied in this paper contains the problems studied in [17, 19, 20, 22].

The remainder of this paper is organized as follows. In Section 2, the definition of the Besov space and priori estimates of solutions to the transport equation are reviewed. Section 3 is devoted to the proofs of Theorems 1.1, 1.2, and 1.3. The proofs of Theorems 1.4 and 1.5 are presented in Section 4.

Notation Let $*$ be the convolution on $\mathbb{S}$. $\|\cdot\|_{L^p(\mathbb{S})}$ stands for the norm in the Lebesgue space $L^p(\mathbb{S})$ $(1\le p\le\infty)$. $\|\cdot\|_{H^s(\mathbb{S})}$ stands for the norm in the Sobolev space $H^s(\mathbb{S})$ $(s\in\mathbb{R})$. $\|\cdot\|_{B^s_{p,r}(\mathbb{S})}$ stands for the norm in the Besov space $B^s_{p,r}(\mathbb{S})$ $(s\in\mathbb{R})$. For $a\lesssim b$, we mean that there exists a uniform constant C, which may be different on different lines such that $a\le Cb$. We assume $a+=a+\varepsilon$, where $\varepsilon>0$ is a sufficiently small number. Since the functions in all spaces are over $\mathbb{S}$, for simplicity, we drop $\mathbb{S}$ in our notations if there is no ambiguity.

2 Preliminary

We recall some basic facts in the Besov space. One may check [31] for more details.

Lemma 2.1 [31] *Let $s\in\mathbb{R}$, $1\le p,r\le\infty$. The nonhomogeneous Besov space is defined by $B^s_{p,r}(\mathbb{S})=\{f\in S'(\mathbb{S})\mid \|f\|_{B^s_{p,r}}<\infty\}$, where*

$$\|f\|_{B^s_{p,r}}=\begin{cases}(\sum_{j=-1}^{\infty}2^{jrs}\|\Delta_j f\|^r_{L^p})^{\frac{1}{r}}, & r<\infty,\\ \sup_{j\ge-1}2^{js}\|\Delta_j f\|_{L^p}, & r=\infty.\end{cases}$$

Moreover, $S_j f=\sum_{q=-1}^{j-1}\Delta_q f$.

We present two related lemmas for the Cauchy problem of the transport equation

$$\begin{cases}f_t+d\cdot\nabla f=F,\\ f|_{t=0}=f_0,\end{cases}\tag{2.1}$$

where $d:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ stands for a given time dependent vector field, $f_0:\mathbb{R}^n\to\mathbb{R}^m$ and $F:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^m$ are known data.

Lemma 2.2 [31] *Let $1\le p\le p_1\le\infty$, $1\le r\le\infty$, $p'=\frac{p}{p-1}$. Assume $s>-n\cdot\min(\frac{1}{p_1},\frac{1}{p'})$ or $s>-1-n\cdot\min(\frac{1}{p_1},\frac{1}{p'})$ if $\nabla\cdot d=0$. Then there exists a constant C depending only on n, p, p_1, r, s such that the following estimate holds*:

$$\|f\|_{\tilde{L}^\infty_t([0,t];B^s_{p,r})}\le e^{C_1\int_0^t Z(\tau)\,d\tau}\left[\|f_0\|_{B^s_{p,r}}+\int_0^t e^{-C_1\int_0^\tau Z(\xi)\,d\xi}\bigl\|F(\tau)\bigr\|_{B^s_{p,r}}\,d\tau\right],\tag{2.2}$$

where

$$Z(t)=\begin{cases}\|\nabla d(t)\|_{B^{\frac{n}{p_1}}_{p_1,\infty}\cap L^\infty}, & s<1+\frac{n}{p_1},\\ \|\nabla d(t)\|_{B^{s-1}_{p_1,r}}, & s>1+\frac{n}{p_1}\text{ or }s=1+\frac{n}{p_1}, r=1.\end{cases}$$

If $f=d$, then for all $s>0$ ($\nabla\cdot d=0$, $s>-1$), (2.2) holds with $Z(t)=\|\nabla d(t)\|_{L^\infty}$.

Let us state the existence result for the transport equation with initial data in the Besov space.

Lemma 2.3 [31] *Let p, p_1, r, s be as in the statement of Lemma 2.2. $f_0 \in B^s_{p,r}$ and $F \in L^1([0,T];B^s_{p,r})$, $d \in L^\rho([0,T];B^{-M}_{\infty,\infty})$ is a time dependent vector field for some $\rho > 1$, $M > 0$. If $s < 1 + \frac{n}{p_1}$, then $\nabla d \in L^1([0,T];B^{\frac{n}{p_1}}_{p_1,\infty} \cap L^\infty)$. If $s > 1 + \frac{n}{p_1}$ or $s = 1 + \frac{n}{p_1}$, $r = 1$, then $\nabla d \in L^1([0,T];B^{s-1}_{p_1,r})$. Thus, problem (2.1) has a unique solution $f \in L^\infty([0,T];B^s_{p,r}) \cap (\bigcap_{s'<s} C([0,T];B^{s'}_{p,1}))$ and (2.2) holds true. If $r < \infty$, then $f \in C([0,T];B^s_{p,r})$.*

3 The proofs of Theorems 1.1, 1.2, and 1.3

3.1 The proof of Theorem 1.1

Applying Lemmas 2.2, 2.3 and using the Littlewood-Paley theory, one may follow similar arguments to [14] to establish the local well-posedness for problem (1.3) with suitable modifications. Here we omit the detail proof. For problem (1.3) with initial value $u_0 \in B^s_{p,r}$ ($s > \max(1 + \frac{1}{p}, \frac{3}{2})$), we see that the corresponding solution $u \in C([0,T];B^s_{p,r}) \cap C^1([0,T];B^{s-1}_{p,r})$. This completes the proof of Theorem 1.1.

3.2 The proof of Theorem 1.2

We investigate the blow-up mechanisms of strong solutions to problem (1.3). Applying Theorem 1.1 and a simple density argument, we only need to show that Theorem 1.2 holds with $s \geq 2$. Here we assume $s = 2$ to prove the theorem.

Multiplying the first equation in (1.3) by u and integrating by parts yield

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}\left(u^2 + u_x^2\right)dx + \int_{\mathbb{S}}\left(\lambda u^2 + \beta u_x^2\right)dx = 0. \tag{3.1}$$

On the other hand, multiplying (1.3) by u_{xx} and integrating by parts again yield

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}}\left(u_x^2 + u_{xx}^2\right)dx + \int_{\mathbb{S}}\left(\lambda u_x^2 + \beta u_{xx}^2\right)dx = -\frac{3}{2}\int_{\mathbb{S}} u_x\left(u_x^2 + u_{xx}^2\right)dx. \tag{3.2}$$

Assume $T < \infty$ and there exists $M_1 > 0$ such that

$$u_x(t,x) \geq -M_1, \quad \text{for all } (t,x) \in [0,T] \times \mathbb{S}. \tag{3.3}$$

We have

$$\frac{d}{dt}\int_{\mathbb{S}}\left(u_x^2 + u_{xx}^2\right)dx \leq (3M_1 - 2\lambda_1)\int_{\mathbb{S}}\left(u_x^2 + u_{xx}^2\right)dx, \tag{3.4}$$

where $\lambda_1 = \min\{\lambda, \beta\}$. Applying the Gronwall inequality to (3.4) yields

$$\left\|u(t)\right\|^2_{H^2} \leq \|u_0\|^2_{H^2} e^{(3M_1 - 2\lambda_1)T}, \quad \text{for all } t \in [0,T], \tag{3.5}$$

which contradicts the assumption that the maximal existence time $T < \infty$.

Conversely, using the Sobolev embedding theorem $H^s \hookrightarrow L^\infty$ ($s > \frac{1}{2}$), we derive that if condition (1.5) in Theorem 1.2 holds, then the corresponding solution blows up in finite time. This completes the proof of Theorem 1.2.

3.3 The proof of Theorem 1.3

Bearing in mind $m(t,x) = u - u_{xx}$, we rewrite the first equation in (1.3) as

$$m_t + um_x + 2u_x m + am_x + (2k + 2a)u_x + (\lambda - \beta)u + \beta m = 0. \tag{3.6}$$

Multiplying (3.6) by m and integrating by parts yield

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} m^2\,dx + \int_{\mathbb{S}} \beta m^2\,dx \\ &\quad = -\frac{3}{2}\int_{\mathbb{S}} u_x m^2\,dx + \int_{\mathbb{S}} (\beta - \lambda)u(u - u_{xx})\,dx \\ &\quad \le -\frac{3}{2}\int_{\mathbb{S}} u_x m^2\,dx + |\beta - \lambda|\|u\|_{L^2}\|m\|_{L^2}. \end{aligned} \tag{3.7}$$

If $g(x) = \frac{\cosh(x-[x]-\frac{1}{2})}{2\sinh\frac{1}{2}}$, $x \in \mathbb{S}$, where $[x]$ stands for the integer part of x, then $(1-\partial_x^2)^{-1}f = g * f$. Using the relation $u = g * m$ and Young's inequality, we have $\|u_x\|_{L^\infty} \le \|g_x\|_{L^2}\|m\|_{L^2} \le \frac{1}{2}\|m\|_{L^2}$ and $\|u\|_{L^\infty} \le \|g\|_{L^2}\|m\|_{L^2} \le \frac{1}{2}(\coth\frac{1}{2})\|m\|_{L^2}$. From (3.7), we obtain

$$\frac{d}{dt}\|m\|_{L^2}^2 + \lambda_2\|m\|_{L^2}^2 \le \frac{3}{2}\left(\|m\|_{L^2}^2\right)^{\frac{3}{2}}, \tag{3.8}$$

where $\lambda_2 = 2\beta - (\coth\frac{1}{2})|\beta - \lambda|$. Multiplying both sides of (3.8) by $e^{\lambda_2 t}$ gives rise to

$$\frac{d}{dt}\left(e^{\lambda_2 t}\|m\|_{L^2}^2\right) \le \frac{3}{2}e^{-\frac{1}{2}\lambda_2 t}\left(e^{\lambda_2 t}\|m\|_{L^2}^2\right)^{\frac{3}{2}}. \tag{3.9}$$

Then we have

$$\frac{d}{dt}\left(e^{\lambda_2 t}\|m\|_{L^2}^2\right)^{-\frac{1}{2}} \ge -\frac{3}{4}e^{-\frac{1}{2}\lambda_2 t}. \tag{3.10}$$

Integrating (3.10) with respect to time variable from 0 to t yields

$$\left(e^{\lambda_2 t}\|m\|_{L^2}^2\right)^{-\frac{1}{2}} \ge \frac{1}{\|m_0\|_{L^2}} + \frac{3}{2\lambda_2}\left(e^{-\frac{1}{2}\lambda_2 t} - 1\right) \ge \frac{1}{\|m_0\|_{L^2}} - \frac{3}{2\lambda_2}. \tag{3.11}$$

Using the assumption $\|m_0\|_{L^2} \le \frac{2\lambda_2}{3}$, we have

$$\|m\|_{L^2} \le e^{-\frac{1}{2}\lambda_2 t}\left(\frac{1}{\|m_0\|_{L^2}} - \frac{3}{2\lambda_2}\right)^{-1}. \tag{3.12}$$

Applying the Sobolev embedding theorem gives rise to $\|u_x\|_{L^\infty} \le \|m\|_{L^2} \le C_2(T)$. Using Theorem 1.2, we complete the proof of Theorem 1.3.

4 The proofs of Theorems 1.4 and 1.5

4.1 The proof of Theorem 1.4

We are in the position to prove the global existence of weak solutions to problem (1.3). Let $u_0(x) \in H^1$ and $u_{\varepsilon 0}(x) = j_\varepsilon(x) * u_0(x)$, where $j_\varepsilon(x)$ is the mollifier. We construct the approximate solution sequence $(u_\varepsilon)_{\varepsilon>0} = (u_\varepsilon(t,x))_{\varepsilon>0}$ as a solution to problem (1.3) with a viscous term $\varepsilon(u_{xx} - u_{xxxx})$. Namely

$$\begin{cases} u_{\varepsilon,t} + (u_\varepsilon + a)\partial_x u_\varepsilon = \varepsilon u_{\varepsilon,xx} - \partial_x P_\varepsilon, \quad t > 0, x \in \mathbb{S}, \\ u_\varepsilon(t,0) = u_\varepsilon(t,1), \qquad u_{\varepsilon,x}(t,0) = u_{\varepsilon,x}(t,1), \qquad u_{\varepsilon,xx}(t,0) = u_{\varepsilon,xx}(t,1), \quad t \geq 0, \\ u_\varepsilon(0,x) = u_{\varepsilon 0}(x), \quad x \in \mathbb{S}, \end{cases} \tag{4.1}$$

where $\partial_x P_\varepsilon = \partial_x(1-\partial_x^2)^{-1}[u_\varepsilon^2 + (2k+2a)u_\varepsilon + \frac{1}{2}u_{\varepsilon,x}^2 - \beta u_{\varepsilon,x}] + (1-\partial_x^2)^{-1}(\lambda u_\varepsilon)$.

We establish the following global well-posedness result for problem (4.1).

Lemma 4.1 *Let $u_{\varepsilon 0} \in H^s$ $(s \geq 2)$. Then there exists a unique solution $u_\varepsilon \in C([0,\infty); H^s)$ to problem* (4.1). *Furthermore, for all $t > 0$, we have*

$$\|u_\varepsilon(t)\|_{H^1}^2 + 2\int_0^t \int_{\mathbb{S}} \left[\lambda u_\varepsilon^2 + (\beta+\varepsilon)u_{\varepsilon,x}^2 + \varepsilon u_{\varepsilon,xx}^2\right] dx\, ds = \|u_{\varepsilon 0}\|_{H^1}^2. \tag{4.2}$$

Proof of Lemma 4.1 Following the standard arguments for the nonlinear parabolic equation and using Theorem 2.1 in [19], we deduce that the problem (4.1) admits a unique solution $u_\varepsilon \in C([0,\infty); H^s)$ $(s \geq 2)$. Multiplying (4.1) by u_ε and using integration by parts yield

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}} \left(u_\varepsilon^2 + u_{\varepsilon,x}^2\right) dx + \int_{\mathbb{S}} \left(\lambda u_\varepsilon^2 + \beta u_{\varepsilon,x}^2\right) dx + \varepsilon \int_{\mathbb{S}} \left(u_{\varepsilon,x}^2 + u_{\varepsilon,xx}^2\right) dx = 0,$$

which completes the proof. □

Using Lemma 4.1 and the Sobolev embedding theorem, we have $\|u_\varepsilon\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|u_\varepsilon\|_{H^1} \leq \frac{1}{\sqrt{2}}\|u_{\varepsilon 0}\|_{H^1} \leq \frac{1}{\sqrt{2}}\|u_0\|_{H^1}$. Differentiating the first equation in (4.1) with respect to x and denoting $q_\varepsilon = \frac{\partial u_\varepsilon}{\partial x}$. Then $q_\varepsilon(t,x)$ satisfies

$$\begin{cases} \frac{\partial q_\varepsilon}{\partial t} + (u_\varepsilon + a)\frac{\partial q_\varepsilon}{\partial x} - \varepsilon\frac{\partial^2 q_\varepsilon}{\partial x^2} + \frac{1}{2}q_\varepsilon^2 + \beta q_\varepsilon = Q_\varepsilon(t,x), \quad t > 0, x \in \mathbb{S}, \\ q_\varepsilon(t,0) = q_\varepsilon(t,1), \qquad q_{\varepsilon,x}(t,0) = q_{\varepsilon,x}(t,1), \quad t \geq 0, \\ q_\varepsilon(0,x) = q_{\varepsilon 0}(x), \quad x \in \mathbb{S}, \end{cases} \tag{4.3}$$

where $Q_\varepsilon(t,x) = [u_\varepsilon^2 + (2k+2a)u_\varepsilon] - (1-\partial_x^2)^{-1}[u_\varepsilon^2 + (2k+2a)u_\varepsilon + \frac{1}{2}q_\varepsilon^2 - \beta q_\varepsilon + \lambda q_\varepsilon]$.

We bear in mind $(1-\partial_x^2)^{-1}f = g * f$. Using (4.2) and Young's inequality yields

$$\begin{aligned} \|Q_\varepsilon(t,x)\|_{L^\infty([0,\infty);L^\infty)} &\lesssim \|u_\varepsilon\|_{L^\infty([0,\infty);L^\infty)}^2 + \|u_\varepsilon\|_{L^\infty([0,\infty);L^\infty)} \\ &\quad + \left\|u_\varepsilon^2 + (2k+2a)u_\varepsilon + \frac{1}{2}q_\varepsilon^2 - \beta q_\varepsilon + \lambda q_\varepsilon\right\|_{L^\infty([0,\infty);L^1)} \\ &\leq L, \end{aligned} \tag{4.4}$$

where L is a constant depending on $\|u_0\|_{H^1}$. Using (4.3), we obtain

$$\frac{\partial q_\varepsilon}{\partial t} + (u_\varepsilon + a)\frac{\partial q_\varepsilon}{\partial x} - \varepsilon\frac{\partial^2 q_\varepsilon}{\partial x^2} + \frac{1}{2}q_\varepsilon^2 + \beta q_\varepsilon = Q_\varepsilon(t,x) \leq L. \tag{4.5}$$

Let $f = f(t)$ satisfy

$$\frac{df}{dt} + \frac{1}{2}f^2 = L, \quad \text{if } t > 0 \text{ and } f(0) = \max\{0, u_{\varepsilon 0,x}\}.$$

From the comparison principle of the parabolic equations, we deduce $q_\varepsilon(t,x) \le f(t)$, for all $t > 0$, $x \in \mathbb{S}$. Taking $F(t) = \frac{2}{t} + \sqrt{2L}$, we have $\frac{dF(t)}{dt} + \frac{1}{2}F^2(t) - L = \frac{2\sqrt{2L}}{t} > 0$. Using the comparison principle of the ODEs yields $f(t) \le F(t)$, for all $t > 0$. Thus we have

$$\partial_x u_\varepsilon(t,x) \le \frac{2}{t} + C_1, \quad \text{for all } t > 0, \tag{4.6}$$

where C_1 is a constant depending on $\|u_0\|_{H^1}$ and the coefficients in problem (1.3).

Lemma 4.2 *Let* $0 < \delta < 1$, $T > 0$, *and* $[a_0, b_0] \subset \mathbb{S}$. *Then there exists a positive constant* C_2 *depending only on* $\|u_0\|_{H^1}$, T, a_0, b_0 *and the coefficients in* (1.3), *such that*

$$\int_0^T \int_{a_0}^{b_0} \left|\partial_x u_\varepsilon(t,x)\right|^{2+\delta} dx\,dt \le C_2, \tag{4.7}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ *is the unique solution to problem* (4.1).

Proof of Lemma 4.2 Let $\chi(x) \in C_c^\infty$ be a cut-off function such that $\chi(x) = 1$, $x \in [a_0, b_0]$. Similar to the proof of Lemma 4.1 in [17], we consider the map $\theta(\xi) = \xi(1 + |\xi|)^\delta$, $\xi \in \mathbb{R}$, $0 < \delta < 1$. Then

$$\begin{aligned}
&\theta'(\xi) = \big(1 + (1+\delta)|\xi|\big)\big(1 + |\xi|\big)^{\delta-1}, \\
&\theta''(\xi) = \delta \operatorname{sign}(\xi)\big(1 + |\xi|\big)^{\delta-2}\big(2 + (1+\delta)|\xi|\big) \\
&\qquad\ = \delta(1+\delta)\operatorname{sign}(\xi)\big(1 + |\xi|\big)^{\delta-1} + (1-\delta)\delta \operatorname{sign}(\xi)\big(1 + |\xi|\big)^{\delta-2}, \\
&\big|\theta(\xi)\big| \le |\xi| + |\xi|^{1+\delta}, \qquad \big|\theta'(\xi)\big| \le 1 + (1+\delta)|\xi|, \qquad \big|\theta''(\xi)\big| \le 2\delta, \\
&\xi\theta(\xi) - \frac{1}{2}\xi^2\theta'(\xi) = \frac{1-\delta}{2}\xi^2\big(1 + |\xi|\big)^\delta + \frac{\delta}{2}\xi^2\big(1 + |\xi|\big)^{\delta-1} \ge \frac{1-\delta}{2}\xi^2\big(1 + |\xi|\big)^\delta.
\end{aligned} \tag{4.8}$$

Multiplying (4.3) by $\chi(x)\theta'(q_\varepsilon)$ and integrating the resultant equation over $\Pi_T = [0,T] \times \mathbb{S}$ yield

$$\begin{aligned}
&\int_{\Pi_T} \chi(x) q_\varepsilon \theta(q_\varepsilon)\,dx\,dt - \frac{1}{2}\int_{\Pi_T} q_\varepsilon^2 \chi(x)\theta'(q_\varepsilon)\,dx\,dt \\
&\quad = \int_{\mathbb{S}} \chi(x)\big[\theta\big(q_\varepsilon(T,x)\big) - \theta\big(q_\varepsilon(0,x)\big)\big]\,dx - \int_{\Pi_T} (u_\varepsilon + a)\chi'(x)\theta(q_\varepsilon)\,dx\,dt \\
&\qquad + \varepsilon \int_{\Pi_T} \partial_x q_\varepsilon \chi'(x)\theta'(q_\varepsilon)\,dx\,dt + \varepsilon \int_{\Pi_T} (\partial_x q_\varepsilon)^2 \chi(x)\theta''(q_\varepsilon)\,dx\,dt \\
&\qquad + \beta \int_{\Pi_T} q_\varepsilon \chi(x)\theta'(q_\varepsilon)\,dx\,dt - \int_{\Pi_T} Q_\varepsilon(t,x)\chi(x)\theta'(q_\varepsilon)\,dx\,dt.
\end{aligned} \tag{4.9}$$

Using (4.2), we have

$$\begin{aligned}
&\int_{\Pi_T} \chi(x) q_\varepsilon \theta(q_\varepsilon)\,dx\,dt - \frac{1}{2}\int_{\Pi_T} q_\varepsilon^2 \chi(x)\theta'(q_\varepsilon)\,dx\,dt \\
&\quad \ge \frac{(1-\delta)}{2}\int_{\Pi_T} \chi(x) q_\varepsilon^2 \big(1 + |q_\varepsilon|\big)^\delta\,dx\,dt
\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}
&\left|\int_{\Pi_T} (u_\varepsilon + a)\chi'(x)\theta(q_\varepsilon)\,dx\,dt\right| \\
&\quad\leq \int_{\Pi_T} \bigl(|u_\varepsilon| + |a|\bigr)\bigl|\chi'(x)\bigr|\bigl(|q_\varepsilon|^{1+\delta} + |q_\varepsilon|\bigr)\,dx\,dt \\
&\quad\leq CT\bigl(\|\chi'\|_{L^{\frac{2}{1-\delta}}}\|u_0\|_{H^1}^{1+\delta} + \|\chi'\|_{L^2}\|u_0\|_{H^1}\bigr).
\end{aligned} \tag{4.11}$$

It follows from some calculations that

$$\begin{aligned}
&\left|\int_{\Pi_T} \beta q_\varepsilon \chi(x)\theta'(q_\varepsilon)\,dx\,dt\right| \\
&\quad\lesssim \int_{\Pi_T} |q_\varepsilon|\bigl|\chi(x)\bigr|\bigl[(\delta+1)|q_\varepsilon| + 1\bigr]\,dx\,dt \\
&\quad\lesssim \int_0^T \int_{\mathbb{S}} \bigl[(\delta+1)|q_\varepsilon|^2\bigl|\chi(x)\bigr| + |q_\varepsilon|\bigl|\chi(x)\bigr|\bigr]\,dx\,dt \\
&\quad\lesssim \int_0^T \bigl(\|\chi\|_{L^\infty}\|q_\varepsilon\|_{L^2}^2 + \|\chi\|_{L^2}\|q_\varepsilon\|_{L^2}\bigr)\,dt \\
&\quad\lesssim \|\chi\|_{L^\infty([0,T];L^\infty)}\|q_\varepsilon\|_{L^2([0,T];L^2)}^2 + \|\chi\|_{L^1([0,T];L^2)}\|q_\varepsilon\|_{L^\infty([0,T];L^2)} \\
&\quad\lesssim C\bigl(\|u_0\|_{H^1}\bigr).
\end{aligned} \tag{4.12}$$

The estimates of other terms are the same as the estimates in [17], we omit the details for simplicity. This completes the proof of Lemma 4.2. □

Lemma 4.3 *There exists a positive constant C_4 depending only on $\|u_0\|_{H^1}$ and the coefficients in* (1.3) *such that $\|\partial_x P_\varepsilon(t)\|_{L^\infty} \leq C_4$.*

Proof of Lemma 4.3 Applying the Sobolev embedding theorem and Young's inequality yields

$$\begin{aligned}
&\bigl\|\partial_x P_\varepsilon(t)\bigr\|_{L^\infty} \\
&\quad\lesssim \left\|\partial_x\bigl(1-\partial_x^2\bigr)^{-1}\left[u_\varepsilon^2 + (2k+2a)u_\varepsilon + \frac{1}{2}u_{\varepsilon,x}^2 - \beta u_{\varepsilon,x}\right] + \bigl(1-\partial_x^2\bigr)^{-1}(\lambda u_\varepsilon)\right\|_{H^{\frac{1}{2}+}} \\
&\quad\lesssim \left\|u_\varepsilon^2 + (2k+2a)u_\varepsilon + \frac{1}{2}u_{\varepsilon,x}^2 - \beta u_{\varepsilon,x}\right\|_{H^{-1+(\frac{1}{2}+)}} + \|\lambda u_\varepsilon\|_{H^{-2+(\frac{1}{2}+)}} \\
&\quad\lesssim \|u_\varepsilon\|_{L^\infty([0,\infty);L^\infty)}^2 + \|u_\varepsilon\|_{L^\infty([0,\infty);L^\infty)} + \|u_\varepsilon\|_{L^\infty([0,\infty);H^1)}^2 + \|u_\varepsilon\|_{L^\infty([0,\infty);H^1)},
\end{aligned} \tag{4.13}$$

which combined with (4.2) completes the proof. □

Lemma 4.4 *There exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}^*} \to 0$ and a function $u \in L^\infty([0,\infty);H^1) \cap H^1([0,T]\times\mathbb{S})$ for all $T > 0$, such that*

$$u_{\varepsilon_j} \rightharpoonup u \quad \text{in } H^1\bigl([0,T]\times\mathbb{S}\bigr), \qquad u_{\varepsilon_j} \to u \quad \text{in } L^\infty\bigl([0,\infty)\times\mathbb{S}\bigr), \tag{4.14}$$

where $u_\varepsilon = u_\varepsilon(t,x)$ is the unique solution to (4.1).

Proof of Lemma 4.4 Using (4.1) and Lemmas 4.1, 4.3, we obtain

$$\begin{aligned}\|\partial_t u_\varepsilon\|_{L^2([0,T]\times\mathbb{S})} &= \big\|(u_\varepsilon + a)\partial_x u_\varepsilon - \varepsilon u_{\varepsilon,xx} + \partial_x P_\varepsilon\big\|_{L^2([0,T]\times\mathbb{S})} \\ &\le C\big(1 + C\|u_0\|_{H^1}^2\big).\end{aligned} \tag{4.15}$$

Hence, $\{u_\varepsilon\}$ is uniformly bounded in $L^\infty([0,\infty);H^1)\cap H^1([0,T]\times\mathbb{S})$. Applying the weakly compactness lemma yields the weak convergence result in (4.14). For each $0 \le s,t \le T$, we have

$$\big\|u_\varepsilon(t) - u_\varepsilon(s)\big\|_{L^2}^2 = \int_{\mathbb{S}}\left(\int_s^t \partial_t u_\varepsilon(\tau,x)\,d\tau\right)^2 dx \le |t-s|\int_{\mathbb{S}}\int_0^T \big(\partial_t u_\varepsilon(\tau,x)\big)^2\,d\tau\,dx.$$

Moreover, bearing in mind $H^1 \hookrightarrow\hookrightarrow L^\infty \hookrightarrow L^2$ and using the Aubin compactness lemma, we deduce the strong convergence result in (4.14). □

Lemma 4.5 *Let $1 < p < \infty$. There exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}^*} \to 0$ and a function $Q \in L^\infty([0,T]\times\mathbb{S})$ such that $Q_{\varepsilon_j} \to Q$ in $L^p([0,T]\times\mathbb{S})$.*

Proof of Lemma 4.5 We fix $T > 0$. Using (4.1), (4.2), and (4.15), we deduce that $\|\partial_t Q_\varepsilon\|_{L^1([0,T];L^1)}$ is bounded. Applying (4.4), we see that Q_ε is uniformly bounded in $W^{1,1}([0,T]\times\mathbb{S})$. Using the Aubin compactness lemma, we complete the proof. □

We use over-bars to denote the weak limits.

Lemma 4.6 *Let $1 < p < 3$, $1 < r < \frac{3}{2}$. Then there exists a sequence $\{\varepsilon_j\}_{j\in\mathbb{N}^*} \to 0$, $q \in L^p([0,\infty)\times\mathbb{S})$, and $\overline{q^2} \in L^r([0,\infty)\times\mathbb{S})$ such that*

$$\begin{aligned} &q_{\varepsilon_j} \rightharpoonup q \quad \textit{in } L^p\big([0,\infty)\times\mathbb{S}\big),\\ &q_{\varepsilon_j} \overset{\star}{\rightharpoonup} q \quad \textit{in } L^\infty\big([0,\infty);L^2\big),\\ &q_{\varepsilon_j}^2 \rightharpoonup \overline{q^2} \quad \textit{in } L^r\big([0,\infty)\times\mathbb{S}\big).\end{aligned} \tag{4.16}$$

Moreover,

$$q^2(t,x) \le \overline{q^2}(t,x), \quad \textit{for a.e. } (t,x) \in [0,\infty)\times\mathbb{S}, \tag{4.17}$$

$$\frac{\partial u}{\partial x} = q \quad \textit{in the sense of distributions on } [0,\infty)\times\mathbb{S}. \tag{4.18}$$

Proof of Lemma 4.6 Using Lemmas 4.1 and 4.2, we obtain (4.16) immediately. From (4.16), we get (4.17). Finally, (4.18) is a consequence of the definition of q_ε, (4.16), and Lemma 4.4. □

We denote the sequences $\{u_{\varepsilon_j}\}_{j\in\mathbb{N}^*}$, $\{q_{\varepsilon_j}\}_{j\in\mathbb{N}^*}$, and $\{Q_{\varepsilon_j}\}_{j\in\mathbb{N}^*}$ by $\{u_\varepsilon\}_{\varepsilon>0}$, $\{q_\varepsilon\}_{\varepsilon>0}$, and $\{Q_\varepsilon\}_{\varepsilon>0}$, respectively. Let $\eta \in C^1$ be convex and η' be Lipschitz continuous on $\mathbb{R}$. Using (4.16), we get

$$\eta(q_\varepsilon) \rightharpoonup \overline{\eta(q)} \quad \text{in } L^p\big([0,\infty)\times\mathbb{S}\big)\ (1 < p < 3), \tag{4.19}$$

$$\eta(q_\varepsilon) \overset{\star}{\rightharpoonup} \overline{\eta(q)} \quad \text{in } L^\infty\big([0,\infty);L^2\big). \tag{4.20}$$

Multiplying (4.3) by $\eta'(q_\varepsilon)$ yields

$$\begin{aligned}&\frac{\partial}{\partial t}\eta(q_\varepsilon)+\frac{\partial}{\partial x}\big[u_\varepsilon\eta(q_\varepsilon)\big]-q_\varepsilon\eta(q_\varepsilon)+a\partial_x\big[\eta(q_\varepsilon)\big]-\varepsilon\frac{\partial^2}{\partial x^2}\eta(q_\varepsilon)+\varepsilon\eta''(q_\varepsilon)\left(\frac{\partial q_\varepsilon}{\partial x}\right)^2\\&\quad=-\beta q_\varepsilon\eta'(q_\varepsilon)-\frac{1}{2}\eta'(q_\varepsilon)q_\varepsilon^2+Q_\varepsilon(t,x)\eta'(q_\varepsilon).\end{aligned}\tag{4.21}$$

Lemma 4.7 *Let $\eta\in C^1$ be convex and η' be Lipschitz continuous on $\mathbb{R}$. Then we have*

$$\frac{\partial\overline{\eta(q)}}{\partial t}+\frac{\partial}{\partial x}\big((u+a)\overline{\eta(q)}\big)\le\overline{q\eta(q)}-\frac{1}{2}\overline{q^2\eta'(q)}-\beta\overline{q\eta'(q)}+Q(t,x)\overline{\eta'(q)},\tag{4.22}$$

in the sense of distributions on $[0,\infty)\times\mathbb{S}$. Here $\overline{\eta(q)}$, $\overline{q\eta(q)}$, $\overline{q^2\eta'(q)}$, $\overline{q\eta'(q)}$, and $\overline{\eta'(q)}$ denote the weak limits of $\eta(q_\varepsilon)$, $q_\varepsilon\eta(q_\varepsilon)$, $q_\varepsilon^2\eta'(q_\varepsilon)$, $q_\varepsilon\eta'(q_\varepsilon)$, and $\eta'(q_\varepsilon)$ in $L^r([0,\infty)\times\mathbb{S})$ $(1<r<\frac{3}{2})$, respectively.

Proof of Lemma 4.7 Using Lemmas 4.4 and 4.6, the convexity of η, and taking the limits as $\varepsilon\to 0$ in (4.21) give rise to (4.22). □

Remark 4.1 From (4.16), we obtain

$$q=q_+ + q_- = \overline{q_+}+\overline{q_-},\qquad q^2=(q_+)^2+(q_-)^2,\qquad \overline{q^2}=\overline{(q_+)^2}+\overline{(q_-)^2},\tag{4.23}$$

a.e. in $[0,\infty)\times\mathbb{S}$, where $\xi_+=\xi\chi_{[0,\infty)}(\xi)$, $\xi_-=\xi\chi_{(-\infty,0)}(\xi)$ for $\xi\in\mathbb{R}$. From (4.6) and Lemma 4.6, we have

$$q_\varepsilon(t,x),q(t,x)\le\frac{2}{t}+C_1,\quad\text{for all }(t,x)\in[0,\infty)\times\mathbb{S},\tag{4.24}$$

where C_1 is a constant depending only on $\|u_0\|_{H^1}$ and the coefficients in (1.3).

Lemma 4.8 *In the sense of distributions on $[0,\infty)\times\mathbb{S}$, we obtain*

$$\frac{\partial q}{\partial t}+\frac{\partial}{\partial x}\big[(u+a)q\big]+\beta q=\frac{1}{2}\overline{q^2}+Q(t,x).\tag{4.25}$$

Proof of Lemma 4.8 Using Lemmas 4.4, 4.5, and 4.6 and taking the limits as $\varepsilon\to 0$ in (4.3) yield (4.25). □

The following lemma contains a generalized formulation of (4.25).

Lemma 4.9 *Let $\eta\in C^1$. We have*

$$\begin{aligned}&\frac{\partial\eta(q)}{\partial t}+\frac{\partial}{\partial x}\big[(u+a)\eta(q)\big]\\&\quad=q\eta(q)+\left(\frac{1}{2}\overline{q^2}-q^2\right)\eta'(q)-\beta q\eta'(q)+Q(t,x)\eta'(q),\end{aligned}\tag{4.26}$$

in the sense of distributions on $[0,\infty)\times\mathbb{S}$.

Proof of Lemma 4.9 Let $\{w_\delta\}_\delta$ be a family of mollifiers defined on $\mathbb{S}$ and $q_\delta(t,x) = (q(t,\cdot) \star w_\delta(\cdot))(x)$. From (4.25), one derives

$$\begin{aligned} &\frac{\partial \eta(q_\delta)}{\partial t} \\ &\quad = \eta'(q_\delta)\frac{\partial q_\delta}{\partial t} \\ &\quad = \eta'(q_\delta)\left[-(u+a)\frac{\partial q}{\partial x} \star w_\delta - q^2 * w_\delta + \frac{1}{2}\overline{q^2} \star w_\delta - \beta q_\delta + Q(t,x) \star w_\delta + \rho_\delta\right], \end{aligned} \tag{4.27}$$

where $\rho_\delta \to 0$ as $\delta \to 0$ in $L^1([0,\infty) \times \mathbb{S})$. Using the boundedness of η' and letting $\delta \to 0$ in (4.27) yield (4.26). □

Strong convergence of q_ε

Following the ideas in [17] and [19], we improve the weak convergence of q_ε in (4.16) to strong convergence. We prove that if the defect measure of $\overline{q^2} - q^2$ is zero initially, then it continues to be zero at all later times.

Lemma 4.10 [17] *Let $u_0(x) \in H^1$. Then we deduce*

$$\lim_{t\to 0}\int_{\mathbb{S}} q^2(t,x)\,dx = \lim_{t\to 0}\int_{\mathbb{S}} \overline{q^2}(t,x)\,dx = \int_{\mathbb{S}} u_{0,x}^2\,dx \tag{4.28}$$

and

$$\lim_{t\to 0}\int_{\mathbb{S}}\left(\overline{\eta_M^{\pm}(q)}(t,x) - \eta_M^{\pm}(q)(t,x)\right)dx = 0, \tag{4.29}$$

where $\eta_M(\xi) = \frac{1}{2}\xi^2$ if $|\xi| \le M$, $\eta_M(\xi) = M|\xi| - \frac{1}{2}M^2$ if $|\xi| > M$ and $\eta_M^+(\xi) = \eta_M(\xi)\chi_{[0,\infty)}(\xi)$, $\eta_M^-(\xi) = \eta_M(\xi)\chi_{(-\infty,0)}(\xi)$, $\xi \in \mathbb{R}$, $M > 0$.

Lemma 4.11 [17] *Let $M > 0$. Then for all $\xi \in \mathbb{R}$,*

$$\begin{aligned} &\eta_M(\xi) = \frac{1}{2}\xi^2 - \frac{1}{2}\big(M - |\xi|\big)^2\chi_{(-\infty,-M)\cup(M,\infty)}(\xi), \\ &\eta_M'(\xi) = \xi + \big(M - |\xi|\big)\operatorname{sign}(\xi)\chi_{(-\infty,-M)\cup(M,\infty)}(\xi), \\ &\eta_M^+(\xi) = \frac{1}{2}(\xi_+)^2 - \frac{1}{2}(M-\xi)^2\chi_{(M,\infty)}(\xi), \\ &\big(\eta_M^+\big)'(\xi) = \xi_+ + (M-\xi)\chi_{(M,\infty)}(\xi), \\ &\eta_M^-(\xi) = \frac{1}{2}(\xi_-)^2 - \frac{1}{2}(M+\xi)^2\chi_{(-\infty,-M)}(\xi), \\ &\big(\eta_M^-\big)'(\xi) = \xi_- - (M+\xi)\chi_{(-\infty,-M)}(\xi). \end{aligned} \tag{4.30}$$

Lemma 4.12 *Let $u_0(x) \in H^1$. For all $t > 0$, we have*

$$\frac{1}{2}\int_{\mathbb{S}}\big(\overline{(q_+)^2} - (q_+)^2\big)(t,x)\,dx \le \int_0^t\int_{\mathbb{S}} Q(s,x)\big[\overline{q_+}(s,x) - q_+(s,x)\big]\,dx\,ds. \tag{4.31}$$

Proof of Lemma 4.12 Let M be sufficiently large and $0 < t < T$ ($T > 0$). Subtracting (4.26) from (4.22) and using Lemma 4.10 yield

$$\begin{aligned}&\frac{\partial}{\partial t}\big(\overline{\eta_M^+(q)} - \eta_M^+(q)\big) + \frac{\partial}{\partial x}\big((u+a)\big[\overline{\eta_M^+(q)} - \eta_M^+(q)\big]\big)\\&\quad\le \big(\overline{q\eta_M^+(q)} - q\eta_M^+(q)\big) - \frac{1}{2}\big(\overline{q^2\big(\eta_M^+\big)'(q)} - q^2\big(\eta_M^+\big)'(q)\big)\\&\qquad - \frac{1}{2}\big(\overline{q^2} - q^2\big)\big(\eta_M^+\big)'(q) - \beta\big(\overline{q\big(\eta_M^+\big)'(q)} - q\big(\eta_M^+\big)'(q)\big)\\&\qquad + Q(t,x)\big(\overline{\big(\eta_M^+\big)'(q)} - \big(\eta_M^+\big)'(q)\big).\end{aligned}\tag{4.32}$$

Using the increasing property of $\eta_M^+(q)$ and (4.17), we derive

$$-\frac{1}{2}\big(\overline{q^2} - q^2\big)\big(\eta_M^+\big)'(q) \le 0.\tag{4.33}$$

It follows from Lemma 4.11 that

$$q\eta_M^+(q) - \frac{1}{2}q^2\big(\eta_M^+\big)'(q) = -\frac{M}{2}q(M-q)\chi_{(M,\infty)}(q),\tag{4.34}$$

$$\overline{q\eta_M^+(q)} - \frac{1}{2}\overline{q^2\big(\eta_M^+\big)'(q)} = -\frac{M}{2}\overline{q(M-q)\chi_{(M,\infty)}(q)}.\tag{4.35}$$

Let $M > C_1$ and $\Omega_M = (\frac{2}{M-C_1}, \infty) \times \mathbb{S}$. Applying Remark 4.1 and (4.24) gives rise to

$$q\eta_M^+(q) - \frac{1}{2}q^2\big(\eta_M^+\big)'(q) = \overline{q\eta_M^+(q)} - \frac{1}{2}\overline{q^2\big(\eta_M^+\big)'(q)} = 0$$

and

$$\begin{aligned}&\eta_M^+(q) = \frac{(q_+)^2}{2}, \qquad \big(\eta_M^+\big)'(q) = q_+, \qquad \overline{\eta_M^+(q)} = \frac{\overline{(q_+)^2}}{2},\\&\overline{\big(\eta_M^+\big)'(q)} = \overline{q_+}, \qquad \overline{q\big(\eta_M^+\big)'(q)} = \overline{(q_+)^2}.\end{aligned}\tag{4.36}$$

Applying (4.17) yields $\beta(\overline{q(\eta_M^+)'(q)} - q(\eta_M^+)'(q)) \ge 0$. From (4.33)-(4.36), in Ω_M, we have

$$\begin{aligned}&\frac{\partial}{\partial t}\big(\overline{\eta_M^+(q)} - \eta_M^+(q)\big) + \frac{\partial}{\partial x}\big((u+a)\big[\overline{\eta_M^+(q)} - \eta_M^+(q)\big]\big)\\&\quad\le Q(t,x)\big(\overline{\big(\eta_M^+\big)'(q)} - \big(\eta_M^+\big)'(q)\big).\end{aligned}\tag{4.37}$$

Integrating (4.37) over $(\frac{2}{M-C_1}, t) \times \mathbb{S}$ yields

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{S}}\big(\overline{(q_+)^2} - (q_+)^2\big)(t,x)\,dx &\le \int_{\mathbb{S}}\Big[\overline{\eta_M^+(q)}\Big(\frac{2}{M-C_1},x\Big) - \eta_M^+(q)\Big(\frac{2}{M-C_1},x\Big)\Big]\,dx\\&\quad + \int_{\frac{2}{M-C_1}}^{t}\int_{\mathbb{S}} Q(s,x)\big[\overline{q_+}(s,x) - q_+(s,x)\big]\,dx\,ds.\end{aligned}\tag{4.38}$$

Taking $M \to \infty$ in (4.38) and using Lemma 4.10 complete the proof. □

Lemma 4.13 *For all $t > 0$ and $M > 0$, we have*

$$\begin{aligned}
&\int_{\mathbb{S}} \bigl(\overline{\eta_M^-(q)} - \eta_M^-(q)\bigr)(t,x)\,dx \\
&\quad\leq \frac{M^2}{2}\int_0^t \int_{\mathbb{S}} \overline{(M+q)\chi_{(-\infty,-M)}(q)}\,dx\,ds \\
&\qquad - \frac{M^2}{2}\int_0^t \int_{\mathbb{S}} (M+q)\chi_{(-\infty,-M)}(q)\,dx\,ds \\
&\qquad + M\int_0^t \int_{\mathbb{S}} \bigl[\overline{\eta_M^-(q)} - \eta_M^-(q)\bigr]\,dx\,ds + \frac{M}{2}\int_0^t \int_{\mathbb{S}} \bigl(\overline{(q_+)^2} - (q_+)^2\bigr)\,dx\,ds \\
&\qquad + \int_0^t \int_{\mathbb{S}} Q(s,x)\bigl(\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\bigr)\,dx\,ds.
\end{aligned} \tag{4.39}$$

Proof of Lemma 4.13 Subtracting (4.26) from (4.22) and using Lemma 4.10, we deduce

$$\begin{aligned}
&\frac{\partial}{\partial t}\bigl(\overline{\eta_M^-(q)} - \eta_M^-(q)\bigr) + \frac{\partial}{\partial x}\bigl((u+a)\bigl[\overline{\eta_M^-(q)} - \eta_M^-(q)\bigr]\bigr) \\
&\quad\leq \bigl(\overline{q\eta_M^-(q)} - q\eta_M^-(q)\bigr) - \frac{1}{2}\bigl(\overline{q^2(\eta_M^-)'(q)} - q^2(\eta_M^-)'(q)\bigr) \\
&\qquad - \frac{1}{2}\bigl(\overline{q^2} - q^2\bigr)(\eta_M^-)'(q) - \beta\bigl(\overline{q(\eta_M^-)'(q)} - q(\eta_M^-)'(q)\bigr) \\
&\qquad + Q(t,x)\bigl(\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\bigr).
\end{aligned} \tag{4.40}$$

Since $-M \leq (\eta_M^-)'(q) \leq 0$, we get

$$-\frac{1}{2}\bigl(\overline{q^2} - q^2\bigr)(\eta_M^-)'(q) \leq \frac{M}{2}\bigl(\overline{q^2} - q^2\bigr). \tag{4.41}$$

Using Lemma 4.11 yields

$$q\eta_M^-(q) - \frac{1}{2}q^2(\eta_M^-)'(q) = -\frac{M}{2}q(M+q)\chi_{(-\infty,-M)}(q), \tag{4.42}$$

$$\overline{q\eta_M^-(q)} - \frac{1}{2}\overline{q^2(\eta_M^-)'(q)} = -\frac{M}{2}\overline{q(M+q)\chi_{(-\infty,-M)}(q)}, \tag{4.43}$$

$$\begin{aligned}
\overline{\eta_M^-(q)} - \eta_M^-(q) &= \frac{1}{2}\bigl(\overline{(q_-)^2} - (q_-)^2\bigr) + \frac{1}{2}(M+q)^2\chi_{(-\infty,-M)}(q) \\
&\quad - \frac{1}{2}\overline{(M+q)^2\chi_{(-\infty,-M)}(q)}.
\end{aligned} \tag{4.44}$$

If $-M \leq q < 0$, $q(\eta_M^-)'(q) = (q_-)^2$, then $\beta(\overline{q(\eta_M^-)'(q)} - q(\eta_M^-)'(q)) \geq 0$. If $q < -M$, $q(\eta_M^-)'(q) = -q_-M$, then $\beta(\overline{q(\eta_M^-)'(q)} - q(\eta_M^-)'(q)) = M(\overline{-q_-} - (-q_-)) \geq 0$. From (4.41)-(4.44), we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\bigl(\overline{\eta_M^-(q)} - \eta_M^-(q)\bigr) + \frac{\partial}{\partial x}\bigl((u+a)\bigl[\overline{\eta_M^-(q)} - \eta_M^-(q)\bigr]\bigr) \\
&\quad\leq -\frac{M}{2}\overline{q(M+q)\chi_{(-\infty,-M)}(q)} + \frac{M}{2}q(M+q)\chi_{(-\infty,-M)}(q) \\
&\qquad + \frac{M}{2}\bigl(\overline{q^2} - q^2\bigr) + Q(t,x)\bigl(\overline{(\eta_M^-)'(q)} - (\eta_M^-)'(q)\bigr).
\end{aligned} \tag{4.45}$$

Integrating (4.45) over $[0,t]\times\mathbb{S}$, we obtain

$$\begin{aligned}
&\int_{\mathbb{S}}\bigl(\overline{\eta_M^-(q)}-\eta_M^-(q)\bigr)(t,x)\,dx\\
&\quad\le -\frac{M}{2}\int_0^t\int_{\mathbb{S}}\overline{q(M+q)\chi_{(-\infty,-M)}(q)}\,dx\,ds+\frac{M}{2}\int_0^t\int_{\mathbb{S}}q(M+q)\chi_{(-\infty,-M)}(q)\,dx\,ds\\
&\qquad+\frac{M}{2}\int_0^t\int_{\mathbb{S}}\bigl(\overline{q^2}-q^2\bigr)\,dx\,ds+\int_0^t\int_{\mathbb{S}}Q(t,x)\bigl(\overline{(\eta_M^-)'(q)}-(\eta_M^-)'(q)\bigr)\,dx\,ds.
\end{aligned}\tag{4.46}$$

Hence

$$\begin{aligned}
&\int_{\mathbb{S}}\bigl(\overline{\eta_M^-(q)}-\eta_M^-(q)\bigr)(t,x)\,dx\\
&\quad\le -\frac{M}{2}\int_0^t\int_{\mathbb{S}}\overline{q(M+q)\chi_{(-\infty,-M)}(q)}\,dx\,ds+\frac{M}{2}\int_0^t\int_{\mathbb{S}}q(M+q)\chi_{(-\infty,-M)}(q)\,dx\,ds\\
&\qquad+M\int_0^t\int_{\mathbb{S}}\bigl[\overline{\eta_M^-(q)}-\eta_M^-(q)\bigr]\,dx\,ds+\frac{M}{2}\int_0^t\int_{\mathbb{S}}\overline{(M+q)^2\chi_{(-\infty,-M)}(q)}\,dx\,ds\\
&\qquad-\frac{M}{2}\int_0^t\int_{\mathbb{S}}(M+q)^2\chi_{(-\infty,-M)}(q)\,dx\,ds+\frac{M}{2}\int_0^t\int_{\mathbb{S}}\bigl(\overline{(q_+)^2}-(q_+)^2\bigr)\,dx\,ds\\
&\qquad+\int_0^t\int_{\mathbb{S}}Q(t,x)\bigl(\overline{(\eta_M^-)'(q)}-(\eta_M^-)'(q)\bigr)\,dx\,ds.
\end{aligned}\tag{4.47}$$

Using $M(M+q)^2-Mq(M+q)=M^2(M+q)$, we obtain (4.39). □

Lemma 4.14 *We deduce that*

$$\overline{q^2}=q^2,\quad \textit{for a.e. } (t,x)\in[0,\infty)\times\mathbb{R}.\tag{4.48}$$

Proof of Lemma 4.14 Applying Lemmas 4.12 and 4.13 gives rise to

$$\begin{aligned}
&\int_{\mathbb{S}}\left(\frac{1}{2}\bigl[\overline{(q_+)^2}-(q_+)^2\bigr]+\bigl[\overline{\eta_M^-(q)}-\eta_M^-(q)\bigr]\right)(t,x)\,dx\\
&\quad\le \frac{M^2}{2}\left(\int_0^t\int_{\mathbb{S}}\overline{(M+q)\chi_{(-\infty,-M)}(q)}\,dx\,ds-\frac{M^2}{2}\int_0^t\int_{\mathbb{S}}(M+q)\chi_{(-\infty,-M)}(q)\,dx\,ds\right)\\
&\qquad+M\int_0^t\int_{\mathbb{S}}\bigl[\overline{\eta_M^-(q)}-\eta_M^-(q)\bigr]\,dx\,ds+\frac{M}{2}\int_0^t\int_{\mathbb{S}}\bigl[\overline{(q_+)^2}-(q_+)^2\bigr]\,dx\,ds\\
&\qquad+\int_0^t\int_{\mathbb{S}}Q(s,x)\bigl([\overline{q_+}-q_+]+\bigl[\overline{(\eta_M^-)'(q)}-(\eta_M^-)'(q)\bigr]\bigr)\,dx\,ds.
\end{aligned}\tag{4.49}$$

Bearing in mind $\|Q(t,x)\|_{L^\infty([0,\infty);L^\infty)}\le L$,

$$q_++(\eta_M^-)'(q)=q-(M+q)\chi_{(-\infty,-M)}(q),\tag{4.50}$$

$$\overline{q_+}+\overline{(\eta_M^-)'(q)}=q-\overline{(M+q)\chi_{(-\infty,-M)}(q)},\tag{4.51}$$

and using the convexity of the map $\xi\to\xi_++(\eta_M^-)'(\xi)$, we obtain

$$\begin{aligned}
0&\le[\overline{q_+}-q_+]+\bigl[\overline{(\eta_M^-)'(q)}-(\eta_M^-)'(q)\bigr]\\
&=(M+q)\chi_{(-\infty,-M)}(q)-\overline{(M+q)\chi_{(-\infty,-M)}(q)}.
\end{aligned}\tag{4.52}$$

Thus

$$\begin{aligned}
&Q(s,x)\big([\overline{q_+}-q_+]+\big[\overline{(\eta_M^-)'(q)}-(\eta_M^-)'(q)\big]\big)\\
&\quad\le -L\big(\overline{(M+q)\chi_{(-\infty,-M)}(q)}-(M+q)\chi_{(-\infty,-M)}(q)\big).
\end{aligned}\tag{4.53}$$

Noting that $\xi\to(M+\xi)\chi_{(-\infty,-M)}(\xi)$ is concave and choosing M large enough yield

$$\begin{aligned}
&\frac{M^2}{2}\big(\overline{(M+q)\chi_{(-\infty,-M)}(q)}-(M+q)\chi_{(-\infty,-M)}(q)\big)\\
&\qquad+Q(s,x)\big([\overline{q_+}-q_+]+\big[\overline{(\eta_M^-)'(q)}-(\eta_M^-)'(q)\big]\big)\\
&\quad\le\left(\frac{M^2}{2}-L\right)\big(\overline{(M+q)\chi_{(-\infty,-M)}(q)}-(M+q)\chi_{(-\infty,-M)}(q)\big)\le 0.
\end{aligned}\tag{4.54}$$

From (4.49)-(4.54), we obtain

$$\begin{aligned}
0&\le\int_{\mathbb{S}}\left(\frac{1}{2}\big[\overline{(q_+)^2}-(q_+)^2\big]+\big[\overline{\eta_M^-(q)}-\eta_M^-(q)\big]\right)(t,x)\,dx\\
&\le CM\int_0^t\int_{\mathbb{S}}\left(\frac{1}{2}\big[\overline{(q_+)^2}-(q_+)^2\big]+\big[\overline{\eta_M^-(q)}-\eta_M^-(q)\big]\right)(s,x)\,dx\,ds.
\end{aligned}\tag{4.55}$$

Using the Gronwall inequality and Lemma 4.10, we have

$$0\le\int_{\mathbb{S}}\left(\frac{1}{2}\big[\overline{(q_+)^2}-(q_+)^2\big]+\big[\overline{\eta_M^-(q)}-\eta_M^-(q)\big]\right)(t,x)\,dx=0,\quad\text{for all } t>0.\tag{4.56}$$

Applying the Fatou lemma and taking $M\to\infty$ in (4.56) yield

$$0\le\int_{\mathbb{S}}\big(\overline{q^2}-q^2\big)(t,x)\,dx=0,\quad\text{for all } t>0,\tag{4.57}$$

which completes the proof. □

Proof Theorem 1.4 Using Lemmas 4.1 and 4.4, we deduce that the condition (i) in Definition 1.1 is satisfied. We need to prove the condition (ii) in Definition 1.1. Applying Lemma 4.14 gives rise to

$$q_\varepsilon\to q\quad\text{in } L^2\big([0,\infty)\times\mathbb{S}\big).\tag{4.58}$$

Applying Lemma 4.6 and (4.58), we deduce that $u(t,x)$ is a distributional solution to problem (1.4). Using (3.1) and bearing in mind $\lambda_1=\min\{\lambda,\beta\}$, we have

$$\big\|u(t)\big\|_{H^1}^2\le\|u_0\|_{H^1}^2e^{-2\lambda_1 t},\quad\text{for all } t>0.$$

Using (1.4) and (4.13) yields

$$\big\|u_t(t)\big\|_{L^2}\le C\big(\|u_0\|_{H^1}e^{-\lambda_1 t}+\|u_0\|_{H^1}^2e^{-2\lambda_1 t}\big)\le Ce^{-\lambda_1 t},\quad\text{for all } t>0.$$

Thus, we derive (1.9). This completes the proof of Theorem 1.4. □

4.2 The proof of Theorem 1.5

First, we present two lemmas which are used to prove the uniqueness of weak solutions to problem (1.3).

Lemma 4.15 [20] *Let $u(t,x) \in C(\mathbb{R}^+ \times \mathbb{S}) \cap L^\infty(\mathbb{R}^+; H^1)$ with $\partial_x u \leq \frac{p}{t} + C$, $t > 0$, $0 \leq p < 2$. Then the problem*

$$\begin{cases} \frac{d}{dt}\rho(t,x) = u(t,\rho(t,x)), & t > 0, x \in \mathbb{S}, \\ \rho(t,x)|_{t=0} = x, & x \in \mathbb{S}, \end{cases} \tag{4.59}$$

admits a unique solution $\rho(t,x) \in L^\infty(\mathbb{R}^+; C^{1-\frac{p}{2}})$. Moreover, if $p = 2$ and $\lim_{t\to 0} \|u(t) - u_0\|_{H^1} = 0$, the problem (4.59) *admits a unique solution $\rho(t,x) \in L^\infty(\mathbb{R}^+; C)$.*

Lemma 4.16 [20] *Let $u(t,x)$ satisfy all the conditions in Lemma* 4.15. *Assume $f(t,x) \in L^\infty(\mathbb{R}^+; H^1)$, $g(t,x) \in L^1(\mathbb{R}^+; L^\infty)$ or $f(t,x) \in L^\infty(\mathbb{R}^+; W^{1,1})$, $g(t,x) \in L^1(\mathbb{R}^+; L^p)$ for all $p \geq p_0$ (p_0 is a sufficiently large number) and $\lim_{t\to 0} \|\partial_x u(t) - \partial_x u_0\|_{L^1} = 0$, where*

$$\begin{cases} f_t + u\partial_x f = g, & t > 0, x \in \mathbb{S}, \\ f|_{t=0} = f_0(x), & x \in \mathbb{S}. \end{cases} \tag{4.60}$$

Then we have

$$\|f(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty} + \int_0^t \|g(s)\|_{L^\infty}\, ds. \tag{4.61}$$

Proof of Theorem 1.5 From [15] and the assumption that $m_0 = u_0 - u_{0,xx}$ is a positive Radon measure, we deduce that there exists a weak solution $u(t,x) \in C([0,\infty); H^1)$ to problem (1.3) and $0 \leq m = u - u_{xx}$ is also a Radon measure. Now we use the duality arguments to give a L^∞ boundedness for $\partial_x u(t,x)$. Taking $\phi(x) \in C_c^\infty$ with $\|\phi\|_{L^1} \leq 1$ and using $(1 - \partial_x^2)^{-1}\phi(x) = \int_{\mathbb{S}} g(x-y)\phi(y)\, dy$, we have

$$\begin{aligned} \left| \int_{\mathbb{S}} \partial_x u \phi\, dx \right| &= \left| \int_{\mathbb{S}} u \partial_x \phi\, dx \right| \\ &= \left| \int_{\mathbb{S}} \int_{\mathbb{S}} g(x-y)\, dm(t,y) \partial_x \phi(x)\, dx \right| \\ &= \left| \int_{\mathbb{S}} \int_{\mathbb{S}} g_x(x-y)\phi(x)\, dx\, dm(t,y) \right| \\ &\leq \frac{1}{2} \|\phi\|_{L^1} \int_{\mathbb{S}} dm(t,y), \end{aligned} \tag{4.62}$$

from which one derives

$$\|\partial_x u(t)\|_{L^\infty} \leq \frac{1}{2} \int_{\mathbb{S}} dm_0(y). \tag{4.63}$$

Let u_1, u_2 be two weak solutions to problem (1.4) with the same initial value u_0, $w = u_1 - u_2$, and $m_0 \in M^+(\mathbb{S})$, where $M^+(\mathbb{R}^+ \times \mathbb{S})$ is the nonnegative Radon measure space. Then

$$\begin{cases} \partial_t w + (u_1 + a)\partial_x w = G(t,x), & t > 0, x \in \mathbb{S}, \\ w(t,0) = w(t,1), \quad w_x(t,0) = w_x(t,1), \quad w_{xx}(t,0) = w_{xx}(t,1), & t \geq 0, \\ w(t,x)|_{t=0} = 0, & x \in \mathbb{S}, \end{cases} \tag{4.64}$$

where $G(t,x) = -\partial_x u_2 w - \partial_x(P_1 - P_2) - \partial_x(1-\partial_x^2)^{-1}[(2k+2a)w - \beta w_x] - (1-\partial_x^2)^{-1}(\lambda w)$ and $P_i = (1-\partial_x^2)^{-1}[u_i^2 + \frac{1}{2}u_{i,x}^2]$ $(i = 1,2)$. Using (4.63) and Lemma 4.15, the problem

$$\begin{cases} \frac{d}{dt}\rho(t,x) = u_1(t,\rho(t,x)) + a, \quad t > 0, x \in \mathbb{S}, \\ \rho(t,x)|_{t=0} = x, \quad x \in \mathbb{S}, \end{cases} \tag{4.65}$$

admits a unique solution $\rho(t,x) \in L^\infty(\mathbb{R}^+;\mathrm{Lip})$. Thus, for problem (4.64), using Lemma 4.16 yields

$$\|w\|_{L^\infty} \le \int_0^t \|G(s)\|_{L^\infty}\,ds. \tag{4.66}$$

From the proof of Corollary in [20], we obtain

$$\begin{aligned} &\|-\partial_x u_2 w\|_{L^\infty} \le C\|\partial_x u_2\|_{L^\infty}\|w\|_{L^\infty} \le C\|w\|_{L^\infty}, \\ &\|-\partial_x(P_1 - P_2)\|_{L^\infty} \le C\|w\|_{L^\infty}, \\ &\|\partial_x(1-\partial_x^2)^{-1}w\|_{L^\infty} \le C\|w\|_{L^\infty}, \\ &\|\partial_x(1-\partial_x^2)^{-1}(-\beta w_x)\|_{L^\infty} \le C\|w\|_{L^\infty}, \\ &\|(1-\partial_x^2)^{-1}\lambda w\|_{L^\infty} \le C\|w\|_{L^\infty}, \end{aligned} \tag{4.67}$$

where the constant C depends on $\|m_i\|_{L^\infty([0,t];L^1)}$ and $\|u_i\|_{L^\infty([0,t];H^1)}$, $i = 1,2$. Hence

$$\|w(t)\|_{L^\infty} \le C_4\int_0^t \|w(s)\|_{L^\infty}\,ds. \tag{4.68}$$

Applying the Gronwall inequality to (4.68) yields $w = 0$. This completes the proof of Theorem 1.5. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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