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Existence of two positive solutions for a class of third-order impulsive singular integro-differential equations on the half-line in Banach spaces

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Abstract
In this paper, we discuss the existence of two positive solutions for an infinite boundary value problem of third-order impulsive singular integro-differential equations on the half-line in Banach spaces by means of the fixed point theorem of cone expansion and compression with norm type.

MSC: 45J05; 47H10

Keywords: impulsive singular integro-differential equation in a Banach space; infinite boundary value problem; fixed point theorem of cone expansion and compression with norm type

1 Introduction

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years (see [1–21]). In papers [22] and [23], we have discussed two infinite boundary value problems for nth-order impulsive nonlinear singular integro-differential equations of mixed type on the half-line in Banach spaces. By constructing a bounded closed convex set, apart from the singularities, and using the Schauder fixed point theorem, we obtain the existence of positive solutions for the infinite boundary value problems. But such equations are sublinear, and there are no results on existence of two positive solutions. In a recent paper [24], we discussed the existence of two positive solutions for a class of second order superlinear singular equations by means of different method, that is, by using the fixed point theorem of cone expansion and compression with norm type, which was established by the author in [25] (see also [26–29]). Now, in this paper, we extend the results of [24] to third-order equations in Banach spaces. The difficulty of this extension appears in two sides: we must introduce a new cone such that we can still use the fixed point theorem of cone expansion and compression with norm type, and, on the other hand, we need to introduce a suitable condition to guarantee the compactness of the corresponding operator. In addition, the construction of an example to show the application of our theorem to an infinite system of scalar equations is also difficult.

Let E be a real Banach space, and P be a cone in E that defines a partial ordering in E by $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, where θ denotes the zero element of E, and the smallest N is called the normal constant of P. If $x \leq y$ and $x \neq y$, then we write $x < y$. Let $P_+ = P\backslash\{\theta\}$, that is, $P_+ = \{x \in P : x > \theta\}$. For details on cone theory, see [27].

Consider the infinite boundary value problem (IBVP) for third-order impulsive singular integro-differential equation of mixed type on the half-line in E:

$$\begin{cases} u'''(t) = f(t, u(t), u'(t), u''(t), (Tu)(t), (Su)(t)), \quad \forall t \in J'_+, \\ \Delta u|_{t=t_k} = I_k(u''(t_k^-)) \quad (k = 1, 2, 3, \ldots), \\ \Delta u'|_{t=t_k} = \bar{I}_k(u''(t_k^-)) \quad (k = 1, 2, 3, \ldots), \\ \Delta u''|_{t=t_k} = \tilde{I}_k(u''(t_k^-)) \quad (k = 1, 2, 3, \ldots), \\ u(0) = \theta, \qquad u'(0) = \theta, \qquad u''(\infty) = \beta u''(0), \end{cases} \tag{1}$$

where $J = [0, \infty)$, $J_+ = (0, \infty)$, $0 < t_1 < \cdots < t_k < \cdots$, $t_k \to \infty$, $J'_+ = J_+\backslash\{t_1, \ldots, t_k, \ldots\}$, $f \in C[J_+ \times P_+ \times P_+ \times P_+ \times P \times P, P]$, $I_k, \bar{I}_k, \tilde{I}_k \in C[P_+, P]$ $(k = 1, 2, 3, \ldots)$, $\beta > 1$, $u''(\infty) = \lim_{t\to\infty} u''(t)$, and

$$(Tu)(t) = \int_0^t K(t, s)u(s)\, ds, \qquad (Su)(t) = \int_0^\infty H(t, s)u(s)\, ds, \tag{2}$$

$K \in C[D, J]$, $D = \{(t, s) \in J \times J : t \geq s\}$, $H \in C[J \times J, J]$. $\Delta u|_{t=t_k}$, $\Delta u'|_{t=t_k}$, and $\Delta u''|_{t=t_k}$ denote the jumps of $u(t)$, $u'(t)$, and $u''(t)$ at $t = t_k$, respectively, that is,

$$\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-), \qquad \Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-), \qquad \Delta u''|_{t=t_k} = u''(t_k^+) - u''(t_k^-),$$

where $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively, and $u'(t_k^+)$ $(u''(t_k^+))$ and $u'(t_k^-)$ $(u''(t_k^-))$ represent the right and left limits of $u'(t)$ $(u''(t))$ at $t = t_k$, respectively. In the following, we always assume that

$$\lim_{t\to 0^+} \|f(t, u, v, w, y, z)\| = \infty, \quad \forall u, v, w \in P_+, y, z \in P, \tag{3}$$

$$\lim_{u\in P_+, \|u\|\to 0} \|f(t, u, v, w, y, z)\| = \infty, \quad \forall t \in J_+, v, w \in P_+, y, z \in P, \tag{4}$$

$$\lim_{v\in P_+, \|v\|\to 0} \|f(t, u, v, w, y, z)\| = \infty, \quad \forall t \in J_+, u, w \in P_+, y, z \in P, \tag{5}$$

and

$$\lim_{w\in P_+, \|w\|\to 0} \|f(t, u, v, w, y, z)\| = \infty, \quad \forall t \in J_+, u, v \in P_+, y, z \in P, \tag{6}$$

that is, $f(t, u, v, w, y, z)$ is singular at $t = 0$, $u = \theta$, $v = \theta$, and $w = \theta$. We also assume that

$$\lim_{w\in P_+, \|w\|\to 0} \|I_k(w)\| = \infty \quad (k = 1, 2, 3, \ldots), \tag{7}$$

$$\lim_{w\in P_+, \|w\|\to 0} \|\bar{I}_k(w)\| = \infty \quad (k = 1, 2, 3, \ldots), \tag{8}$$

and

$$\lim_{w\in P_+,\|w\|\to 0}\left\|\tilde{I}_k(w)\right\| = \infty \quad (k = 1,2,3,\ldots), \tag{9}$$

that is, $I_k(w)$, $\bar{I}_k(w)$, and $\tilde{I}_k(w)$ $(k = 1,2,3,\ldots)$ are singular at $w = \theta$. Let $PC[J,E] = \{u : u$ is a map from J into E such that $u(t)$ is continuous at $t \neq t_k$, left-continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1,2,3,\ldots\}$ and $PC^1[J,E] = \{u \in PC[J,E] : u'(t)$ is continuous at $t \neq t_k$, and $u'(t_k^+)$ and $u'(t_k^-)$ exist for $k = 1,2,3,\ldots\}$. Let $u \in PC^1[J,E]$. For $0 < h < t_k - t_{k-1}$, by the mean value theorem ([30], Theorem 1.1.1) we have

$$u(t_k) - u(t_k - h) \in h\overline{co}\{u'(t) : t_k - h < t < t_k\}; \tag{10}$$

hence, it is easy to see that the left derivative of $u(t)$ at $t = t_k$, which is denoted by $u'_-(t_k)$, exists, and

$$u'_-(t_k) = \lim_{h\to 0^+}\frac{u(t_k) - u(t_k - h)}{h} = u'\left(t_k^-\right). \tag{11}$$

In what follows, it is understood that $u'(t_k) = u'_-(t_k)$. So, for $u \in PC^1[J,E]$, we have $u' \in PC[J,E]$. Let $PC^2[J,E] = \{u \in PC[J,E] : u''(t)$ is continuous at $t \neq t_k$, and $u''(t_k^+)$ and $u''(t_k^-)$ exist for $k = 1,2,3,\ldots\}$. For $u \in PC^2[J,E]$, we have

$$u'(t_k - h) = u'(t) + \int_t^{t_k-h} u''(s)\,ds, \quad \forall t_{k-1} < t < t_k - h < t_k\ (h > 0),$$

so, observing the existence of $u''(t_k^-)$ and taking limits as $h \to 0^+$ in this equality, we see that $u'(t_k^-)$ exists and

$$u'\left(t_k^-\right) = u'(t) + \int_t^{t_k} u''(s)\,ds, \quad \forall t_{k-1} < t < t_k.$$

Similarly, we can show that $u'(t_k^+)$ exists. Hence, $u \in PC^1[J,E]$. Consequently, $PC^2[J,E] \subset PC^1[J,E]$. For $u \in PC^2[J,E]$, by using $u'(t)$ and $u''(t)$ instead of $u(t)$ and $u'(t)$ in (10) and (11) we get the conclusion: the left derivative of $u'(t)$ at $t = t_k$, which is denoted by $u''_-(t_k)$, exists, and $u''_-(t_k) = u''(t_k^-)$. In what follows, it is understood that $u''(t_k) = u''_-(t_k)$. Hence, for $u \in PC^2[J,E]$, we have $u' \in PC^1[J,E]$ and $u'' \in PC[J,E]$.

A map $u \in PC^2[J,E] \cap C^3[J'_+,E]$ is called a positive solution of IBVP (1) if $u(t) > \theta$ for $t \in J_+$ and $u(t)$ satisfies (1). Now, we need to introduce a new space $DPC^2[J,E]$ and a new cone Q in it. Let

$$DPC^2[J,E] = \left\{u \in PC^2[J,E] : \sup_{t\in J_+}\frac{\|u(t)\|}{t^2} < \infty, \sup_{t\in J_+}\frac{\|u'(t)\|}{t} < \infty, \sup_{t\in J}\left\|u''(t)\right\| < \infty\right\}.$$

It is easy to see that $DPC^2[J,E]$ is a Banach space with norm

$$\|u\|_D = \max\left\{\|u\|_S, \left\|u'\right\|_T, \left\|u''\right\|_B\right\},$$

where

$$\|u\|_S = \sup_{t\in J_+} \frac{\|u(t)\|}{t^2}, \qquad \|u'\|_T = \sup_{t\in J_+} \frac{\|u'(t)\|}{t}, \qquad \|u''\|_B = \sup_{t\in J} \|u''(t)\|.$$

Let $W = \{u \in DPC^2[J,E] : u(t) \geq \theta, u'(t) \geq \theta, u''(t) \geq \theta, \forall t \in J\}$ and

$$Q = \Big\{ u \in W : \frac{u(t)}{t^2} \geq (2\beta - 1)^{-1} \frac{u(s)}{s^2}, \forall t, s \in J_+; \frac{u'(t)}{t} \geq \beta^{-1} \frac{u'(s)}{s}, \forall t, s \in J_+;$$
$$u''(t) \geq \beta^{-1} u''(s), \forall t, s \in J \Big\}.$$

Obviously, W and Q are two cones in the space $DPC^2[J,E]$, and $Q \subset W$. Let $Q_+ = \{u \in Q : \|u\|_D > 0\}$ and $Q_{pq} = \{u \in Q : p \leq \|u\|_D \leq q\}$ for $q > p > 0$.

2 Several lemmas

In the following, we always assume that the cone P is normal with normal constant N.

Remark 1 For $u \in DPC^2[J,E]$, we have $u(0) = \theta$ and $u'(0) = \theta$. This is clear since $u(0) \neq \theta$ implies

$$\sup_{t\in J_+} \frac{\|u(t)\|}{t^2} = \infty,$$

and $u'(0) \neq \theta$ implies

$$\sup_{t\in J_+} \frac{\|u'(t)\|}{t} = \infty.$$

Lemma 1 *For $u \in Q$, we have*

$$\frac{1}{2} N^{-2} \beta^{-1} (2\beta - 1)^{-1} \|u\|_D \leq \frac{\|u(t)\|}{t^2} \leq \|u\|_D, \quad \forall t \in J_+, \tag{12}$$

$$N^{-2} \beta^{-2} \beta' \|u\|_D \leq \frac{\|u'(t)\|}{t} \leq \|u\|_D, \quad \forall t \in J_+, \tag{13}$$

and

$$N^{-2} \beta^{-2} \beta' \|u\|_D \leq \|u''(t)\| \leq \|u\|_D, \quad \forall t \in J, \tag{14}$$

where

$$\beta' = \min\{2(2\beta - 1)^{-1}, 1\}. \tag{15}$$

Proof The method of the proof is similar to that of Lemma 1 in [24], but it is more complicate. For $u \in Q$, we need to establish six inequalities:

$$\|u\|_S \geq \frac{1}{2} N^{-1} \beta^{-1} \|u''\|_B, \quad \|u''\|_B \geq 2N^{-1} \beta^{-1} (2\beta - 1)^{-1} \|u\|_S, \quad \|u\|_S \geq \frac{1}{2} N^{-1} \beta^{-1} \|u'\|_T,$$

$$\|u'\|_T \geq 2N^{-1}\beta^{-1}(2\beta-1)^{-1}\|u\|_S, \quad \|u'\|_T \geq N^{-1}\beta^{-1}\|u''\|_B, \quad \|u''\|_B \geq N^{-1}\beta^{-1}\|u'\|_T.$$

For example, we establish the first one. By Remark 1, $u(0) = \theta$ and $u'(0) = \theta$, so

$$\frac{u(t_1)}{t_1^2} = t_1^{-2}\int_0^{t_1} u'(s)\,ds = t_1^{-2}\int_0^{t_1} ds \int_0^s u''(r)\,dr.$$

Since

$$u''(r) \geq \beta^{-1}u''(t), \quad \forall r, t \in J,$$

we have

$$\frac{u(t_1)}{t_1^2} \geq t_1^{-2}\beta^{-1}\left(\int_0^{t_1} s\,ds\right)u''(t) = \frac{1}{2}\beta^{-1}u''(t), \quad \forall t \in J,$$

and hence

$$\|u\|_S \geq \frac{1}{2}N^{-1}\beta^{-1}\|u''\|_B.$$

From these six inequalities it is easy to prove inequalities (12)-(14). For example, we prove (12). We have

$$\frac{\|u(t)\|}{t^2} \geq N^{-1}(2\beta-1)^{-1}\frac{\|u(s)\|}{s^2}, \quad \forall t, s \in J_+,$$

so,

$$\frac{\|u(t)\|}{t^2} \geq N^{-1}(2\beta-1)^{-1}\|u\|_S, \quad \forall t \in J_+,$$

and therefore

$$\frac{\|u(t)\|}{t^2} \geq \begin{cases} \frac{1}{2}N^{-2}\beta^{-1}(2\beta-1)^{-1}\|u''\|_B, \\ \frac{1}{2}N^{-2}\beta^{-1}(2\beta-1)^{-1}\|u'\|_T, \end{cases} \quad \forall t \in J_+,$$

and hence, (12) holds. □

Corollary *For $u \in Q_+$, we have $u(t) > \theta$ and $u'(t) > \theta$ for $t \in J_+$ and $u''(t) > \theta$ for $t \in J$.*

This follows from (12)-(14).

Let us list some conditions.

(H_1) $\sup_{t\in J}\int_0^t K(t,s)s^2\,ds < \infty$, $\sup_{t\in J}\int_0^\infty H(t,s)s^2\,ds < \infty$, and

$$\lim_{t'\to t}\int_0^\infty \left|H(t',s) - H(t,s)\right|s^2\,ds = 0, \quad \forall t \in J.$$

In this case, let

$$k^* = \sup_{t\in J}\int_0^t K(t,s)s^2\,ds, \qquad h^* = \sup_{t\in J}\int_0^\infty H(t,s)s^2\,ds.$$

(H_2) There exist $a, b \in C[J_+, J]$, $g \in C[J_+ \times P_+, J]$, and $G \in C[J_+ \times J \times J, J]$ such that

$$\big\|f(t,u,v,w,y,z)\big\| \le a(t)g\big(\|u\|,\|v\|\big) + b(t)G\big(\|w\|,\|y\|,\|z\|\big),$$
$$\forall t \in J_+, u, v, w \in P_+, y, z \in P,$$

and

$$a^*_{p,q} = \int_0^\infty a(t)g_{p,q}(t)\,dt < \infty$$

for any $q \ge p > 0$, where

$$g_{p,q}(t) = \max\left\{g(s_1,s_2) : \frac{1}{2}N^{-2}\beta^{-1}(2\beta-1)^{-1}pt^2 \le s_1 \le qt^2, N^{-2}\beta^{-2}\beta' pt \le s_2 \le qt\right\},$$
$$\forall t \in J_+,$$

and

$$b^* = \int_0^\infty b(t)\,dt < \infty.$$

In this case, let (for $q \ge p > 0$)

$$G_{p,q} = \max\big\{G(x_1,x_2,x_3) : N^{-2}\beta^{-2}\beta' p \le x_1 \le q, 0 \le x_2 \le k^* q, 0 \le x_3 \le h^* q\big\},$$

where β' is defined by (15).

(H_3) $I_k(w) \le t_k \bar{I}_k(w)$ and $\bar{I}_k(w) \le t_k \tilde{I}_k(w)$, $\forall w \in P_+$ ($k = 1,2,3,\ldots$), and there exist $\gamma_k \in J$ ($k = 1,2,3,\ldots$) and $F \in C[J_+, J]$ such that

$$\big\|\tilde{I}_k(w)\big\| \le \gamma_k F\big(\|w\|\big), \quad \forall w \in P_+ \ (k = 1,2,3,\ldots),$$

and

$$\tilde{\gamma} = \sum_{k=1}^\infty t_k^2 \gamma_k < \infty,$$

and, consequently,

$$\gamma^* = \sum_{k=1}^\infty \gamma_k \le t_1^{-2}\tilde{\gamma} < \infty, \qquad \bar{\gamma} = \sum_{k=1}^\infty t_k\gamma_k \le t_1^{-1}\tilde{\gamma} < \infty.$$

In this case, let (for $q \ge p > 0$)

$$N_{p,q} = \max\big\{F(s) : N^{-2}\beta^{-2}\beta' p \le s \le q\big\}.$$

(H_4) For any $t \in J_+$, $r > p > 0$, and $q > 0$, $f(t, P_{pr}, P_{pr}, P_{pr}, P_q, P_q) = \{f(t,u,v,w,y,z) : u, v, w \in P_{pr}, y, z \in P_q\}$, $I_k(P_{pr}) = \{I_k(w) : w \in P_{pr}\}$, $\bar{I}_k(P_{pr}) = \{\bar{I}_k(w) : w \in P_{pr}\}$, and $\tilde{I}_k(P_{pr}) = \{\tilde{I}_k(w) :$

$w \in P_{pr}\}$ $(k = 1,2,3,\ldots)$ are relatively compact in E, where $P_{pr} = \{w \in P : p \leq \|w\| \leq r\}$ and $P_q = \{w \in P : \|w\| \leq q\}$.

(H_5) There exist $w_0 \in P_+$, $c \in C[J_+, J]$, and $\tau \in C[P_+, J]$ such that

$$f(t,u,v,w,y,z) \geq c(t)\tau(w)w_0, \quad \forall t \in J_+, u,v,w \in P_+, y,z \in P,$$

and

$$\frac{\tau(w)}{\|w\|} \to \infty \quad \text{as } w \in P_+, \|w\| \to \infty,$$

and

$$c^* = \int_0^\infty c(t)\,dt < \infty.$$

(H_6) There exist $w_1 \in P_+$, $d \in C[J_+, J]$, and $\sigma \in C[P_+, J]$ such that

$$f(t,u,v,w,y,z) \geq d(t)\sigma(w)w_1, \quad \forall t \in J_+, u,v,w \in P_+, y,z \in P,$$

and

$$\sigma(w) \to \infty \quad \text{as } w \in P_+, \|w\| \to 0,$$

and

$$d^* = \int_0^\infty d(t)\,dt < \infty.$$

Remark 2 It is clear: if condition (H_1) is satisfied, then the operators T and S defined by (2) are bounded linear operators from $DPC^2[J,E]$ into $BC[J,E]$ (the Banach space of all bounded continuous maps from J into E with norm $\|u\|_B = \sup_{t\in J}\|u(t)\|$), and $\|T\| \leq k^*$, $\|S\| \leq h^*$; moreover, we have $T(DPC^2[J,P]) \subset BC[J,P]$ and $S(DPC^2[J,P]) \subset BC[J,P]$, $DPC^2[J,P] = \{u \in DPC^2[J,E] : u(t) \geq \theta, \forall t \in J\}$ and $BP[J,P] = \{u \in BP[J,E] : u(t) \geq \theta, \forall t \in J\}$.

Remark 3 Condition (H_5) means that the function $f(t,u,v,w,y,z)$ is superlinear with respect to w.

Remark 4 Condition (H_6) means that the function $f(t,u,v,w,y,z)$ is singular at $w = \theta$, and it is stronger than (6).

Remark 5 If condition (H_3) is satisfied, then (7) implies (8), and (8) implies (9).

Remark 6 Condition (H_4) is satisfied automatically when E is finite-dimensional.

Remark 7 In what follows, we need the following three formulas (see [6], Lemma 2):

(a) If $u \in PC[J,E] \cap C^1[J'_+, E]$, then

$$u(t) = u(0) + \int_0^t u'(s)\,ds + \sum_{0<t_k<t} \bigl[u\bigl(t_k^+\bigr) - u\bigl(t_k^-\bigr)\bigr], \quad \forall t \in J. \tag{16}$$

(b) If $u \in PC^1[J,E] \cap C^2[J'_+,E]$, then

$$u(t) = u(0) + tu'(0) + \int_0^t (t-s)u''(s)\,ds + \sum_{0<t_k<t} \bigl\{ \bigl[u(t_k^+) - u(t_k^-)\bigr] + (t-t_k)\bigl[u'(t_k^+) - u'(t_k^-)\bigr] \bigr\}, \quad \forall t \in J. \tag{17}$$

(c) If $u \in PC^2[J,E] \cap C^3[J'_+,E]$, then

$$u(t) = u(0) + tu'(0) + \frac{t^2}{2}u''(0) + \frac{1}{2}\int_0^t (t-s)^2 u'''(s)\,ds + \sum_{0<t_k<t} \biggl\{ \bigl[u(t_k^+) - u(t_k^-)\bigr] + (t-t_k)\bigl[u'(t_k^+) - u'(t_k^-)\bigr] + \frac{1}{2}(t-t_k)^2\bigl[u''(t_k^+) - u''(t_k^-)\bigr] \biggr\}, \quad \forall t \in J. \tag{18}$$

We shall reduce IBVP (1) to an impulsive integral equation. To this end, we first consider the operator A defined by

$$\begin{aligned}(Au)(t) = {} & \frac{t^2}{2(\beta-1)} \biggl\{ \int_0^\infty f\bigl(s,u(s),u'(s),u''(s),(Tu)(s),(Su)(s)\bigr)\,ds + \sum_{k=1}^\infty \tilde{I}_k\bigl(u''(t_k^-)\bigr) \biggr\} \\ & + \frac{1}{2}\int_0^t (t-s)^2 f\bigl(s,u(s),u'(s),u''(s),(Tu)(s),(Su)(s)\bigr)\,ds \\ & + \sum_{0<t_k<t} \biggl\{ I_k\bigl(u''(t_k^-)\bigr) + (t-t_k)\bar{I}_k\bigl(u''(t_k^-)\bigr) \\ & + \frac{1}{2}(t-t_k)^2\tilde{I}_k\bigl(u''(t_k^-)\bigr) \biggr\}, \quad \forall t \in J. \end{aligned} \tag{19}$$

In what follows, we write $J_1 = [0,t_1]$, $J_k = (t_{k-1},t_k]$ $(k = 2,3,4,\ldots)$.

Lemma 2 *If conditions* (H_1)-(H_4) *are satisfied, then operator A defined by* (19) *is a continuous operator from Q_+ into Q; moreover, for any $q > p > 0$, $A(Q_{pq})$ is relatively compact.*

Proof Let $u \in Q_+$ and $\|u\|_D = r$. Then $r > 0$, and, by (12)-(14) and Remark 1,

$$\frac{1}{2}N^{-2}\beta^{-1}(2\beta-1)^{-1}rt^2 \le \bigl\|u(t)\bigr\| \le rt^2, \quad \forall t \in J,$$

and

$$N^{-2}\beta^{-2}\beta' rt \le \bigl\|u'(t)\bigr\| \le rt, \qquad N^{-2}\beta^{-2}\beta' r \le \bigl\|u''(t)\bigr\| \le r, \quad \forall t \in J,$$

so, conditions (H_1) and (H_2) imply (for k^*, h^*, $a(t)$, $g_{p,q}(t)$, $a^*_{p,q}$, $G_{p,q}$, $b(t)$, b^*, see conditions (H_1) and (H_2))

$$\bigl\|f\bigl(t,u(t),u'(t),u''(t),(Tu)(t),(Su)(t)\bigr)\bigr\| \le a(t)g_{r,r}(t) + G_{r,r}b(t), \quad \forall t \in J_+, \tag{20}$$

where $g_{r,r}(t)$ and $G^*_{r,r}$ are $g_{p,q}(t)$ and $G^*_{p,q}$ for $p = r$ and $q = r$, respectively. By (20) and condition (H_2) we know that the infinite integral $\int_0^\infty f(t,u(t),u'(t),u''(t),(Tu)(t),(Su)(t))\,dt$ is convergent and

$$\int_0^\infty \left\| f\bigl(t, u(t), u'(t), u''(t), (Tu)(t), (Su)(t)\bigr) \right\| dt \le a_{r,r}^* + G_{r,r} b^*. \tag{21}$$

On the other hand, by condition (H_3) we have

$$\left\| \tilde{I}_k\bigl(u''(t_k^-)\bigr) \right\| \le N_{r,r} \gamma_k \quad (k = 1, 2, 3, \ldots), \tag{22}$$

where $N_{r,r}$ is $N_{p,q}$ for $p = r$ and $q = r$, which implies the convergence of the infinite series $\sum_{k=1}^\infty \tilde{I}_k(u''(t_k^-))$ and

$$\sum_{k=1}^\infty \left\| \tilde{I}_k\bigl(u''(t_k^-)\bigr) \right\| \le N_{r,r} \gamma^*. \tag{23}$$

From (19) we get

$$\frac{(Au)(t)}{t^2} \ge \frac{1}{2(\beta - 1)} \left\{ \int_0^\infty f\bigl(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\bigr) ds + \sum_{k=1}^\infty \tilde{I}_k\bigl(u''(t_k^-)\bigr) \right\}, \quad \forall t \in J_+. \tag{24}$$

Moreover, by condition (H_3) we have

$$\begin{aligned}
&I_k\bigl(u''(t_k^-)\bigr) + (t - t_k)\bar{I}_k\bigl(u''(t_k^-)\bigr) + \frac{1}{2}(t - t_k)^2 \tilde{I}_k\bigl(u''(t_k^-)\bigr) \\
&\quad \le t\bar{I}_k\bigl(u''(t_k^-)\bigr) + \frac{1}{2}(t - t_k)^2 \tilde{I}_k\bigl(u''(t_k^-)\bigr) \\
&\quad \le t t_k \tilde{I}_k\bigl(u''(t_k^-)\bigr) + \frac{1}{2}(t - t_k)^2 \tilde{I}_k\bigl(u''(t_k^-)\bigr) \\
&\quad = \frac{1}{2}\bigl(t^2 + t_k^2\bigr)\tilde{I}_k\bigl(u''(t_k^-)\bigr) \le t^2 \tilde{I}_k\bigl(u''(t_k^-)\bigr), \quad \forall 0 < t_k < t,
\end{aligned}$$

so, (19) gives

$$\begin{aligned}
\frac{(Au)(t)}{t^2} &\le \frac{1}{2(\beta - 1)} \left\{ \int_0^\infty f\bigl(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\bigr) ds \right. \\
&\quad \left. + \sum_{k=1}^\infty \tilde{I}_k\bigl(u''(t_k^-)\bigr) \right\} + \frac{1}{2} \int_0^\infty f\bigl(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\bigr) ds \\
&\quad + \sum_{k=1}^\infty \tilde{I}_k\bigl(u''(t_k^-)\bigr) \\
&\le \frac{1}{2(\beta - 1)} \left\{ \int_0^\infty f\bigl(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\bigr) ds + \sum_{k=1}^\infty \tilde{I}_k\bigl(u''(t_k^-)\bigr) \right\} \\
&\quad + \int_0^\infty f\bigl(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\bigr) ds + \sum_{k=1}^\infty \tilde{I}_k\bigl(u''(t_k^-)\bigr)
\end{aligned}$$

$$= \frac{2\beta - 1}{2(\beta - 1)} \left\{ \int_0^\infty f\big(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\big)\, ds + \sum_{k=1}^{\infty} \tilde{I}_k\big(u''\big(t_k^-\big)\big) \right\}, \quad \forall t \in J_+. \tag{25}$$

On the other hand, by (19) we have

$$\begin{aligned}(Au)'(t) = {} & \frac{t}{\beta - 1} \left\{ \int_0^\infty f\big(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\big)\, ds + \sum_{k=1}^{\infty} \tilde{I}_k\big(u''\big(t_k^-\big)\big) \right\} \\ & + \int_0^t (t - s) f\big(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\big)\, ds + \sum_{0<t_k<t} \big\{\bar{I}_k\big(u''\big(t_k^-\big)\big) \\ & + (t - t_k)\tilde{I}_k\big(u''\big(t_k^-\big)\big)\big\}, \quad \forall t \in J,\end{aligned} \tag{26}$$

so,

$$\frac{(Au)'(t)}{t} \geq \frac{1}{\beta - 1} \left\{ \int_0^\infty f\big(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\big)\, ds + \sum_{k=1}^{\infty} \tilde{I}_k\big(u''\big(t_k^-\big)\big) \right\}, \quad \forall t \in J_+, \tag{27}$$

and, by condition (H_3),

$$\begin{aligned}\frac{(Au)'(t)}{t} \leq {} & \frac{1}{\beta - 1} \left\{ \int_0^\infty f\big(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\big)\, ds + \sum_{k=1}^{\infty} \tilde{I}_k\big(u''\big(t_k^-\big)\big) \right\} \\ & + \int_0^\infty f\big(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\big)\, ds + \sum_{k=1}^{\infty} \tilde{I}_k\big(u''\big(t_k^-\big)\big) \\ = {} & \frac{\beta}{\beta - 1} \left\{ \int_0^\infty f\big(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\big)\, ds + \sum_{k=1}^{\infty} \tilde{I}_k\big(u''\big(t_k^-\big)\big) \right\}, \\ & \forall t \in J_+.\end{aligned} \tag{28}$$

In addition, (26) gives

$$\begin{aligned}(Au)''(t) = {} & \frac{1}{\beta - 1} \left\{ \int_0^\infty f\big(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\big)\, ds + \sum_{k=1}^{\infty} \tilde{I}_k\big(u''\big(t_k^-\big)\big) \right\} \\ & + \int_0^t f\big(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\big)\, ds + \sum_{0<t_k<t} \tilde{I}_k\big(u''\big(t_k^-\big)\big), \\ & \forall t \in J,\end{aligned} \tag{29}$$

so,

$$(Au)''(t) \geq \frac{1}{\beta - 1} \left\{ \int_0^\infty f\big(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s)\big)\, ds + \sum_{k=1}^{\infty} \tilde{I}_k\big(u''\big(t_k^-\big)\big) \right\}, \quad \forall t \in J, \tag{30}$$

and

$$
\begin{aligned}
(Au)''(t) \le{}& \frac{1}{\beta-1}\left\{\int_0^{\infty} f\bigl(s,u(s),u'(s),u''(s),(Tu)(s),(Su)(s)\bigr)\,ds + \sum_{k=1}^{\infty}\tilde{I}_k\bigl(u''\bigl(t_k^-\bigr)\bigr)\right\} \\
&+ \int_0^{\infty} f\bigl(s,u(s),u'(s),u''(s),(Tu)(s),(Su)(s)\bigr)\,ds + \sum_{k=1}^{\infty}\tilde{I}_k\bigl(u''\bigl(t_k^-\bigr)\bigr) \\
={}& \frac{\beta}{\beta-1}\left\{\int_0^{\infty} f\bigl(s,u(s),u'(s),u''(s),(Tu)(s),(Su)(s)\bigr)\,ds + \sum_{k=1}^{\infty}\tilde{I}_k\bigl(u''\bigl(t_k^-\bigr)\bigr)\right\}, \\
&\forall t\in J. \qquad (31)
\end{aligned}
$$

It follows from (25), (28), (31), (21), and (23) that $Au\in DPC^2[J,E]$ and

$$\|Au\|_S \le \frac{N(2\beta-1)}{2(\beta-1)}\bigl(a_{r,r}^* + G_{r,r}b^* + N_{r,r}\gamma^*\bigr), \qquad (32)$$

$$\bigl\|(Au)'\bigr\|_T \le \frac{N\beta}{\beta-1}\bigl(a_{r,r}^* + G_{r,r}b^* + N_{r,r}\gamma^*\bigr), \qquad (33)$$

$$\bigl\|(Au)''\bigr\|_B \le \frac{N\beta}{\beta-1}\bigl(a_{r,r}^* + G_{r,r}b^* + N_{r,r}\gamma^*\bigr). \qquad (34)$$

Moreover, (24), (25), (27), (28), (30), and (31) imply

$$\frac{(Au)(t)}{t^2} \ge (2\beta-1)^{-1}\frac{(Au)(s)}{s^2}, \qquad \frac{(Au)'(t)}{t} \ge \beta^{-1}\frac{(Au)'(s)}{s}, \quad \forall t,s\in J_+,$$

and

$$(Au)''(t) \ge \beta^{-1}(Au)''(s), \quad \forall t,s\in J,$$

and hence, $Au\in Q$. Thus, we have proved that A maps Q_+ into Q.

Now, we are going to show that A is continuous. Let $u_n,\bar{u}\in Q_+$ and $\|u_n-\bar{u}\|_D\to 0$ ($n\to\infty$). Write $\|\bar{u}\|_D = 2\bar{r}$ ($\bar{r}>0$), and we may assume that $\bar{r}\le\|u_n\|_D\le 3\bar{r}$ ($n=1,2,3,\ldots$). So, (12)-(14) imply

$$
\begin{aligned}
&\frac{1}{2}N^{-2}\beta^{-1}(2\beta-1)^{-1}\bar{r}t^2 \le \bigl\|u_n(t)\bigr\| \le 3\bar{r}t^2, \qquad \frac{1}{2}N^{-2}\beta^{-1}(2\beta-1)^{-1}\bar{r}t^2 \le \bigl\|\bar{u}(t)\bigr\| \le 3\bar{r}t^2, \\
&\quad \forall t\in J\ (n=1,2,3,\ldots), \qquad (35) \\
&N^{-2}\beta^{-2}\beta'\bar{r}t \le \bigl\|u_n'(t)\bigr\| \le 3\bar{r}t, \qquad N^{-2}\beta^{-2}\beta'\bar{r}t \le \bigl\|\bar{u}'(t)\bigr\| \le 3\bar{r}t, \\
&\quad \forall t\in J\ (n=1,2,3,\ldots), \qquad (36)
\end{aligned}
$$

and

$$
\begin{aligned}
&N^{-2}\beta^{-2}\beta'\bar{r} \le \bigl\|u_n''(t)\bigr\| \le 3\bar{r}, \qquad N^{-2}\beta^{-2}\beta'\bar{r} \le \bigl\|\bar{u}''(t)\bigr\| \le 3\bar{r}, \\
&\quad \forall t\in J\ (n=1,2,3,\ldots). \qquad (37)
\end{aligned}
$$

By (19) we have

$$\begin{aligned}
&\frac{\|(Au_n)(t)-(A\bar{u})(t)\|}{t^2} \\
&\quad\leq \frac{1}{2(\beta-1)}\left\{\int_0^\infty \big\|f\big(s,u_n(s),u_n'(s),u''(s),(Tu_n)(s),(Su_n)(s)\big)\right. \\
&\qquad\left.-f\big(s,\bar{u}(s),\bar{u}'(s),\bar{u}''(s),(T\bar{u})(s),(S\bar{u})(s)\big)\big\|\,ds+\sum_{k=1}^{\infty}\big\|\tilde{I}_k\big(u_n''\big(t_k^-\big)\big)-\tilde{I}_k\big(\bar{u}''\big(t_k^-\big)\big)\big\|\right\} \\
&\qquad+\frac{1}{2}\int_0^t \big\|f\big(s,u_n(s),u_n'(s),u_n''(s),\big(Tu_n(s),(Su_n)(s)\big) \\
&\qquad-f\big(s,\bar{u}(s),\bar{u}'(s),\bar{u}''(s),(T\bar{u})(s),(T\bar{u})(s)\big)\big\|\,ds \\
&\qquad+\frac{1}{t^2}\sum_{0<t_k<t}\big\|I_k\big(u_n''\big(t_k^-\big)\big)-I_k\big(\bar{u}''\big(t_k^-\big)\big)\big\|+\frac{1}{t}\sum_{0<t_k<t}\big\|\bar{I}_k\big(u_n''\big(t_k^-\big)\big)-\bar{I}_k\big(\bar{u}''\big(t_k^-\big)\big)\big\| \\
&\qquad+\frac{1}{2}\sum_{0<t_k<t}\big\|\tilde{I}_k\big(u_n''\big(t_k^-\big)\big)-\tilde{I}_k\big(\bar{u}''\big(t_k^-\big)\big)\big\|,\quad \forall t\in J_+\ (n=1,2,3,\ldots).
\end{aligned}\tag{38}$$

When $0<t\leq t_1$, we have

$$\sum_{0<t_k<t}\big\|I_k\big(u_n''\big(t_k^-\big)\big)-I_k\big(\bar{u}''\big(t_k^-\big)\big)\big\|=0,\qquad \sum_{0<t_k<t}\big\|\bar{I}_k\big(u_n''\big(t_k^-\big)\big)-\bar{I}_k\big(\bar{u}''\big(t_k^-\big)\big)\big\|=0,$$

so,

$$\begin{aligned}
\sup_{t\in J_+}\frac{1}{t^2}\sum_{0<t_k<t}\big\|I_k\big(u_n''\big(t_k^-\big)\big)-I_k\big(\bar{u}''\big(t_k^-\big)\big)\big\| &= \sup_{t_1<t<\infty}\frac{1}{t^2}\sum_{0<t_k<t}\big\|I_k\big(u_n''\big(t_k^-\big)\big)-I_k\big(\bar{u}''\big(t_k^-\big)\big)\big\| \\
&\leq \frac{1}{t_1^2}\sum_{k=1}^{\infty}\big\|I_k\big(u_n''\big(t_k^-\big)\big)-I_k\big(\bar{u}''\big(t_k^-\big)\big)\big\|
\end{aligned}\tag{39}$$

and

$$\begin{aligned}
\sup_{t\in J_+}\frac{1}{t}\sum_{0<t_k<t}\big\|\bar{I}_k\big(u_n''\big(t_k^-\big)\big)-\bar{I}_k\big(\bar{u}''\big(t_k^-\big)\big)\big\| &= \sup_{t_1<t<\infty}\frac{1}{t}\big\|\bar{I}_k\big(u_n''\big(t_k^-\big)\big)-\bar{I}_k\big(\bar{u}''\big(t_k^-\big)\big)\big\| \\
&\leq \frac{1}{t_1}\sum_{k=1}^{\infty}\big\|\bar{I}_k\big(u_n''\big(t_k^-\big)\big)-\bar{I}_k\big(\bar{u}''\big(t_k^-\big)\big)\big\|.
\end{aligned}\tag{40}$$

It follows from (38)-(40) that

$$\begin{aligned}
\|Au_n-A\bar{u}\|_S = \sup_{t\in J_+}\frac{\|(Au_n)(t)-(A\bar{u})(t)\|}{t^2} &\leq \frac{1}{t_1^2}\sum_{k=1}^{\infty}\big\|I_k\big(u_n''\big(t_k^-\big)\big)-I_k\big(\bar{u}''\big(t_k^-\big)\big)\big\| \\
&\quad+\frac{1}{t_1}\sum_{k=1}^{\infty}\big\|\bar{I}_k\big(u_n''\big(t_k^-\big)\big)-\bar{I}_k\big(\bar{u}''\big(t_k^-\big)\big)\big\| \\
&\quad+\frac{\beta}{2(\beta-1)}\left\{\int_0^\infty \big\|f\big(s,u_n(s),u_n'(s),u_n''(s),\right.
\end{aligned}$$

$$(Tu_n)(s),(Su_n)(s)\big)-f\big(s,\bar{u}(s),\bar{u}'(s),\bar{u}''(s),(T\bar{u})(s),(S\bar{u})(s)\big)\big\|\,ds \\ +\sum_{k=1}^{\infty}\big\|\tilde{I}_k\big(u_n''(t_k^-)\big)-\tilde{I}_k\big(\bar{u}''(t_k^-)\big)\big\|\Bigg\}\quad (n=1,2,3,\ldots). \tag{41}$$

It is clear that

$$f\big(t,u_n(t),u_n'(t),u_n''(t),(Tu_n)(t),(Su_n)(t)\big)\to f\big(t,\bar{u}(t),\bar{u}'(t),\bar{u}''(t),(T\bar{u})(t),(S\bar{u})(t)\big) \\ \text{as } n\to\infty, \forall t\in J_+,$$

and, similarly to (20) and observing (35)-(37), we have

$$\big\|f\big(t,u_n(t),u_n'(t),u_n''(t),(Tu_n)(t),(Su_n)(t)\big)-f\big(t,\bar{u}(t),\bar{u}'(t),\bar{u}''(t),(T\bar{u})(t),(S\bar{u})(t)\big)\big\| \\ \le 2\big[a(t)g_{\bar{r},3\bar{r}}(t)+G_{\bar{r},3\bar{r}}b(t)\big]=\sigma(t),\quad \forall t\in J_+\ (n=1,2,3,\ldots),$$

and by condition (H_2) we see that $\int_0^\infty \sigma(t)\,dt<\infty$. Hence, the dominated convergence theorem implies

$$\lim_{n\to\infty}\int_0^\infty \big\|f\big(t,u_n(t),u_n'(t),u_n''(t),(Tu_n)(t),(Su_n)(t)\big) \\ -f\big(t,\bar{u}(t),\bar{u}'(t),\bar{u}''(t),(T\bar{u})(t),(S\bar{u})(t)\big)\big\|\,dt=0. \tag{42}$$

On the other hand, similarly to (22) and observing (37), we have

$$\big\|\tilde{I}_k\big(u_n''(t_k^-)\big)\big\|\le N_{\bar{r},3\bar{r}}\gamma_k,\qquad \big\|\tilde{I}_k\big(\bar{u}''(t_k^-)\big)\big\|\le N_{\bar{r},3\bar{r}}\gamma_k\quad (k,n=1,2,3,\ldots). \tag{43}$$

By (43) and condition (H_3), using a similar method in the proof of Lemma 2 of [24], we can prove

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\big\|I_k\big(u_n''(t_k^-)\big)-I_k\big(\bar{u}''(t_k^-)\big)\big\|=0,\quad \lim_{n\to\infty}\sum_{k=1}^{\infty}\big\|\bar{I}_k\big(u_n''(t_k^-)\big)-\bar{I}_k\big(\bar{u}''(t_k^-)\big)\big\|=0 \tag{44}$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\big\|\tilde{I}_k\big(u_n''(t_k^-)\big)-\tilde{I}_k\big(\bar{u}''(t_k^-)\big)\big\|=0. \tag{45}$$

It follows from (41), (42), (44), and (45) that

$$\lim_{n\to\infty}\|Au_n-A\bar{u}\|_S=0. \tag{46}$$

On the other hand, similarly to (41) and observing (26) and (29), we easily get

$$\big\|(Au_n)'-(A\bar{u})'\big\|_T\le\frac{1}{t_1}\sum_{k=1}^{\infty}\big\|\bar{I}_k\big(u_n''(t_k^-)\big)-\bar{I}_k\big(\bar{u}''(t_k^-)\big)\big\|+\frac{\beta}{\beta-1}\Bigg\{\int_0^\infty\big\|f\big(s,u_n(s), \\ u_n'(s),u_n''(s),(Tu_n)(s),(Su_n)(s)\big)$$

$$- t\big(s,\bar{u}(s),\bar{u}'(s),\bar{u}''(s),(T\bar{u})(s),(S\bar{u})(s)\big)\big\| \, ds$$

$$+ \sum_{k=1}^{\infty}\big\|\tilde{I}_k\big(u_n''(t_k^-)\big) - \tilde{I}_k\big(\bar{u}''(t_k^-)\big)\big\|\Bigg\} \quad (n = 1,2,3,\ldots), \tag{47}$$

and

$$\big\|(Au_n)'' - (A\bar{u})''\big\|_B \le \frac{\beta}{\beta-1}\Bigg\{\int_0^{\infty}\big\|f\big(s,u_n(s),u_n'(s),u_n''(s),(Tu_n)(s),(Su_n)(s)\big)$$

$$- f\big(s,\bar{u}(s),\bar{u}'(s),\bar{u}''(s),(T\bar{u})(s),(S\bar{u})(s)\big)\big\| \, ds$$

$$+ \sum_{k=1}^{\infty}\big\|\tilde{I}_k\big(u_n''(t_k^-)\big) - \tilde{I}_k\big(\bar{u}''(t_k^-)\big)\big\|\Bigg\}. \tag{48}$$

So, (47), (48) and (42), (44), (45) imply

$$\lim_{n\to\infty}\big\|(Au_n)' - (A\bar{u})'\big\|_T = 0 \tag{49}$$

and

$$\lim_{n\to\infty}\big\|(Au_n)'' - (A\bar{u})''\big\|_B = 0. \tag{50}$$

It follows from (46), (49), and (50) that $\|Au_n - A\bar{u}\|_D \to 0$ as $n \to \infty$, and the continuity of A is proved.

Finally, we prove that $A(Q_{pq})$ is relatively compact, where $q > p > 0$ are arbitrarily given. Let $\bar{u}_n \in Q_{pq}$ ($n = 1,2,3,\ldots$). Then, by (12)-(14) and Remark 1, we have

$$\begin{aligned}&\frac{1}{2}N^{-2}\beta^{-1}(2\beta-1)^{-1}pt^2 \le \big\|\bar{u}_n(t)\big\| \le qt^2, \qquad N^{-2}\beta^{-2}\beta' pt \le \big\|\bar{u}_n'(t)\big\| \le qt,\\&N^{-2}\beta^{-2}\beta' p \le \big\|\bar{u}_n''(t)\big\| \le q, \quad \forall t \in J \ (n = 1,2,3,\ldots).\end{aligned} \tag{51}$$

Similarly to (20), (22), (32)-(34) and observing (51), we get

$$\big\|f\big(t,\bar{u}_n(t),\bar{u}_n'(t),\bar{u}_n''(t),(T\bar{u}_n)(t),(S\bar{u}_n)(t)\big)\big\| \le a(t)g_{p,q}(t) + G_{p,q}b(t),$$

$$\forall t \in J_+ \ (n = 1,2,3,\ldots), \tag{52}$$

$$\big\|\tilde{I}_k\big(\bar{u}_n''(t_k^-)\big)\big\| \le N_{p,q}\gamma_k \quad (k,n = 1,2,3,\ldots), \tag{53}$$

$$\|A\bar{u}_n\|_S \le \frac{N(2\beta-1)}{2(\beta-1)}\big(a_{p,q}^* + G_{p,q}b^* + N_{p,q}\gamma^*\big) \quad (n = 1,2,3,\ldots), \tag{54}$$

$$\big\|(A\bar{u}_n)'\big\|_T \le \frac{N\beta}{\beta-1}\big(a_{p,q}^* + G_{p,q}b^* + N_{p,q}\gamma^*\big) \quad (n = 1,2,3,\ldots), \tag{55}$$

and

$$\big\|(A\bar{u}_n)''\big\|_B \le \frac{N\beta}{\beta-1}\big(a_{p,q}^* + G_{p,q}b^* + N_{p,q}\gamma^*\big) \quad (n = 1,2,3,\ldots). \tag{56}$$

From (54) we see that the functions $\{(A\bar{u}_n)(t)\}$ $(n = 1, 2, 3, \ldots)$ are uniformly bounded on $[0, r]$ for any $r > 0$. Consider $J_i = (t_{i-1}, t_i]$ for any fixed i. By (19) we have

$$\begin{aligned}
&(A\bar{u}_n)(t') - (A\bar{u}_n)(t) \\
&\quad = \frac{(t')^2 - t^2}{2(\beta - 1)} \left\{ \int_0^\infty f(s, \bar{u}_n(s), \bar{u}_n'(s), \bar{u}_n''(s), (T\bar{u}_n(s), (S\bar{u}_n(s)) \, ds + \sum_{k=1}^\infty \tilde{I}_k(\bar{u}_n''(t_k^-)) \right\} \\
&\qquad + \frac{1}{2} \int_0^t \left[(t' - s)^2 - (t - s)^2 \right] f(s, \bar{u}_n(s), \bar{u}_n'(s), \bar{u}_n''(s), (T\bar{u}_n)(s), (S\bar{u}_n)(s)) \, ds \\
&\qquad + \frac{1}{2} \int_t^{t'} (t' - s)^2 f(s, \bar{u}_n(s), \bar{u}_n'(s), \bar{u}_n''(s), (T\bar{u}_n)(s), (S\bar{u}_n)(s)) \, ds \\
&\qquad + \sum_{k=1}^{i-1} \left\{ (t' - t) \bar{I}_k(\bar{u}_n''(t_k^-)) + \frac{1}{2} \left[(t' - t_k)^2 - (t - t_k)^2 \right] \tilde{I}_k(\bar{u}''(t_k^-)) \right\}, \\
&\qquad \forall t, t' \in J_i, t' > t \; (n = 1, 2, 3, \ldots). \qquad (57)
\end{aligned}$$

Since

$$\begin{aligned}
&(t' - s)^2 - (t - s)^2 = (t' - t)(t' + t - 2s) \leq (t' - t)(t' + t) = (t')^2 - t^2, \\
&\quad \forall t, t' \in J_i, t' > t, 0 \leq s \leq t
\end{aligned}$$

and, similarly,

$$(t' - t_k)^2 - (t - t_k)^2 \leq (t')^2 - t^2, \quad \forall t, t' \in J_i, t' > t \; (k = 1, 2, \ldots, i - 1),$$

(57) implies

$$\begin{aligned}
&\theta \leq (A\bar{u}_n)(t') - (A\bar{u}_n)(t) \\
&\quad \leq \frac{(t')^2 - t^2}{2(\beta - 1)} \left\{ \int_0^\infty f(s, \bar{u}_n(s), \bar{u}_n'(s), \bar{u}_n''(s), (T\bar{u}_n)(s), (S\bar{u}_n)(s)) \, ds + \sum_{k=1}^\infty \tilde{I}_k(\bar{u}_n''(t_k^-)) \right\} \\
&\qquad + \frac{1}{2} \left[(t')^2 - t^2 + (t' - t)^2 \right] \int_0^\infty f(s, \bar{u}_n(s), \bar{u}_n'(s), \bar{u}_n''(s), (T\bar{u}_n)(s), (S\bar{u}_n)(s)) \, ds \\
&\qquad + (t' - t) \sum_{k=1}^\infty \bar{I}_k(\bar{u}_n''(t_k^-)) + \frac{1}{2} \left[(t')^2 - t^2 \right] \sum_{k=1}^\infty \tilde{I}_k(\bar{u}_n''(t_k^-)), \\
&\qquad \forall t, t' \in J_i, t' > t \; (n = 1, 2, 3, \ldots), \qquad (58)
\end{aligned}$$

and, consequently, by (52), (53), condition (H_3), and (58) we get

$$\begin{aligned}
&\left\| (A\bar{u}_n)(t') - (A\bar{u}_n)(t) \right\| \\
&\quad \leq \frac{N[(t')^2 - t^2]}{2(\beta - 1)} \left(a_{p,q}^* + G_{p,q} b^* + N_{p,q} \gamma^* \right) + \frac{N}{2} \left[(t')^2 - t^2 + (t' - t)^2 \right] \left(a_{p,q}^* + G_{p,q} b^* \right) \\
&\qquad + N(t' - t) N_{p,q} \bar{\gamma} + \frac{N}{2} \left[(t')^2 - t^2 \right] N_{p,q} \gamma^*, \quad \forall t, t' \in J_i, t' > t \; (n = 1, 2, 3, \ldots). \qquad (59)
\end{aligned}$$

From (59) we know that the maps $\{w_n(t)\}$ $(n = 1,2,3,\ldots)$ defined by

$$w_n(t) = \begin{cases} (A\bar{u}_n)(t), & \forall t \in J_i = (t_{i-1}, t_i], \\ (A\bar{u}_n)(t_{i-1}^+), & \forall t = t_{i-1} \end{cases} \quad (n = 1,2,3,\ldots) \tag{60}$$

($(A\bar{u}_n)(t_{i-1}^+)$ denotes the right limit of $(A\bar{u}_n)(t)$ at $t = t_{i-1}$) are equicontinuous on $\bar{J}_i = [t_{i-1}, t_i]$. On the other hand, for any $\epsilon > 0$, choose a sufficiently large $\tau > 0$ and a sufficiently large positive integer j such that

$$\int_\tau^\infty a(t)g_{p,q}(t)\,dt + G_{p,q}\int_\tau^\infty b(t)\,dt < \epsilon, \qquad N_{p,q}\sum_{k=j+1}^\infty \gamma_k < \epsilon. \tag{61}$$

We have, by (60), (19), (52), (53), and (61),

$$\begin{aligned} w_n(t) = {} & \frac{t^2}{2(\beta-1)}\Bigg\{\int_0^\tau f\big(s,\bar{u}_n(s),\bar{u}_n'(s),\bar{u}_n''(s),(T\bar{u}_n),(S\bar{u}_n)(s)\big)\,ds + \int_\tau^\infty f\big(s,\bar{u}_n(s),\bar{u}_n'(s), \\ & \bar{u}_n''(s),(T\bar{u}_n)(s),(S\bar{u}_n)(s)\big)\,ds + \sum_{k=1}^j \tilde{I}_k\big(\bar{u}_n''(t_k^-)\big) + \sum_{k=j+1}^\infty \tilde{I}_k\big(\bar{u}_n''(t_k^-)\big)\Bigg\} \\ & + \frac{1}{2}\int_0^t (t-s)^2 f\big(s,\bar{u}_n(s),\bar{u}_n'(s),\bar{u}''(s),(T\bar{u}_n)(s),(S\bar{u}_n)(s)\big)\,ds \\ & + \sum_{k=1}^{i-1}\Big\{I_k\big(\bar{u}_n''(t_k^-)\big) + (t-t_k)\bar{I}_k\big(\bar{u}_n''(t_k^-)\big) + \frac{1}{2}(t-t_k)^2\tilde{I}_k\big(\bar{u}''(t_k^-)\big)\Big\}, \\ & \forall t \in \bar{J}_i \ (n = 1,2,3,\ldots), \end{aligned} \tag{62}$$

and

$$\left\|\int_\tau^\infty f\big(s,\bar{u}_n(s),\bar{u}_n'(s),\bar{u}_n''(s),(T\bar{u}_n)(s),(S\bar{u}_n)(s)\big)\,ds\right\| < \epsilon \quad (n = 1,2,3,\ldots), \tag{63}$$

$$\left\|\sum_{k=j+1}^\infty \tilde{I}_k\big(\bar{u}_n''(t_k^-)\big)\right\| < \epsilon \quad (n = 1,2,3,\ldots). \tag{64}$$

It follows from (62)-(64) and [30], Theorem 1.2.3, that

$$\begin{aligned} \alpha\big(W(t)\big) \le {} & \frac{t^2}{2(\beta-1)}\Bigg\{2\int_0^\tau \alpha\big(f\big(s,V(s),V'(s),V''(s),(TV)(s),(SV)(s)\big)\big)\,ds + 2\epsilon \\ & + \sum_{k=1}^j \alpha\big(\tilde{I}_k\big(V''(t_k^-)\big)\big) + 2\epsilon\Bigg\} \\ & + \int_0^t (t-s)^2\alpha\big(f\big(s,V(s),V'(s),V''(s),(TV)(s),(SV)(s)\big)\big)\,ds \\ & + \sum_{k=1}^{i-1}\Big\{\alpha\big(I_k\big(V''(t_k^-)\big)\big) + (t-t_k)\alpha\big(\bar{I}_k\big(V''(t_k^-)\big)\big) \\ & + \frac{1}{2}(t-t_k)^2\alpha\big(\tilde{I}_k\big(V''(t_k^-)\big)\big)\Big\}, \quad \forall t \in \bar{J}_i, \end{aligned} \tag{65}$$

where $W(t) = \{w_n(t) : n = 1,2,3,\ldots\}$, $V(s) = \{\bar{u}_n(s) : n = 1,2,3,\ldots\}$, $V'(s) = \{\bar{u}_n'(s) : n = 1,2,3,\ldots\}$, $V''(s) = \{\bar{u}_n''(s) : n = 1,2,3,\ldots\}$, $(TV)(s) = \{(T\bar{u}_n)(s) : n = 1,2,3,\ldots\}$, $(SV)(s) = \{(S\bar{u}_n)(s) : n = 1,2,3,\ldots\}$, and $\alpha(U)$ denotes the Kuratowski measure of noncompactness of a bounded set $U \subset E$ (see [30], Section 1.2). Since $V(s), V'(s), V''(s) \subset P_{p^*r^*}$ for $s \in J_+$ and $(TV)(s) \subset P_{q^*}$, $(SV)(s) \subset P_{q^*}$ for $s \in J$, where $p^* = \min\{\frac{1}{2}N^{-2}\beta^{-1}(2\beta - 1)^{-1}ps^2, N^{-2}\beta^{-2}\beta' ps, N^{-2}\beta^{-2}\beta' p\}$, $r^* = \max\{qs^2, qs, q\}$, and $q^* = \max\{k^*q, h^*q\}$, we see that, by condition (H_4),

$$\alpha\big(f\big(s, V(s), V'(s), V''(s), (TV)(s), (SV)(s)\big)\big) = 0, \quad \forall s \in J_+, \tag{66}$$

and

$$\alpha\big(I_k\big(V''\big(t_k^-\big)\big)\big) = 0, \quad \alpha\big(\bar{I}_k\big(V''\big(t_k^-\big)\big)\big) = 0, \quad \alpha\big(\tilde{I}_k\big(V''\big(t_k^-\big)\big)\big) = 0 \quad (k = 1,2,3,\ldots). \tag{67}$$

It follows from (65)-(67) that

$$\alpha\big(W(t)\big) \le \frac{2t^2\epsilon}{\beta - 1}, \quad \forall t \in \bar{J}_i,$$

which implies by virtue of the arbitrariness of ϵ that $\alpha(W(t)) = 0$ for $t \in \bar{J}_i$. Hence, by Ascoli-Arzela theorem (see [30], Theorem 1.2.5) we conclude that $W = \{w_n : n = 1,2,3,\ldots\}$ is relatively compact in $C[\bar{J}_i, E]$, and therefore, $\{w_n(t)\}$ $(n = 1,2,3,\ldots)$ has a subsequence that is convergent uniformly on $\bar{J}_i$, so, $\{(A\bar{u}_n)(t)\}$ $(n = 1,2,3,\ldots)$ has a subsequence that is convergent uniformly on J_i. Since i may be any positive integer, by the diagonal method, we can choose a subsequence $\{(A\bar{u}_{n_j})(t)\}$ $(j = 1,2,3,\ldots)$ of $\{(A\bar{u}_n)(t)\}$ $(n = 1,2,3,\ldots)$ such that $\{(A\bar{u}_{n_j})(t)\}$ $(j = 1,2,3,\ldots)$ is convergent uniformly on each J_i $(i = 1,2,3,\ldots)$. Let

$$\lim_{j\to\infty} (A\bar{u}_{n_j})(t) = \bar{w}(t), \quad \forall t \in J. \tag{68}$$

By a similar method, we can prove that $\{(A\bar{u}_{n_j})'(t)\}$ $(j = 1,2,3,\ldots)$ has a subsequence that is convergent uniformly on each J_i $(i = 1,2,3,\ldots)$. For simplicity of notation, we may assume that $\{(A\bar{u}_{n_j})'(t)\}$ $(j = 1,2,3,\ldots)$ itself converges uniformly on each J_i $(i = 1,2,3,\ldots)$. Let

$$\lim_{j\to\infty} (A\bar{u}_{n_j})'(t) = \bar{y}(t), \quad \forall t \in J. \tag{69}$$

Again by a similar method, we can prove that $\{(A\bar{u}_{n_j})''(t)\}$ $(j = 1,2,3,\ldots)$ has a subsequence that is convergent uniformly on each J_i $(i = 1,2,3,\ldots)$. Again for simplicity of notation, we may assume that $\{(A\bar{u}_{n_j})''(t)\}$ $(j = 1,2,3,\ldots)$ itself converges uniformly on each J_i $(i = 1,2,3,\ldots)$. Let

$$\lim_{j\to\infty} (A\bar{u}_{n_j})''(t) = \bar{z}(t). \tag{70}$$

By (68)-(70) and the uniformity of convergence we have

$$\bar{w}'(t) = \bar{y}(t), \qquad \bar{y}'(t) = \bar{z}(t), \quad \forall t \in J, \tag{71}$$

and so, $\bar{w} \in PC^2[J,E]$. From (54)-(56) we get

$$\|\bar{w}\|_S \leq \frac{N(2\beta-1)}{2(\beta-1)}\left(a^*_{p,q} + G_{p,q}b^* + N_{p,q}\gamma^*\right)$$

and

$$\left\|\bar{w}'\right\|_T \leq \frac{N\beta}{\beta-1}\left(a^*_{p,q} + G_{p,q}b^* + N_{p,q}\gamma^*\right),$$

$$\left\|\bar{w}''\right\|_B \leq \frac{N\beta}{\beta-1}\left(a^*_{p,q} + G_{p,q}b^* + N_{p,q}\gamma^*\right).$$

Consequently, $\bar{w} \in DPC^2[J,E]$, and

$$\|\bar{w}\|_D \leq \frac{N\beta}{\beta-1}\left(a^*_{p,q} + G_{p,q}b^* + N_{p,q}\gamma^*\right).$$

Let $\epsilon > 0$ be arbitrarily given. Choose a sufficiently large positive number η such that

$$\int_\eta^\infty a(t)g_{p,q}(t)\,dt + G_{p,q}\int_\eta^\infty b(t)\,dt + N_{p,q}\sum_{t_k \geq \eta}\gamma_k < \epsilon. \tag{72}$$

For any $\eta < t < \infty$, we have, by (29),

$$(A\bar{u}_{n_j})''(t) - (A\bar{u}_{n_j})''(\eta) = \int_\eta^t f\left(s, \bar{u}_{n_j}(s), \bar{u}'_{n_j}(s), \bar{u}''_{n_j}(s), (T\bar{u}_{n_j})(s), (S\bar{u}_{n_j})(s)\right) ds + \sum_{\eta \leq t_k < t} \tilde{I}_k\left(\bar{u}''_{n_j}\left(t_k^-\right)\right) \quad (j = 1,2,3,\ldots),$$

so, from (52) and (53) we get

$$\left\|(A\bar{u}_{n_j})''(t) - (A\bar{u}_{n_j})''(\eta)\right\| \leq \int_\eta^t a(s)g_{p,q}(s)\,ds + G_{p,q}\int_\eta^t b(s)\,ds + N_{p,q}\sum_{\eta \leq t_k < t}\gamma_k \quad (j = 1,2,3,\ldots),$$

which implies by (72) that

$$\left\|(A\bar{u}_{n_j})''(t) - (A\bar{u}_{n_j})''(\eta)\right\| < \epsilon, \quad \forall t > \eta\ (j = 1,2,3,\ldots),$$

and, letting $j \to \infty$ and observing (70) and (71), we get

$$\left\|\bar{w}''(t) - \bar{w}''(\eta)\right\| \leq \epsilon, \quad \forall t > \eta.$$

On the other hand, since $\{(A\bar{u}_{n_j})''(t)\}$ converges uniformly to $\bar{w}''(t)$ on $[0,\eta]$ as $j \to \infty$, there exists a positive integer j_0 such that

$$\left\|(A\bar{u}_{n_j})''(t) - \bar{w}''(t)\right\| < \epsilon, \quad \forall t \in [0,\eta], j > j_0,$$

and hence

$$\begin{aligned}\big\|(A\bar{u}_{n_j})''(t)-\bar{w}''(t)\big\| &\le \big\|(A\bar{u}_{n_j})''(t)-(A\bar{u}_{n_j})''(\eta)\big\| + \big\|(A\bar{u}_{n_j})''(\eta)-\bar{w}''(\eta)\big\| \\ &\quad + \big\|\bar{w}''(\eta)-\bar{w}''(t)\big\| < 3\epsilon, \quad \forall t>\eta, j>j_0.\end{aligned}$$

Consequently,

$$\big\|(A\bar{u}_{n_j})''-\bar{w}''\big\|_B \le 3\epsilon, \quad \forall j>j_0,$$

and hence

$$\lim_{j\to\infty}\big\|(A\bar{u}_{n_j})''-\bar{w}''\big\|_B = 0. \tag{73}$$

It is clear that (19) and (26) imply

$$(A\bar{u}_{n_j})\big(t_k^+\big)-(A\bar{u}_{n_j})\big(t_k^-\big) = I_k\big(\bar{u}''_{n_j}\big(t_k^-\big)\big) \quad (k,j=1,2,3,\ldots) \tag{74}$$

and

$$(A\bar{u}_{n_j})'\big(t_k^+\big)-(A\bar{u}_{n_j})'\big(t_k^-\big) = \bar{I}_k\big(\bar{u}''_{n_j}\big(t_k^-\big)\big) \quad (k,j=1,2,3,\ldots). \tag{75}$$

By the uniformity of convergence of $\{(A\bar{u}_{n_j})(t)\}$ and $\{(A\bar{u}_{n_j})'(t)\}$ we see that

$$\lim_{j\to\infty}(A\bar{u}_{n_j})\big(t_k^-\big)=\bar{w}\big(t_k^-\big), \qquad \lim_{j\to\infty}(A\bar{u}_{n_i})\big(t_k^+\big)=\bar{w}\big(t_k^+\big) \quad (k=1,2,3,\ldots)$$

and

$$\lim_{j\to\infty}(A\bar{u}_{n_j})'\big(t_k^-\big)=\bar{w}'\big(t_k^-\big), \qquad \lim_{j\to\infty}(A\bar{u}_{n_j})'\big(t_k^+\big)=\bar{w}'\big(t_k^+\big) \quad (k=1,2,3,\ldots),$$

so, (74) and (75) imply that limits $\lim_{j\to\infty} I_k(\bar{u}''_{n_j}(t_k^-))$ $(k=1,2,3,\ldots)$ and $\lim_{j\to\infty} \bar{I}_k(\bar{u}''_{n_j}(t_k^-))$ $(k=1,2,3,\ldots)$ exist and

$$\bar{w}\big(t_k^+\big)-\bar{w}\big(t_k^-\big)=\lim_{j\to\infty} I_k\big(\bar{u}''_{n_j}\big(t_k^-\big)\big), \qquad \bar{w}'\big(t_k^+\big)-\bar{w}'\big(t_k^-\big)=\lim_{j\to\infty} \bar{I}_k\big(\bar{u}''_{n_j}\big(t_k^-\big)\big) \quad (k=1,2,3,\ldots).$$

Let

$$\lim_{j\to\infty} I_k\big(\bar{u}''_{n_j}\big(t_k^-\big)\big)=z_k, \qquad \lim_{j\to\infty} \bar{I}_k\big(\bar{u}''_{n_j}\big(t_k^-\big)\big)=\bar{z}_k \quad (k=1,2,3,\ldots).$$

Then $z_k \ge \theta$, $\bar{z}_k \ge \theta$ $(k=1,2,3,\ldots)$, and

$$\bar{w}\big(t_k^+\big)-\bar{w}\big(t_k^-\big)=z_k, \qquad \bar{w}'\big(t_k^+\big)-\bar{w}'\big(t_k^-\big)=\bar{z}_k \quad (k=1,2,3,\ldots). \tag{76}$$

By (53) and condition (H_3) we have

$$\big\|I_k\big(\bar{u}''_{n_j}\big(t_k^-\big)\big)\big\| \le NN_{p,q}t_k^2\gamma_k, \qquad \big\|\bar{I}_k\big(\bar{u}''_{n_j}\big(t_k^-\big)\big)\big\| \le NN_{p,q}t_k\gamma_k \quad (k,j=1,2,3,\ldots), \tag{77}$$

so,

$$\|z_k\| \le NN_{p,q}t_k^2\gamma_k, \qquad \|\bar{z}_k\| \le NN_{p,q}t_k\gamma_k \quad (k = 1,2,3,\ldots). \tag{78}$$

For any given $\epsilon > 0$, by condition (H_3) we can choose a sufficiently large positive integer k_0 such that

$$N_{p,q}\sum_{k=k_0+1}^{\infty} t_k\gamma_k < \epsilon, \qquad N_{p,q}\sum_{k=k_0+1}^{\infty} t_k^2\gamma_k < \epsilon, \tag{79}$$

and then, choose another sufficiently large positive integer j_1 such that

$$\bigl\|I_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr) - z_k\bigr\| < \frac{\epsilon}{k_0}, \qquad \bigl\|\bar{I}_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr) - \bar{z}_k\bigr\| < \frac{\epsilon}{k_0}, \quad \forall j > j_1\ (k = 1,2,\ldots,k_0). \tag{80}$$

It follows from (77)-(80) that

$$\begin{aligned}\sum_{k=1}^{\infty}\bigl\|I_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr) - z_k\bigr\| &\le \sum_{k=1}^{k_0}\bigl\|I_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr) - z_k\bigr\| \\ &\quad + \sum_{k=k_0+1}^{\infty}\bigl\|I_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr)\bigr\| + \sum_{k=k_0+1}^{\infty}\|z_k\| < 3\epsilon, \quad \forall j > j_1,\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{\infty}\bigl\|\bar{I}_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr) - \bar{z}_k\bigr\| &\le \sum_{k=1}^{k_0}\bigl\|\bar{I}_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr) - \bar{z}_k\bigr\| \\ &\quad + \sum_{k=k_0+1}^{\infty}\bigl\|\bar{I}_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr)\bigr\| + \sum_{k=k_0+1}^{\infty}\|\bar{z}_k\| < 3\epsilon, \quad \forall j > j_1,\end{aligned}$$

and hence

$$\lim_{j\to\infty}\sum_{k=1}^{\infty}\bigl\|I_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr) - z_k\bigr\| = 0, \qquad \lim_{j\to\infty}\sum_{k=1}^{\infty}\bigl\|\bar{I}_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr) - \bar{z}_k\bigr\| = 0. \tag{81}$$

By (16), (17), and (74)-(76) we have

$$\begin{aligned}&(A\bar{u}_{n_j})'(t) = \int_0^t (A\bar{u}_{n_j})''(s)\,ds + \sum_{0<t_k<t}\bar{I}_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr), \quad \forall t \in J\ (j = 1,2,3,\ldots), \\ &\bar{w}'(t) = \int_0^t \bar{w}''(s)\,ds + \sum_{0<t_k<t}\bar{z}_k, \quad \forall t \in J,\end{aligned}$$

and

$$\begin{aligned}&(A\bar{u}_{n_j})(t) = \int_0^t (t-s)(A\bar{u}_{n_j})''(s)\,ds + \sum_{0<t_k<t}\bigl\{I_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr) + (t-t_k)\bar{I}_k\bigl(\bar{u}''_{n_j}\bigl(t_k^-\bigr)\bigr)\bigr\}, \\ &\quad \forall t \in J\ (j = 1,2,3,\ldots),\end{aligned}$$

$$\bar{w}(t) = \int_0^t (t-s)\bar{w}''(s)\,ds + \sum_{0<t_k<t} \{ z_k + (t-t_k)\bar{z}_k \}, \quad \forall t \in J,$$

which imply

$$\begin{aligned} \| (A\bar{u}_{n_j})'(t) - \bar{w}'(t) \| &\le t \| (A\bar{u}_{n_j})'' - \bar{w}'' \|_B \\ &\quad + \sum_{0<t_k<t} \| \bar{I}_k(\bar{u}''_{n_j}(t_k^-)) - \bar{z}_k \|, \quad \forall t \in J \ (j = 1,2,3,\ldots), \end{aligned} \tag{82}$$

and

$$\begin{aligned} \| (A\bar{u}_{n_j})(t) - \bar{w}(t) \| &\le t^2 \| (A\bar{u}_{n_j})'' - \bar{w}'' \|_B + \sum_{0<t_k<t} \{ \| I_k(\bar{u}''_{n_j}(t_k^-)) - z_k \| \\ &\quad + t \| \bar{I}_k(\bar{u}''_{n_j}(t_k^-)) - \bar{z}_k \| \}, \quad \forall t \in J. \end{aligned} \tag{83}$$

Since

$$\sum_{0<t_k<t} \| I_k(\bar{u}''_{n_j}(t_k^-)) - z_k \| = 0, \qquad \sum_{0<t_k<t} \| \bar{I}_k(\bar{u}''_{n_j}(t_k^-)) - \bar{z}_k \| = 0, \quad \forall 0 < t \le t_1 \ (j = 1,2,3,\ldots),$$

(82) and (83) imply

$$\| (A\bar{u}_{n_j})' - \bar{w}' \|_T \le \| (A\bar{u}_{n_j})'' - \bar{w}'' \|_B + t_1^{-1} \sum_{k=1}^{\infty} \| \bar{I}_k(\bar{u}''_{n_j}(t_k^-)) - \bar{z}_k \| \quad (j = 1,2,3,\ldots) \tag{84}$$

and

$$\begin{aligned} \| A\bar{u}_{n_j} - \bar{w} \|_S &\le \| (A\bar{u}_{n_j})'' - \bar{w}'' \|_B + t_1^{-2} \sum_{k=1}^{\infty} \| I_k(\bar{u}''_{n_j}(t_k^-)) - z_k \| \\ &\quad + t_1^{-1} \sum_{k=1}^{\infty} \| \bar{I}_k(\bar{u}''_{n_j}(t_k^-)) - \bar{z}_k \| \quad (j = 1,2,3,\ldots). \end{aligned} \tag{85}$$

By (84), (85), (73), and (81) we get

$$\lim_{j\to\infty} \| (A\bar{u}_{n_j})' - \bar{w}' \|_T = 0, \qquad \lim_{j\to\infty} \| A\bar{u}_{n_j} - \bar{w} \|_S = 0. \tag{86}$$

It follows from (73) and (86) that $\| A\bar{u}_{n_j} - \bar{w} \|_D \to 0$ as $j \to \infty$, and the relative compactness of $A(Q_{pq})$ is proved. □

Lemma 3 *Let conditions* (H_1)-(H_4) *be satisfied. Then* $u \in Q_+ \cap C^3[J'_+, E]$ *is a positive solution of IBVP* (1) *if and only if* $u \in Q_+$ *is a solution of the following impulsive integral equation*:

$$\begin{aligned} u(t) &= \frac{t^2}{2(\beta-1)} \left\{ \int_0^{\infty} f(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s))\,ds + \sum_{k=1}^{\infty} \tilde{I}_k(u''(t_k^-)) \right\} \\ &\quad + \frac{1}{2} \int_0^t (t-s)^2 f(s, u(s), u'(s), u''(s), (Tu)(s), (Su)(s))\,ds \end{aligned}$$

$$+\sum_{0<t_k<t}\left\{I_k\big(u''(t_k^-)\big)+(t-t_k)\bar{I}_k\big(u''(t_k^-)\big)\right.$$
$$\left.+\frac{1}{2}(t-t_k)^2\tilde{I}_k\big(u''(t_k^-)\big)\right\},\quad \forall t\in J, \tag{87}$$

that is, u is a fixed point of the operator A defined by (19).

Proof The method of the proof is similar to that of Lemma 3 in [24], the difference is in using formula (18) instead of formula (17) with discussion in a Banach space. We omit the proof. □

Lemma 4 (Fixed point theorem of cone expansion and compression with norm type; see [25], Corollary 1, or [26], Theorem 3, or [27], Theorem 2.3.4; see also [28, 29]) *Let P be a cone in a real Banach space E, and Ω_1, Ω_2 be two bounded open sets in E such that $\theta\in\Omega_1$, $\bar{\Omega}_1\subset\Omega_2$, where θ denotes the zero element of E, and $\bar{\Omega}_i$ denotes the closure of Ω_i* ($i=1,2$). *Let an operator $A:P\cap(\bar{\Omega}_2\backslash\Omega_1)\to P$ be completely continuous* (*i.e., continuous and compact*). *Suppose that one of the following two conditions is satisfied*:

$$\text{(a)}\quad \|Ax\|\le\|x\|,\quad \forall x\in P\cap\partial\Omega_1;\qquad \|Ax\|\ge\|x\|,\quad \forall x\in P\cap\partial\Omega_2,$$

where $\partial\Omega_i$ denotes the boundary of Ω_i ($i=1,2$);

$$\text{(b)}\quad \|Ax\|\ge\|x\|,\quad \forall x\in P\cap\partial\Omega_1;\qquad \|Ax\|\le\|x\|,\quad \forall x\in P\cap\partial\Omega_2.$$

Then A has at least one fixed point in $P\cap(\bar{\Omega}_2\backslash\Omega_1)$.

3 Main theorem

Theorem *Let conditions* (H_1)-(H_6) *be satisfied. Assume that there exists* $r>0$ *such that*

$$\frac{N\beta}{\beta-1}\big(a^*_{r,r}+G_{r,r}b^*+N_{r,r}\gamma^*\big)<r, \tag{88}$$

where $a^*_{r,r}$, $G_{r,r}$, and $N_{r,r}$ are $a^*_{p,q}$, $G_{p,q}$, and $N_{p,q}$ for $p=r$ and $q=r$, respectively (*for $a^*_{p,q}$, $G_{p,q}$, $N_{p,q}$, b^*, and γ^*; see conditions* (H_2) *and* (H_3)). *Then IBVP* (1) *has at least two positive solutions $u^*,u^{**}\in Q_+\cap C^3[J'_+,E]$ such that*

$$0<\inf_{t\in J_+}\frac{\|u^*(t)\|}{t^2}\le\sup_{t\in J_+}\frac{\|u^*(t)\|}{t^2}<r,$$
$$0<\inf_{t\in J_+}\frac{\|(u^*)'(t)\|}{t}\le\sup_{t\in J_+}\frac{\|(u^*)'(t)\|}{t}<r,$$
$$0<\inf_{t\in J}\big\|\big(u^*\big)''(t)\big\|\le\sup_{t\in J}\big\|\big(u^*\big)''(t)\big\|<r,$$
$$\frac{1}{2}N^{-2}\beta^{-1}(2\beta-1)^{-1}r<\inf_{t\in J_+}\frac{\|u^{**}(t)\|}{t^2}\le\sup_{t\in J_+}\frac{\|u^{**}(t)\|}{t^2}<\infty,$$
$$N^{-2}\beta^{-2}\beta' r<\inf_{t\in J_+}\frac{\|(u^{**})'(t)\|}{t}\le\sup_{t\in J_+}\frac{\|(u^{**})'(t)\|}{t}<\infty,$$

and

$$N^{-2}\beta^{-2}\beta' r < \inf_{t\in J}\left\|\left(u^{**}\right)''(t)\right\| \le \sup_{t\in J}\left\|\left(u^{**}\right)''(t)\right\| < \infty.$$

Proof By Lemma 2 and Lemma 3 the operator A defined by (19) is continuous from Q_+ into Q and we need to prove that A has two fixed points u^* and u^{**} in Q_+ such that $0 < \|u^*\|_D < r < \|u^{**}\|_D$.

By condition (H_5) there exists $r_1 > 0$ such that

$$f(t,u,v,w,y,z) \ge \frac{N^3\beta^2(\beta-1)}{c^*\|w_0\|}c(t)\|w\|w_0, \quad \forall t\in J_+, u,v,w\in P_+, \|w\|\ge r_1, y,z\in P. \tag{89}$$

Choose

$$r_2 > \max\left\{N^2\beta^2\left(\beta'\right)^{-1}r_1, r\right\}. \tag{90}$$

For $u\in Q$, $\|u\|_D = r_2$, we have, by (14) and (90), $\|u''(t)\| \ge N^{-2}\beta^{-2}\beta' r_2 > r_1$, $\forall t\in J$, so, (29) and (89) imply

$$\begin{aligned}(Au)''(t) &\ge \frac{1}{\beta-1}\int_0^\infty f\left(s,u(s),u'(s),u''(s),(Tu)(s),(Su)(s)\right)ds\\ &\ge \frac{N^3\beta^2(\beta')^{-1}}{c^*\|w_0\|}\left(\int_0^\infty c(s)\left\|u''(s)\right\|ds\right)w_0\\ &\ge \frac{Nr_2}{c^*\|w_0\|}\left(\int_0^\infty c(s)\,ds\right)w_0 = \frac{Nr_2}{\|w_0\|}w_0, \quad \forall t\in J,\end{aligned}$$

and, consequently, $\|(Au)''(t)\| \ge r_2$, $\forall t\in J$; hence,

$$\|Au\|_D \ge \|u\|_D, \quad \forall u\in Q, \|u\|_D = r_2. \tag{91}$$

By condition (H_6) there exists $r_3 > 0$ such that

$$f(t,u,v,w,y,z) \ge \frac{N(\beta-1)r}{d^*\|w_1\|}d(t)w_1, \quad \forall t\in J_+, u,v,w\in P_+, 0<\|w\|<r_3, y,z\in P. \tag{92}$$

Choose

$$0 < r_4 < \min\{r_3, r\}. \tag{93}$$

For $u\in Q$, $\|u\|_D = r_4$, we have, by (14) and (93), $r_3 > r_4 \ge \|u''(t)\| \ge N^{-2}\beta^{-2}\beta' r_4 > 0$, so, we get, by (29) and (92),

$$\begin{aligned}(Au)''(t) &\ge \frac{1}{\beta-1}\int_0^\infty f\left(s,u(s),u'(s),u''(s),(Tu)(s),(Su)(s)\right)ds\\ &\ge \frac{Nr}{d^*\|w_1\|}\left(\int_0^\infty d(s)\,ds\right)w_1 = \frac{Nr}{\|w_1\|}w_1, \quad \forall t\in J,\end{aligned}$$

and, consequently, $\|(Au)''(t)\| \geq r > r_4$, $\forall t \in J$; hence,

$$\|Au\|_D > \|u\|_D, \quad \forall u \in Q, \|u\|_D = r_4. \tag{94}$$

On the other hand, for $u \in Q$, $\|u\|_D = r$, (32)-(34) imply

$$\|Au\|_D \leq \frac{N\beta}{\beta - 1}\left(a^*_{r,r} + G_{r,r}b^* + N_{r,r}\gamma^*\right),$$

so, (88) implies

$$\|Au\|_D < \|u\|_D, \quad \forall u \in Q, \|u\|_D = r. \tag{95}$$

By (90) and (93) we know that $0 < r_4 < r < r_2$, and, by Lemma 2 the operator A is completely continuous from $Q_{r_4 r_2}$ into Q; hence, (91), (94), (95), and Lemma 4 imply that A has two fixed points $u^*, u^{**} \in Q_{r_4 r_2}$ such that $r_4 < \|u^*\|_D < r < \|u^{**}\|_D \leq r_2$. The proof is complete. □

Example Consider the infinite system of scalar third-order impulsive singular integro-differential equations of mixed type on the half-line:

$$\begin{cases} u'''_n(t) = \frac{e^{-2t}}{30n^2 t^{\frac{1}{5}}}\{\frac{1}{2[u_n(t)+6u_{2n-1}(t)+\sum_{m=1}^{\infty} u_m(t)]^{\frac{1}{5}}} + \frac{1}{6[u'_{2n+1}(t)+\sum_{m=1}^{\infty} u'_m(t)]^{\frac{1}{5}}} \\ \qquad + \frac{1}{8}[u''_{n+1}(t) + \sum_{m=1}^{\infty} u''_m(t)]^2 + \frac{1}{9\sum_{m=1}^{\infty} u''_m(t)}\} \\ \qquad + \frac{e^{-3t}}{40n^3 t^{\frac{1}{5}}}\{(\int_0^t e^{-(t+2)s} u_n(s)\,ds)^2 \\ \qquad + (\int_0^{\infty} \frac{u_{2n}(s)\,ds}{(1+t+s)^4})^3\}, \quad \forall 0 < t < \infty, t \neq k\ (k = 1,2,3,\ldots; n = 1,2,3,\ldots), \\ \Delta u_n|_{t=k} = n^{-2}3^{-k-4}\frac{1}{4[u''_{3n}(k^-)]^2+\sqrt{\sum_{m=1}^{\infty} u''_m(k^-)}} \quad (k = 1,2,3,\ldots; n = 1,2,3,\ldots), \\ \Delta u'_n|_{t=k} = n^{-2}k^{-1}3^{-k-4}\frac{1}{2[u''_{3n}(k^-)]^2+\sqrt{\sum_{m=1}^{\infty} u''_m(k^-)}} \quad (k = 1,2,3,\ldots; n = 1,2,3,\ldots), \\ \Delta u''_n|_{t=k} = n^{-2}k^{-2}3^{-k-4}\frac{1}{u''_{3n}(k^-)]^2+\sqrt{\sum_{m=1}^{\infty} u''_m(k^-)}} \quad (k = 1,2,3,\ldots; n = 1,2,3,\ldots), \\ u_n(0) = 0, \qquad u'_n(0) = 0, \qquad u''_n(\infty) = 2u''_n(0) \quad (n = 1,2,3,\ldots). \end{cases} \tag{96}$$

Conclusion The infinite system (96) has at least two positive solutions $\{u^*_n(t)\}$ ($n = 1,2,3,\ldots$) and $\{u^{**}_n(t)\}$ ($n = 1,2,3,\ldots$) such that

$$\begin{aligned} &0 < \inf_{0<t<\infty} \frac{\sum_{m=1}^{\infty} u^*_m(t)}{t^2} \leq \sup_{0<t<\infty} \frac{\sum_{m=1}^{\infty} u^*_m(t)}{t^2} < 1, \\ &0 < \inf_{0<t<\infty} \frac{\sum_{m=1}^{\infty} (u^*_m)'(t)}{t} \leq \sup_{0<t<\infty} \frac{\sum_{m=1}^{\infty} (u^*_m)'(t)}{t} < 1, \\ &0 < \inf_{0\leq t<\infty} \sum_{m=1}^{\infty} \left(u^*_m\right)''(t) \leq \sup_{0\leq t<\infty} \sum_{m=1}^{\infty} \left(u^*_m\right)''(t) < 1, \\ &\frac{1}{12} < \inf_{0<t<\infty} \frac{\sum_{m=1}^{\infty} u^{**}_m(t)}{t^2} \leq \sup_{0<t<\infty} \frac{\sum_{m=1}^{\infty} u^{**}_m(t)}{t^2} < \infty, \\ &\frac{1}{6} < \inf_{0<t<\infty} \frac{\sum_{m=1}^{\infty} (u^{**}_m)'(t)}{t} \leq \sup_{0<t<\infty} \frac{\sum_{m=1}^{\infty} (u^{**}_m)'(t)}{t} < \infty, \end{aligned}$$

and

$$\frac{1}{6} < \inf_{0\le t<\infty} \sum_{m=1}^{\infty} (u_m^{**})''(t) \le \sup_{0\le t<\infty} \sum_{m=1}^{\infty} (u_m^{**})''(t) < \infty.$$

Proof Let $E = l^1 = \{u = (u_1,\dots,u_n,\dots) : \sum_{n=1}^{\infty} |u_n| < \infty\}$ with norm $\|u\| = \sum_{n=1}^{\infty} |u_n|$ and $P = \{u = (u_1,\dots,u_n,\dots) \in E : u_n \ge 0, n = 1,2,3,\dots\}$. Then P is a normal cone in E with normal constant $N = 1$, and the infinite system (96) can be regarded as an IBVP of the form (1). In this situation, $u = (u_1,\dots,u_n,\dots)$, $v = (v_1,\dots,v_n,\dots)$, $w = (w_1,\dots,w_n,\dots)$, $y = (y_1,\dots,y_n,\dots)$, $z = (z_1,\dots,z_n,\dots)$, $t_k = k$ $(k = 1,2,3,\dots)$, $K(t,s) = e^{-(t+2)s}$, $H(t,s) = (1+t+s)^{-4}$, $\beta = 2$, $\beta' = \frac{2}{3}$, $f = (f_1,\dots,f_n,\dots)$, $I_k = (I_{k1},\dots,I_{kn},\dots)$, $\bar{I}_k = (\bar{I}_{k1},\dots,\bar{I}_{kn},\dots)$, and $\tilde{I}_k = (\tilde{I}_{k1},\dots,\tilde{I}_{kn},\dots)$, in which

$$\begin{aligned} f_n(t,u,v,w,y,z) = {}& \frac{e^{-2t}}{30n^2 t^{\frac{1}{5}}} \Biggl\{ \frac{1}{2}\Biggl(u_n + 6u_{2n-1} + \sum_{m=1}^{\infty} u_m\Biggr)^{-\frac{1}{5}} + \frac{1}{6}\Biggl(v_{2n+1} + \sum_{m=1}^{\infty} v_m\Biggr)^{-\frac{1}{5}} \\ & + \frac{1}{8}\Biggl(w_{n+1} + \sum_{m=1}^{\infty} w_m\Biggr)^{2} + \frac{1}{9}\Biggl(\sum_{m=1}^{\infty} w_m\Biggr)^{-1} \Biggr\} \\ & + \frac{e^{-3t}}{40n^3 t^{\frac{1}{5}}}\bigl(y_n^2 + z_{2n}^3\bigr), \quad \forall t \in J_+, u,v,w \in P_+, y,z \in P, \end{aligned} \tag{97}$$

$$I_{kn}(w) = n^{-2}3^{-k-4}\Biggl(4w_{3n}^2 + \sqrt{\sum_{m=1}^{\infty} w_m}\Biggr)^{-1}, \quad \forall w \in P_+ \ (k = 1,2,3,\dots; n = 1,2,3,\dots), \tag{98}$$

$$\begin{aligned} \bar{I}_{kn}(w) = {}& n^{-2}k^{-1}3^{-k-4}\Biggl(2w_{3n}^2 + \sqrt{\sum_{m=1}^{\infty} w_m}\Biggr)^{-1}, \\ & \forall w \in P_+ \ (k = 1,2,3,\dots; n = 1,2,3,\dots), \end{aligned} \tag{99}$$

and

$$\begin{aligned} \tilde{I}_{kn}(w) = {}& n^{-2}k^{-2}3^{-k-4}\Biggl(w_{2n}^2 + \sqrt{\sum_{m=1}^{\infty} w_m}\Biggr)^{-1}, \\ & \forall w \in P_+ \ (k = 1,2,3,\dots; n = 1,2,3,\dots). \end{aligned} \tag{100}$$

It is easy to see that $f \in C[J_+ \times P_+ \times P_+ \times P_+ \times P \times P, P]$, $I_k, \bar{I}_k, \tilde{I}_k \in C[P_+, P]$ $(k = 1,2,3,\dots)$, and condition (H_1) is satisfied with $k^* \le 4e^{-2}$ (by using the fact that the function $\phi(s) = s^2 e^{-s}$ $(0 \le s < \infty)$ attains its maximum at $s = 2$) and $h^* \le 1$. We have by (97)

$$\begin{aligned} 0 \le f_n(t,u,v,w,y,z) \le {}& \frac{e^{-2t}}{30n^2 t^{\frac{1}{5}}}\biggl(\frac{1}{2}\|u\|^{-\frac{1}{5}} + \frac{1}{6}\|v\|^{-\frac{1}{5}} + \frac{1}{8}\bigl(2\|w\|\bigr)^2 + \frac{1}{9}\|w\|^{-1}\biggr) \\ & + \frac{e^{-3t}}{40n^3 t^{\frac{1}{5}}}\bigl(\|y\|^2 + \|z\|^3\bigr) \le \frac{e^{-2t}}{n^2 t^{\frac{1}{5}}}\biggl(\frac{1}{60}\|u\|^{-\frac{1}{5}} + \frac{1}{180}\|v\|^{\frac{1}{5}} + \frac{1}{60}\|w\|^2 + \frac{1}{270}\|w\|^{-1} \\ & + \frac{1}{40}\|y\|^2 + \frac{1}{40}\|z\|^3\biggr), \quad \forall t \in J_+, u,v,w \in P_+, y,z \in P, \end{aligned} \tag{101}$$

so, observing the inequality $\sum_{n=1}^{\infty}\frac{1}{n^2}<2$, we get

$$\begin{aligned}\|f(t,u,v,w,z)\| = \sum_{n=1}^{\infty} f_n(t,u,v,w,z) &< \frac{e^{-2t}}{t^{\frac{1}{5}}}\biggl(\frac{1}{30}\|u\|^{-\frac{1}{5}}+\frac{1}{90}\|v\|^{-\frac{1}{5}}+\frac{1}{30}\|w\|^2 \\ &\quad +\frac{1}{135}\|w\|^{-1}+\frac{1}{20}\|y\|^2+\frac{1}{20}\|z\|^3\biggr), \quad \forall t\in J_+, u,v,w\in P_+, y,z\in P,\end{aligned}$$

which implies that condition (H_2) is satisfied for

$$a(t)=b(t)=\frac{e^{-2t}}{t^{\frac{1}{5}}}, \qquad g(s_1,s_2)=\frac{1}{30}s_1^{-\frac{1}{5}}+\frac{1}{90}s_2^{-\frac{1}{5}}$$

and

$$G(x_1,x_2,x_3)=\frac{1}{30}x_1^2+\frac{1}{135}x_1^{-1}+\frac{1}{20}x_2^2+\frac{1}{20}x_3^3 \tag{102}$$

with (for $q\geq p>0$)

$$\begin{aligned}g_{p,q}(t) &= \max\left\{\frac{1}{30}s_1^{-\frac{1}{5}}+\frac{1}{90}s_2^{-\frac{1}{5}} : \frac{pt^2}{12}\leq s_1\leq qt^2, \frac{pt}{6}\leq s_2\leq qt\right\} \\ &= \frac{1}{30}\left(\frac{12}{p}\right)^{\frac{1}{5}} t^{-\frac{2}{5}}+\frac{1}{90}\left(\frac{6}{q}\right)^{\frac{1}{5}} t^{-\frac{1}{5}}, \quad \forall t\in J_+,\end{aligned}$$

$$a_{p,q}^{*}=\int_0^{\infty} a(t)g_{p,q}(t)\,dt=\frac{1}{30}\left(\frac{12}{p}\right)^{\frac{1}{5}}\int_0^{\infty}\frac{e^{-2t}}{t^{\frac{3}{5}}}\,dt+\frac{1}{90}\left(\frac{6}{q}\right)^{\frac{1}{5}}\int_0^{\infty}\frac{e^{-2t}}{t^{\frac{2}{5}}}\,dt<\infty, \tag{103}$$

and

$$b^{*}=\int_0^{\infty}\frac{e^{-2t}}{t^{\frac{1}{5}}}\,dt<\infty. \tag{104}$$

By (98)-(100) it is obvious that

$$0\leq I_{kn}(w)\leq k\bar{I}_{kn}(w), \qquad \bar{I}_{kn}(w)\leq \tilde{I}_{kn}(w), \quad \forall w\in P_+\ (k=1,2,3,\ldots; n=1,2,3,\ldots),$$

so,

$$I_k(w)\leq k\bar{I}_k(w), \qquad \bar{I}_k(w)\leq k\tilde{I}_k(w), \quad \forall w\in P_+\ (k=1,2,3,\ldots).$$

Moreover, from (100) we get

$$0\leq \tilde{I}_{kn}(w)\leq n^{-2}k^{-2}3^{-k-4}\frac{1}{\sqrt{\|w\|}}, \quad \forall w\in P_+\ (k=1,2,3,\ldots; n=1,2,3,\ldots),$$

so,

$$\|\tilde{I}_k(w)\|=\sum_{n=1}^{\infty}\tilde{I}_{kn}(w)\leq k^{-2}3^{-k-4}\frac{2}{\sqrt{\|w\|}}, \quad \forall w\in P_+\ (k=1,2,3,\ldots),$$

which implies that condition (H_3) is satisfied for $\gamma_k = k^{-2}3^{-k-4}$ and

$$F(s) = 2s^{-\frac{1}{2}} \tag{105}$$

with

$$\tilde{\gamma} = \sum_{k=1}^{\infty} k^2\gamma_k = \frac{1}{162}, \qquad \gamma^* = \sum_{k=1}^{\infty} \gamma_k < \tilde{\gamma}. \tag{106}$$

On the other hand, (97) implies

$$f_n(t,u,v,w,y,z) \geq \frac{e^{-2t}}{240n^2t^{\frac{1}{5}}}\|w\|^2, \quad \forall t \in J_+, u,v,w \in P_+, y,z \in P,$$

and

$$f_n(t,u,v,w,y,z) \geq \frac{e^{-2t}}{270n^2t^{\frac{1}{5}}}\|w\|^{-1}, \quad \forall t \in J_+, u,v,w \in P_+, y,z \in P, \tag{107}$$

so, we see that condition (H_5) is satisfied for

$$w_0 = \left(1, \frac{1}{2^2}, \dots, \frac{1}{n^2}, \dots\right), \qquad c(t) = \frac{e^{-2t}}{t^{\frac{1}{5}}}, \qquad \tau(w) = \frac{1}{240}\|w\|^2,$$

and condition (H_6) is satisfied for

$$w_1 = \left(1, \frac{1}{2^2}, \dots, \frac{1}{n^2}, \dots\right), \qquad d(t) = \frac{e^{-2t}}{t^{\frac{1}{5}}}, \qquad \sigma(w) = \frac{1}{270}\|w\|^{-1}.$$

In addition, by (97) we have

$$f_n(t,u,v,w,y,z) \geq \frac{e^{-2t}}{60n^2t^{\frac{1}{5}}}\big(8\|u\|\big)^{-\frac{1}{5}} = \frac{e^{-2t}}{60(8^{\frac{1}{5}})n^2t^{\frac{1}{5}}}\|u\|^{-\frac{1}{5}},$$
$$\forall t \in J_+, u,v,w \in P_+, y,z \in P, \tag{108}$$

and

$$f_n(t,u,v,w,y,z) \geq \frac{e^{-2t}}{180n^2t^{\frac{1}{5}}}\big(2\|v\|\big)^{-\frac{1}{5}} = \frac{e^{-2t}}{180(2^{\frac{1}{5}})n^2t^{\frac{1}{5}}}\|v\|^{-\frac{1}{5}},$$
$$\forall t \in J_+, u,v,w \in P_+, y,z \in P. \tag{109}$$

It follows from (108), (109), and (107) that

$$\big\|f(t,u,v,w,y,z)\big\| \geq \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)\frac{e^{-2t}}{60(8^{\frac{1}{5}})t^{\frac{1}{5}}}\|u\|^{-\frac{1}{5}} > \frac{e^{-2t}}{60(8^{\frac{1}{5}})t^{\frac{1}{5}}}\|u\|^{-\frac{1}{5}},$$
$$\forall t \in J_+, u,v,w \in P_+, y,z \in P,$$

$$\|f(t,u,v,w,y,z)\| \geq \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)\frac{e^{-2t}}{180(2^{\frac{1}{5}})t^{\frac{1}{5}}}\|v\|^{-\frac{1}{5}} > \frac{e^{-2t}}{180(2^{\frac{1}{5}})t^{\frac{1}{5}}}\|v\|^{-\frac{1}{5}},$$
$$\forall t \in J+, u,v,w \in P_+, y,z \in P,$$

and

$$\|f(t,u,v,w,y,z)\| \geq \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)\frac{e^{-2t}}{270t^{\frac{1}{5}}}\|w\|^{-1} > \frac{e^{-2t}}{270t^{\frac{1}{5}}}\|w\|^{-1},$$
$$\forall t \in J_+, u,v,w \in P_+, y,z \in P;$$

hence, (3), (4), (5), and (6) are satisfied, that is, $f(t,u,v,w,y,z)$ is singular at $t=0$, $u=\theta$, $v=\theta$, and $w=\theta$ ($\theta=(0,0,\ldots,0,\ldots)$). On the other hand, from (98)-(100) we get

$$I_{kn}(w) \geq n^{-2}3^{-k-4}\left(4\|w\|^2+\sqrt{\|w\|}\right)^{-1}, \quad \forall w \in P_+ \ (k=1,2,3,\ldots;n=1,2,3,\ldots),$$
$$\bar{I}_{kn}(w) \geq n^{-2}k^{-1}3^{-k-4}\left(2\|w\|^2+\sqrt{\|w\|}\right)^{-1}, \quad \forall w \in P_+ \ (k=1,2,3,\ldots;n=1,2,3,\ldots),$$

and

$$\tilde{I}_{kn}(w) \geq n^{-2}k^{-2}3^{-k-4}\left(\|w\|^2+\sqrt{\|w\|}\right)^{-1}, \quad \forall w \in P_+ \ (k=1,2,3,\ldots;n=1,2,3,\ldots),$$

so,

$$\|I_k(w)\| \geq \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)3^{-k-4}\left(4\|w\|^2+\sqrt{\|w\|}\right)^{-1} > 3^{-k-4}\left(4\|w\|^2+\sqrt{\|w\|}\right)^{-1},$$
$$\forall w \in P_+ \ (k=1,2,3,\ldots),$$
$$\|\bar{I}_k(w)\| \geq \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)k^{-1}3^{-k-4}\left(2\|w\|^2+\sqrt{\|w\|}\right)^{-1} > k^{-1}3^{-k-4}\left(2\|w\|^2+\sqrt{\|w\|}\right)^{-1},$$
$$\forall w \in P_+ \ (k=1,2,3,\ldots),$$

and

$$\|\tilde{I}_k(w)\| \geq \left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)k^{-2}3^{-k-4}\left(\|w\|+\sqrt{\|w\|}\right)^{-1} > k^{-2}3^{-k-4}\left(\|w\|+\sqrt{\|w\|}\right)^{-1},$$
$$\forall w \in P_+ \ (k=1,2,3,\ldots),$$

which imply that (7), (8), and (9) are satisfied, that is, $I_k(w)$, $\bar{I}_k(w)$, and $\tilde{I}_k(w)$ are singular at $w=\theta$. Now, we check that condition (H_4) is satisfied. Let $t \in J_+$, $r > p > 0$, and $q > 0$ be fixed, and $\{y^{(m)}\}$ be any sequence in $f(t,P_{pr},P_{pr},P_{pr},P_q,P_q)$, where $y^{(m)}=(y_1^{(m)},\ldots,y_n^{(m)},\ldots)$. Then, by (101) we have

$$0 \leq y_n^{(m)} \leq \frac{e^{-2t}}{n^2t^{\frac{1}{5}}}\left(\frac{1}{60}p^{-\frac{1}{5}}+\frac{1}{180}p^{-\frac{1}{5}}+\frac{1}{60}r^2+\frac{1}{270}p^{-1}+\frac{1}{40}q^2+\frac{1}{40}q^3\right)$$
$$(n,m=1,2,3,\ldots). \tag{110}$$

So, $\{y_n^{(m)}\}$ is bounded, and, by the diagonal method we can choose a subsequence $\{m_i\} \subset \{m\}$ such that

$$y_n^{(m_i)} \to \bar{y}_n \quad \text{as } i \to \infty \ (n = 1, 2, 3, \ldots), \tag{111}$$

which implies by (110) that

$$0 \le \bar{y}_n \le \frac{e^{-2t}}{n^2 t^{\frac{1}{5}}} \left(\frac{1}{60} p^{-\frac{1}{5}} + \frac{1}{180} p^{-\frac{1}{5}} + \frac{1}{60} r^2 + \frac{1}{270} p^{-1} + \frac{1}{40} q^2 + \frac{1}{40} q^3 \right) \\ (n = 1, 2, 3, \ldots). \tag{112}$$

Consequently, $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n, \ldots) \in l^1 = E$. Let $\epsilon > 0$ be given. Choose a positive integer n_0 such that

$$\frac{e^{-2t}}{t^{\frac{1}{5}}} \left(\sum_{n=n_0+1}^{\infty} \frac{1}{n^2} \right) \left(\frac{1}{60} p^{-\frac{1}{5}} + \frac{1}{180} p^{-\frac{1}{5}} + \frac{1}{60} r^2 + \frac{1}{270} p^{-1} + \frac{1}{40} q^2 + \frac{1}{40} q^3 \right) < \frac{\epsilon}{3}. \tag{113}$$

By (111) we can choose a positive integer i_0 such that

$$\left| y_n^{(m_i)} - \bar{y}_n \right| < \frac{\epsilon}{3n_0}, \quad \forall i > i_0 \ (n = 1, 2, \ldots, n_0). \tag{114}$$

It follows from (110)-(114) that

$$\begin{aligned} \left\| y^{(m_i)} - \bar{y} \right\| &= \sum_{n=1}^{\infty} \left| y_n^{(m_i)} - \bar{y}_n \right| \le \sum_{n=1}^{n_0} \left| y_n^{(m_i)} - \bar{y}_n \right| + \sum_{n=n_0+1}^{\infty} \left| y_n^{(m_i)} \right| \\ &\quad + \sum_{n=n_0+1}^{\infty} |\bar{y}_n| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \quad \forall i > i_0; \end{aligned}$$

hence, $y^{(m_i)} \to \bar{y}$ in E as $i \to \infty$. Thus, we have proved that $f(t, P_{pr}, P_{pr}, P_{pr}, P_q, P_q)$ is relatively compact in E. Similarly, we can prove that $I_k(P_{pr})$, $\bar{I}_k(P_{pr})$, and $\tilde{I}_k(P_{pr})$ $(k = 1, 2, 3, \ldots)$ are relatively compact in E. Hence, condition (H_4) is satisfied. Finally, we check that inequality (88) is satisfied for $r = 1$, that is,

$$2\left(a_{1,1}^* + G_{1,1} b^* + N_{1,1} \gamma^*\right) < 1. \tag{115}$$

Since

$$6^{\frac{1}{5}} = 1.43 \cdots < \frac{3}{2}, \qquad 12^{\frac{1}{5}} = 1.64 \cdots < \frac{17}{10},$$
$$e^{-2} = 0.13 \cdots < \frac{7}{50}, \qquad \sqrt{6} = 2.44 \cdots < \frac{5}{2},$$

and

$$\int_0^{\infty} \frac{e^{-2t}}{t^{\alpha}} \, dt < \int_0^1 \frac{dt}{t^{\alpha}} + \int_1^{\infty} e^{-2t} \, dt = \frac{1}{1-\alpha} + \frac{1}{2} e^{-2}, \quad \forall 0 < \alpha < 1,$$

we have, by (103) and (104),

$$\begin{aligned} a_{1,1}^* &< \frac{1}{30}(12)^{\frac{1}{5}}\left(\frac{5}{2}+\frac{1}{2}e^{-2}\right)+\frac{1}{90}(6)^{\frac{1}{5}}\left(\frac{5}{3}+\frac{1}{2}e^{-2}\right) \\ &< \frac{1}{30}\left(\frac{17}{10}\right)\left(\frac{5}{2}+\frac{7}{100}\right)+\frac{1}{90}\left(\frac{3}{2}\right)\left(\frac{5}{3}+\frac{7}{100}\right) \\ &= \frac{4{,}369}{30{,}000}+\frac{521}{18{,}000} < 0.15+0.04 = 0.19 \end{aligned}$$

and

$$b^* < \frac{5}{4}+\frac{7}{100} = \frac{33}{25}.$$

Moreover, (102) implies

$$\begin{aligned} G_{1,1} &\le \max\left\{\frac{1}{30}x_1^2+\frac{1}{135}x_1^{-1}+\frac{1}{20}x_2^2+\frac{1}{20}x_3^3 : \frac{1}{6}\le x_1\le 1, 0\le x_2\le\frac{14}{25}, 0\le x_3\le 1\right\} \\ &\le \frac{1}{30}+\frac{6}{135}+\frac{1}{20}\left(\frac{14}{25}\right)^2+\frac{1}{20} = \frac{23}{180}+\frac{49}{3{,}125}, \end{aligned}$$

and (105) implies

$$N_{1,1} = \max\left\{2s^{-\frac{1}{2}} : \frac{1}{6}\le s\le 1\right\} = 2\sqrt{6} < 5,$$

so,

$$G_{1,1}b^* < \left(\frac{23}{180}+\frac{49}{3{,}125}\right)\left(\frac{33}{25}\right) = \frac{253}{1{,}500}+\frac{1{,}617}{78{,}125} < 0.17+0.03 = 0.2,$$

and, observing (106), we have

$$N_{1,1}\gamma^* < \frac{5}{162} < 0.04.$$

Consequently,

$$2\left(a_{1,1}^*+G_{1,1}b^*+N_{1,1}\gamma^*\right) < 2(0.19+0.2+0.04) = 0.86 < 1.$$

Hence, (115) holds, and our conclusion follows from the theorem. □

Competing interests
The author declares that they have no competing interests.

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