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Existence and multiplicity of positive solutions for p-Laplacian elliptic equations

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Abstract
We study a p-Laplacian elliptic equation with Hardy term and Hardy-Sobolev critical exponent, where the nonlinearity is $(p-1)$-sublinear near zero and $(p^*(s)-1)$-sublinear near infinity ($p^*(s)=\frac{p(N-s)}{N-p}$ is the Hardy-Sobolev critical exponent). By using variational methods and some analysis techniques, we obtain the existence and multiplicity of positive solutions for the p-Laplacian elliptic equation. To the best of our knowledge, no result has been published concerning the existence and multiplicity of positive solutions for the p-Laplacian elliptic equation.

MSC: 35A15; 35J91

Keywords: p-Laplacian elliptic equations; Hardy term; Hardy-Sobolev critical exponent; variational methods

1 Introduction and main results

In this paper, we will study the existence and multiplicity of positive solutions for the following p-Laplacian elliptic equation:

$$\begin{cases} -\triangle_p u-\mu\frac{|u|^{p-2}u}{|x|^p}=\frac{|u|^{p^*(s)-2}}{|x|^s}u+\lambda f(x,u), & x\in\Omega\setminus\{0\},\\ u=0, & x\in\partial\Omega. \end{cases} \tag{1.1}$$

Here, $\Omega\subset\mathbb{R}^N$ $(N\geq 3)$ is an open bounded domain with smooth boundary $\partial\Omega$ and $0\in\Omega$, $p\in(1,N)$, $s\in[0,p)$, $\lambda,\mu\in\mathbb{R}^+$, $\triangle_p u:=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian differential operator, $p^*(s)=\frac{p(N-s)}{N-p}$ is the Hardy-Sobolev critical exponent, $p^*=p^*(0)=\frac{Np}{N-p}$ is the Sobolev critical exponent, and we have the function $f:\Omega\times\mathbb{R}\to\mathbb{R}$.

Let

$$\|u\|:=\left(\int_\Omega\left(|\nabla u|^p-\mu\frac{|u|^p}{|x|^p}\right)dx\right)^{\frac{1}{p}},\quad u\in W_0^{1,p}(\Omega),$$

which is well defined on the Sobolev space $W_0^{1,p}(\Omega)$ by the Hardy inequality [1]. From [2], we know $\|u\|$ is comparable with the standard Sobolev norm of $W_0^{1,p}(\Omega)$, but it is not a norm since the triangle inequality or subadditivity may fail. The following best Hardy-

Sobolev constant will be useful in this paper:

$$A_{\mu,s}(\Omega) := \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|u\|^p}{(\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\,dx)^{\frac{p}{p^*(s)}}}. \tag{1.2}$$

In recent decades, there were many authors [1, 3–17] who have studied the existence or multiplicity of solutions for elliptic equations with the operator $-\triangle - \frac{\mu}{|x|^2}$ $(0 \le \mu < (\frac{N-2}{2})^2)$. But most of the authors only considered the case $s = 0$.

Next we only state some most related results of (1.1). Han [18] obtained the existence of multiplicity of positive solutions for the following equation:

$$\begin{cases} -\triangle_p u - \mu\frac{|u|^{p-2}u}{|x|^p} = Q(x)|u|^{p^*-2}u + \lambda|u|^{p-2}u, & x\in\Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where $Q(x) \ge 0$ is a bounded function on $\overline{\Omega}$. The authors [19] only studied (1.3) in the special cases where $Q(x) \equiv 1$ and $\mu = 0$. The authors [2] studied the following equation:

$$\begin{cases} -\triangle_p u - \mu\frac{u^{p-1}}{|x|^p} = \frac{|u|^{p^*(s)-1}}{|x|^s} + |u|^{p^*-1}, & x\in\mathbb{R}^N, \\ u \in D_1^p(\mathbb{R}^N), \end{cases} \tag{1.4}$$

where $D_1^p(\mathbb{R}^N)$ is defined as the completion of $C_c^\infty(\mathbb{R}^N)$, and they obtained a positive solution $u \in D_1^p(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$ for any $0 < s < p$ and $\mu \in (-\infty, \mu_1)$, where $\mu_1 := (\frac{N-p}{p})^p$. Later, the authors [20] obtained a nontrivial solution of a more general case than (1.4) by the ideas in [2]. Kang [21] obtained one positive solution for the following equation:

$$\begin{cases} -\triangle_p u - \mu\frac{|u|^{p-2}u}{|x|^p} = \frac{|u|^{p^*(s)-2}}{|x|^s}u + \lambda\frac{|u|^{q-2}u}{|x|^t}, & x\in\Omega\setminus\{0\}, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $0 \le t < p$, $p \le q < p^*(t)$.

Inspired by the above results, we shall study the existence and multiplicity of positive solutions for (1.1) with the nonlinearity f being $(p-1)$-sublinear at zero and $(p^*(s)-1)$-sublinear at infinity (see the following (A_1)), which is different from the above results. Due to the lack of compactness of the embeddings in $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega, |x|^{-p}\,dx)$, and $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*(s)}(\Omega, |x|^{-s}\,dx)$, we cannot use the standard variational argument directly. The corresponding energy functional fails to satisfy the classical Palais-Smale ((PS)) condition in $W_0^{1,p}(\Omega)$. But we can establish a local (PS) condition in a suitable range, so the existence result can be obtained by constructing a minimax level within this range and the mountain pass lemma in [3, 22].

Let $\|\cdot\|_p$ be the norm in $L^p(\Omega)$ and $F(x,t) := \int_0^t f(x,s)\,ds$, $x\in\Omega$, $t\in\mathbb{R}$. Let $a(\mu)$ and $b(\mu)$ be zeros of the function

$$f(t) = (p-1)t^p - (N-p)t^{p-1} + \mu, \quad t \ge 0, 0 \le \mu < \mu_1 := \left(\frac{N-p}{p}\right)^p$$

satisfying $0 \le a(\mu) < \frac{N-p}{p} < b(\mu)$; see [23]. To state our results, we make the following assumptions:

(A₁) $f \in C(\overline{\Omega} \times \mathbb{R}^+, \mathbb{R}), f(x,0) \equiv 0$, and

$$\lim_{t\to 0^+} \frac{f(x,t)}{t^{p-1}} = +\infty, \qquad \lim_{t\to\infty} \frac{f(x,t)}{t^{p^*(s)-1}} = 0 \quad \text{uniformly for } x \in \overline{\Omega}.$$

(A_2) $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is nondecreasing with respect to the second variable.
(A_3) $2 \le p < N$, $N < \min\{pb(\mu), p(1+p)\}$ and $0 \le s \le N - \frac{(N-p)(1+p)}{p}$.
(A_3') $2 \le p < N$, $pb(\mu) \le N < p + \frac{p^2 b(\mu)}{1+p}$ and $N - pb(\mu) < s \le N - \frac{(N-p)(1+p)}{p}$.

Remark 1.1 In (A_3) and (A_3'), we can easily check that $N < p(1+p)$ implies $N - \frac{(N-p)(1+p)}{p} > 0$, $N < p + \frac{p^2 b(\mu)}{1+p}$ implies $N - pb(\mu) < N - \frac{(N-p)(1+p)}{p}$. Besides, $N - \frac{(N-p)(1+p)}{p} < p$ holds.

Now our results read as follows.

Theorem 1.1 *If* $N \ge 3$, $0 \le s < p$, $0 \le \mu < \mu_1$, $1 < p < N$ *and* (A_1) *hold, then there exists* $\lambda^* > 0$ *such that* (1.1) *has at least one nontrivial positive solution* u_λ *for any* $\lambda \in (0, \lambda^*)$.

Theorem 1.2 *If* $N \ge 3$, $0 \le s < p$, $0 \le \mu < \mu_1$, (A_1), (A_2) *and* $((A_3)$ *or* $(A_3'))$ *hold, then there exists* $\lambda^* > 0$ *such that* (1.1) *has at least two nontrivial positive solutions for every* $\lambda \in (0, \lambda^*)$.

Remark 1.2 We should mention that the above p-Laplacian problems studied in [2, 18–21] are all not $(p-1)$-sublinear at zero. Besides, our nonlinearity f is more general. To the best of our knowledge, our Theorems 1.1 and 1.2 are new.

Let $D^{1,p}(\mathbb{R}^N) := \{u \in L^{p^*}(\mathbb{R}^N); |\nabla u| \in L^p(\mathbb{R}^N)\}$. A typical model of (1.1) is the following equation:

$$\begin{cases} -\triangle_p u - \mu \frac{u^{p-1}}{|x|^p} = u^{p^*-1}, & \text{in } \mathbb{R}^N \setminus \{0\}, \\ u > 0, & \text{in } \mathbb{R}^N \setminus \{0\}, \\ u \in D^{1,p}(\mathbb{R}^N), & \mu \in [0, \mu_1). \end{cases}$$

From [23], we see that this problem has radially symmetric ground states,

$$V_\varepsilon(x) = \varepsilon^{-\frac{N-p}{p}} U_{p,\mu}\left(\frac{x}{\varepsilon}\right) = \varepsilon^{-\frac{N-p}{p}} U_{p,\mu}\left(\frac{|x|}{\varepsilon}\right), \quad \forall \varepsilon > 0,$$

and they satisfy

$$\int_{\mathbb{R}^N} \left(\left|\nabla V_\varepsilon(x)\right|^p - \mu \frac{|V_\varepsilon(x)|^p}{|x|^p} \right) dx = \int_{\mathbb{R}^N} \left|V_\varepsilon(x)\right|^{p^*} dx = A_{\mu,0}^{\frac{N}{p}},$$

where $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ is the unique radial solution of this problem, satisfying

$$U_{p,\mu}(1) = \left(\frac{N(\mu_1 - \mu)}{N-p}\right)^{\frac{1}{p^*-p}}.$$

Moreover, $U_{p,\mu}$ has the following properties:

$$\lim_{r\to 0} r^{a(\mu)} U_{p,\mu}(r) = c_1 > 0, \qquad \lim_{r\to+\infty} r^{b(\mu)} U_{p,\mu}(r) = c_2 > 0,$$

$$\lim_{r\to 0} r^{a(\mu)+1} U'_{p,\mu}(r) = c_1 a(\mu) \ge 0, \qquad \lim_{r\to +\infty} r^{b(\mu)+1} U'_{p,\mu}(r) = c_2 b(\mu) > 0,$$

where c_1 and c_2 are positive constants depending on p and N; $a(\mu)$ and $b(\mu)$ are zeros of the function

$$f(t) = (p-1)t^p - (N-p)t^{p-1} + \mu, \quad t \ge 0, 0 \le \mu < \mu_1,$$

satisfying $0 \le a(\mu) < \frac{N-p}{p} < b(\mu)$; see [23]. The above results are useful in studying equation (1.1).

Remark 1.3 As $\mu = 0$ and $s = 0$, then $b(\mu) = b(0) = \frac{N-p}{p-1}$. When $p = 2$ and $0 \le \mu < \mu_2 := (\frac{N-2}{2})^2$, it is well known that $a(\mu) = \sqrt{\mu_2} - \sqrt{\mu_2 - \mu}$ and $b(\mu) = \sqrt{\mu_2} + \sqrt{\mu_2 - \mu}$.

In Section 2, we will give the proof of Theorem 1.1. In Section 3, we first of all give some preliminary lemmas, and then we will complete the proof of Theorem 1.2.

2 Proof of Theorem 1.1

Let $X := W_0^{1,p}(\Omega)$ and $u^\pm := \max\{\pm u, 0\}$. Note that the values of $f(x,t)$ for $t < 0$ are irrelevant in Theorems 1.1-1.2, so we define

$$f(x,t) \equiv 0, \quad x \in \Omega, t \le 0.$$

The functional corresponding of (1.1) is

$$\begin{aligned} I(u) = {}& \frac{1}{p} \int_\Omega \left(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p} \right) dx - \frac{1}{p^*(s)} \int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s} dx \\ & - \lambda \int_\Omega F\left(x, u^+\right) dx, \quad u \in W_0^{1,p}(\Omega). \end{aligned}$$

By (A_1) and the Hardy inequalities (see [1]), we have $I \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$. Now it is well known that there is a one-to-one correspondence between the weak solutions of (1.1) and the critical points of I on $W_0^{1,p}(\Omega)$. More precisely, we say $u \in W_0^{1,p}(\Omega)$ is a weak solution of (1.1) if

$$\begin{aligned} \left\langle I'(u), v \right\rangle = {}& \int_\Omega \left(|\nabla u|^{p-2} \nabla u \nabla v - \mu \frac{|u|^{p-2} u v}{|x|^p} \right) dx - \int_\Omega \frac{(u^+)^{p^*(s)-1}}{|x|^s} v \, dx - \lambda \int_\Omega f\left(x, u^+\right) v \, dx \\ = {}& 0 \end{aligned}$$

for any $v \in W_0^{1,p}(\Omega)$.

Proof of Theorem 1.1 By the Sobolev and Hardy-Sobolev inequalities, we get

$$\begin{aligned} & \|u\|_p^p \le C\|u\|^p, \qquad \int_\Omega \frac{|u|^{p^*(s)}}{|x|^s} dx \le C\|u\|^{p^*(s)} \quad \text{and} \\ & \|u\|_{p^*}^{p^*} \le C\|u\|^{p^*}, \quad \forall u \in X, \end{aligned} \tag{2.1}$$

and it follows from (A_1) that

$$\exists\delta > 0 \quad \text{such that} \quad \bigl|F(x,t)\bigr| < \frac{t^{p^*(s)}}{p^*(s)|x^s|} \quad \text{for } t > \delta,$$
$$\exists M_1 > 0 \quad \text{such that} \quad \bigl|F(x,t)\bigr| \le M_1, \quad \forall t \in (0,\delta],$$

uniformly for all $x \in \overline{\Omega} \setminus \{0\}$. Thus, we get

$$\bigl|F(x,t)\bigr| \le M_1 + \frac{t^{p^*(s)}}{p^*(s)|x|^s}, \quad \forall t \in \mathbb{R}, x \in \overline{\Omega} \setminus \{0\}. \tag{2.2}$$

By (2.1) and (2.2), we have

$$I(u) = \frac{1}{p}\|u\|^p - \frac{1}{p^*(s)}\int_\Omega \frac{(u^+)^{p^*(s)}}{|x|^s}\,dx - \lambda\int_\Omega F\bigl(x,u^+\bigr)\,dx \ge \frac{1}{p}\|u\|^p - C_1\|u\|^{p^*(s)} - \lambda M_1|\Omega|$$

for all $\lambda \in (0,1]$ and some $C_1 = \frac{C\mu}{p^*(s)}$, so there are $\rho > 0$ and $\lambda^* \in (0,1]$ such that

$$I(u) > 0 \quad \text{if } \|u\| = \rho \quad \text{and} \quad I(u) \ge -C_2 \quad \text{if } \|u\| \le \rho$$

for any $0 < \lambda < \lambda^*$, where $C_2 = C_1\rho^{p^*(s)} + \lambda^* M_1|\Omega|$. We choose $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $u_0^+ \not\equiv 0$. Let $M_2 := \|u_0\|^p/(\lambda\|u_0^+\|_p^p)$. By (A_1), there is δ_1 such that

$$\bigl|F(x,t)\bigr| \ge \frac{2M_2}{p}|t|^p, \quad 0 < t < \delta_1.$$

Hence, we get

$$\begin{aligned} I(ru_0) &= \frac{r^p}{p}\|u_0\|^p - \frac{r^{p^*(s)}}{p^*(s)}\int_\Omega \frac{(u_0^+)^{p^*(s)}}{|x|^s}\,dx - \lambda\int_\Omega F\bigl(x,ru_0^+\bigr)\,dx \\ &\le \frac{r^p}{p}\|u_0\|^p - \frac{2r^p}{p}\lambda M_2\bigl\|u_0^+\bigr\|_p^p = -\frac{r^p}{p}\|u_0\|^p < 0 \end{aligned}$$

for any $0 < \lambda < \lambda^*$ and $0 < r < \min\{\rho, \delta_1/\|u_0^+\|_\infty\}$. So there is u small enough such that $I(u) < 0$. We deduce that

$$\inf_{u\in \overline{B_\rho(0)}} I(u) < 0 < \inf_{u\in\partial B_\rho(0)} I(u).$$

By Ekeland's variational principle in [24], there is a minimizing sequence $\{u_n\} \subset \overline{B_\rho(0)}$ such that

$$I(u_n) \le \inf_{u\in\overline{B_\rho(0)}} I(u) + \frac{1}{n}, \qquad I(\omega) \ge I(u_n) - \frac{1}{n}\|\omega - u_n\|, \quad \omega \in \overline{B_\rho(0)}.$$

So, we have

$$\bigl\|I'(u_n)\bigr\| \to 0 \quad \text{and} \quad I(u_n) \to c_\lambda \quad \text{as } n \to \infty,$$

where c_λ stands for the infimum of $I(u)$ on $\overline{B_\rho(0)}$. Note that $\{u_n\}$ is bounded and $\overline{B_\rho(0)}$ is a closed convex set, so there is $u_\lambda \in \overline{B_\rho(0)} \subset W_0^{1,p}(\Omega)$. By [1], we have

$$\begin{aligned}
&u_n \rightharpoonup u_\lambda \quad \text{weakly in } W_0^{1,p}(\Omega),\\
&u_n \to u_\lambda \quad \text{strongly in } L^\gamma(\Omega), 1 < \gamma < p^*,\\
&u_n \to u_\lambda \quad \text{a.e. in } \Omega,\\
&\nabla u_n \to \nabla u_\lambda \quad \text{a.e. in } \Omega,\\
&\frac{u_n}{x} \rightharpoonup \frac{u_\lambda}{x} \quad \text{weakly in } L^p(\Omega),\\
&\int_\Omega \frac{|u_n|^{p^*(s)-2}u_n}{|x|^s} v\,dx \to \int_\Omega \frac{|u_\lambda|^{p^*(s)-2}u_\lambda}{|x|^s} v\,dx, \quad \forall v \in W_0^{1,p}(\Omega).
\end{aligned}$$

Thus, passing to the limit in $\langle I'(u_n), v\rangle$, as $n \to \infty$, we have

$$\int_\Omega \left(|\nabla u_\lambda|^{p-2}\nabla u_\lambda \nabla v - \mu \frac{|u_\lambda|^{p-2}u_\lambda v}{|x|^p} \right) dx - \int_\Omega \frac{(u_\lambda^+)^{p^*(s)-1}v}{|x|^s}\,dx - \lambda \int_\Omega f\left(x, u_\lambda^+\right) v\,dx = 0$$

for all $v \in W_0^{1,p}(\Omega)$. That is, $\langle I'(u_\lambda), v\rangle = 0$. Therefore, u_λ is a critical point of I. Since $\|u_\lambda^-\|^p = -\langle I'(u_\lambda), u_\lambda^-\rangle = 0$, $u_\lambda = u_\lambda^+ \geq 0$. Moreover, by (A_1) and the boundedness of Ω, we have

$$\begin{aligned}
&\exists M_3 > 0 \quad \text{such that} \quad \left|f(x,t)\right| < \frac{1}{\lambda}\frac{t^{p^*(s)-1}}{|x|^s} \quad \text{for } t > M_3,\\
&\exists \delta_2 \in (0, M_3) \quad \text{such that} \quad \left|f(x,t)\right| > 0 \quad \text{for } 0 < t < \delta_2,\\
&\exists M_4 > 0 \quad \text{such that} \quad \left|f(x,t)\right| \leq M_4 \quad \text{for all } t \in [\delta_2, M_3]
\end{aligned}$$

for all $x \in \overline{\Omega} \setminus \{0\}$. Therefore, we deduce that

$$f(x,t) \geq -\frac{1}{\lambda}\frac{t^{p^*(s)-1}}{|x|^s} - M_4 t \delta_2^{-1}, \quad \forall t \in \mathbb{R}^+, x \in \overline{\Omega} \setminus \{0\}. \tag{2.3}$$

From (1.1) and (2.3), we have $-\triangle_p u_\lambda + \lambda M_4 \delta_2^{-1} u_\lambda \geq 0$. By the strong maximum principle, we have $u_\lambda > 0$. So the proof of Theorem 1.1 is finished. □

3 Proof of Theorem 1.2

In this section, we will look for the second positive solution by a translated functional as in [3]. For fixed $\lambda \in (0, \lambda^*)$, we will look for the second solution of (1.1) of the form $u = u_\lambda + v$, where u_λ is the first positive solution obtained in the previous section. The corresponding equation for v is

$$\begin{cases} -\triangle_p v - \mu \frac{|v|^{p-2}v}{|x|^p} \\ \quad = \frac{(u_\lambda + v)^{p^*(s)-1}}{|x|^s} - \frac{u_\lambda^{p^*(s)-1}}{|x|^s} + \lambda f(x, u_\lambda + v) - \lambda f(x, u_\lambda), & x \in \Omega \setminus \{0\}, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Let us define

$$g(x,t)=\begin{cases}\frac{(u_\lambda+t)^{p^*(s)-1}}{|x|^s}-\frac{u_\lambda^{p^*(s)-1}}{|x|^s}+\lambda f(x,u_\lambda+t)-\lambda f(x,u_\lambda), & t\geq 0,\\ 0, & t<0,\end{cases} \tag{3.2}$$

$$G(x,t)=\int_0^t g(x,s)\,ds,$$

and

$$\begin{aligned}J(v)&=\frac{1}{p}\int_\Omega\left(|\nabla v|^p-\mu\frac{|v|^p}{|x|^p}\right)dx-\int_\Omega G\big(x,v^+\big)\,dx\\&=\frac{1}{p}\|v\|^p-\frac{1}{p^*(s)}\int_\Omega\left(\frac{(u_\lambda+v^+)^{p^*(s)}}{|x|^s}-\frac{u_\lambda^{p^*(s)}}{|x|^s}-p^*(s)\frac{u_\lambda^{p^*(s)-1}v^+}{|x|^s}\right)dx\\&\quad-\lambda\int_\Omega\big(F\big(x,u_\lambda+v^+\big)-F(x,u_\lambda)-f(x,u_\lambda)v^+\big)\,dx.\end{aligned}$$

Now, we have one-to-one correspondence between critical points of J in $W_0^{1,p}(\Omega)$ and solutions of (3.1). That is, if $v\in W_0^{1,p}(\Omega)$, $v\not\equiv 0$ is a critical point of J, then v is a solution of (3.1). Since $\|v^-\|^p=-\langle J'(v),v^-\rangle=0$, $v=v^+\geq 0$. Besides, by the maximum principle, $v>0$ in Ω. Here, $u=u_\lambda+v$ is a positive solution of (1.1) and $u\neq u_\lambda$. If $v=0$ is the only critical point of J in $W_0^{1,p}(\Omega)$, we will get a contradiction. Then the existence of the second positive solution of (1.1) can be proved.

Lemma 3.1 $v=0$ *is a local minimum of* J *in* $W_0^{1,p}(\Omega)$.

Proof For any $v\in W_0^{1,p}(\Omega)$, we write $v=v^+-v^-$. By J and direct computation, we have

$$J(v)=\frac{1}{p}\big\|v^-\big\|^p+I\big(u_\lambda+v^+\big)-I(u_\lambda). \tag{3.3}$$

Since u_λ is a local minimizer of I in $W_0^{1,p}(\Omega)$, we have $J(v)\geq\frac{1}{p}\|v^-\|^p$ for $\|v\|\leq\varepsilon$ with ε being small enough. □

Lemma 3.2 *Suppose that* $1<p<N$, (A_1) *and* (A_2) *hold, moreover,* $v=0$ *is the only critical point of* J. *Let* $\{v_n\}$ *be a* $(PS)_c$ *sequence with* $0<c<\frac{p-s}{p(N-s)}A_{\mu,s}^{\frac{N-s}{p-s}}$, *then we have*

$$v_n\to 0\quad\text{in } W_0^{1,p}(\Omega) \text{ as } n\to\infty.$$

Proof Let $\{v_n\}$ be a sequence in $W_0^{1,p}(\Omega)$ such that

$$J(v_n)\to c<\frac{p-s}{p(N-s)}A_{\mu,s}^{\frac{N-s}{p-s}}\quad\text{and}\quad J'(v_n)\to 0\quad\text{in }\big(W_0^{1,p}(\Omega)\big)^*. \tag{3.4}$$

By (3.3) and (3.4), we have

$$J(v_n)=\frac{1}{p}\big\|v_n^-\big\|^p+I\big(u_\lambda+v_n^+\big)-I(u_\lambda)=c+o(1), \tag{3.5}$$

$$\langle J'(v_n), u_\lambda + v_n^+\rangle = \int_\Omega |\nabla v_n^-|^{p-2}\nabla v_n^- \nabla u_\lambda\,dx + \langle I'(u_\lambda + v_n^+), u_\lambda + v_n^+\rangle = o(1)\|u_\lambda + v_n^+\|,$$

which yields

$$\begin{aligned}
&J(v_n) - \frac{1}{p}\langle J'(v_n), u_\lambda + v_n^+\rangle \\
&= \frac{1}{p}\left(\|v_n^-\|^p - \int_\Omega |\nabla v_n^-|^{p-2}\nabla v_n^-\nabla u_\lambda\,dx - \langle I'(u_\lambda + v_n^+), u_\lambda + v_n^+\rangle\right) + I(u_\lambda + v_n^+) - I(u_\lambda) \\
&\le c + 1 + o(1)\|u_\lambda + v_n^+\|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{1}{p}\left(\|v_n^-\|^p - \int_\Omega |\nabla v_n^-|^{p-2}\nabla v_n^-\nabla u_\lambda\,dx\right) + \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)\int_\Omega \frac{(u_\lambda + v_n^+)^{p^*(s)}}{|x|^s}\,dx \\
&\quad + \lambda\int_\Omega \left[\frac{1}{p}f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+) - F(x, u_\lambda + v_n^+)\right]dx \\
&\quad \le I(u_\lambda) + c + 1 + o(1)\|u_\lambda + v_n^+\|.
\end{aligned} \tag{3.6}$$

By (A_1) and the boundedness of Ω, for any $\varepsilon > 0$, there is $M_5 = M_5(\varepsilon) > 0$ such that

$$\begin{aligned}
&|f(x,t)t| \le \varepsilon\frac{|t|^{p^*(s)}}{|x|^s} \quad \text{for } x \in \Omega\setminus\{0\} \text{ and } |t| > M_5, \\
&|f(x,t)t| \le C_3(\varepsilon) \quad \text{for } x \in \Omega \text{ and } |t| \in [0, M_5]; \\
&|F(x,t)| \le \frac{\varepsilon}{p}\frac{|t|^{p^*(s)}}{|x|^s} \quad \text{for } x \in \Omega\setminus\{0\} \text{ and } |t| > M_5, \\
&|F(x,t)| \le C_4(\varepsilon) \quad \text{for } x \in \Omega \text{ and } |t| \in [0, M_5],
\end{aligned}$$

where $C_3(\varepsilon), C_4(\varepsilon) > 0$. Thus, we have

$$|f(x,t)t| \le C_3(\varepsilon) + \varepsilon\frac{|t|^{p^*(s)}}{|x|^s}, \quad (x,t) \in (\Omega\setminus\{0\}) \times \mathbb{R}, \tag{3.7}$$

$$|F(x,t)| \le C_4(\varepsilon) + \frac{\varepsilon}{p}\frac{|t|^{p^*(s)}}{|x|^s}, \quad (x,t) \in (\Omega\setminus\{0\}) \times \mathbb{R}. \tag{3.8}$$

Let $C(\varepsilon) = \frac{1}{p}C_3(\varepsilon) + C_4(\varepsilon)$, by (3.7) and (3.8), we have

$$F(x,t) - \frac{1}{p}f(x,t)t \le C(\varepsilon) + \frac{2\varepsilon}{p}\frac{|t|^{p^*(s)}}{|x|^s}, \quad (x,t) \in (\Omega\setminus\{0\}) \times \mathbb{R}. \tag{3.9}$$

By (3.6) and (3.9), we have

$$\begin{aligned}
&\left(\frac{p-s}{p(N-s)} - \frac{2\lambda\varepsilon}{p}\right)\int_\Omega \frac{(u_\lambda + v_n^+)^{p^*(s)}}{|x|^s}\,dx \\
&\quad \le \lambda C(\varepsilon)|\Omega| - \frac{1}{p}\|v_n^-\|^p + C_5\|v_n^-\|^{p-1} + C_6 + o(1)\|u_\lambda + v_n^+\|,
\end{aligned}$$

where $C_5 = \frac{1}{p}\|u_\lambda\|$ and $C_6 = I(u_\lambda) + c + 1$. Let $\varepsilon = \frac{p-s}{4(N-s)\lambda}$, then we have

$$\int_\Omega \frac{(u_\lambda + v_n^+)^{p^*(s)}}{|x|^s}\,dx \le C_7\|v_n^-\|^{p-1} + C_8 + o(1)\|u_\lambda + v_n^+\|,$$

where $C_7 = \frac{2p(N-s)}{p-s}C_5$ and $C_8 = \frac{2p(N-s)}{p-s}(\lambda C(\varepsilon)|\Omega| + C_6)$. Together with (3.3), (3.5), and (3.8), we have

$$\begin{aligned}
&\frac{1-\varepsilon}{p}\|v_n^-\|^p + \frac{1}{p}\big[(1-\varepsilon)\|v_n^+\|^p - \overline{C_\varepsilon}\|u_\lambda\|^p - (1-\varepsilon)\|v_n^+\|^{p-1}\big] \\
&\quad\le \frac{1}{p}\|v_n^-\|^p + \frac{1}{p}\big[(1-\varepsilon)\|v_n^+\|^p - \overline{C_\varepsilon}\|u_\lambda\|^p\big] \\
&\quad\le \frac{1}{p}\|v_n^-\|^p + \frac{1}{p}\big|\big(\|v_n^+\| - \|u_\lambda\|\big)\big|^p \\
&\quad\le \frac{1}{p}\|v_n^-\|^p + \frac{1}{p}\|u_\lambda + v_n^+\|^p \\
&\quad= \frac{1}{p^*(s)}\int_\Omega \frac{(u_\lambda + v_n^+)^{p^*(s)}}{|x|^s}\,dx + \lambda\int_\Omega F\big(x, u_\lambda + v_n^+\big)\,dx + J(v_n) + I(u_\lambda) + o(1) \\
&\quad\le C_9\|v_n^-\|^{p-1} + C_{10} + o(1)\|u_\lambda + v_n^+\|,
\end{aligned}$$

where the second inequality is due to the elementary inequality

$$|a-b|^t \ge (1-\varepsilon)a^t - \overline{C_\varepsilon}b^t, \quad t \ge 1, a, b > 0.$$

Here, $C_9 = (\frac{1}{p^*(s)} + \frac{\lambda\varepsilon}{p})C_7$ and $C_{10} = \lambda C_4(\varepsilon)|\Omega| + (\frac{1}{p^*(s)} + \frac{\lambda\varepsilon}{p})C_8 + I(u_\lambda) + c + o(1)$. Since $\|v_n^-\|^p + \|v_n^+\|^p = \|v_n\|^p$, we get

$$\|v_n\|^p - C_{11}\|v_n^+\|^{p-1} - C'_{11}\|v_n^-\|^{p-1} \le C_{12} + o(1)\|u_\lambda\|,$$

where $C_{11} = 1 + o(1)$, $C'_{11} = \frac{C_9 p}{1-\varepsilon}$, $C_{12} = \frac{\overline{C_\varepsilon}\|u_\lambda\|^p + pC_{10}}{1-\varepsilon}$. So we get

$$\|v_n\|^p - C_{13}\|v_n\|^{p-1} \le C_{12} + o(1)\|u_\lambda\|,$$

where $C_{13} = C_{11} + C'_{11}$. It shows that $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$, going if necessary to a subsequence, we have

$$\begin{aligned}
&v_n \rightharpoonup v_0 \quad \text{weakly in } W_0^{1,p}(\Omega), \\
&v_n \to v_0 \quad \text{strongly in } L^\gamma(\Omega), 1 < \gamma < p^*, \\
&v_n \to v_0 \quad \text{a.e. in } \Omega,
\end{aligned} \tag{3.10}$$

as $n \to \infty$.

Since v_n is bounded in $W_0^{1,p}(\Omega)$, it follows from the Sobolev embedding theorem that there is $M' > 0$ such that $\|u_\lambda + v_n^+\|_{p^*(s)}^{p^*(s)} \le M'$. Let meas E denote the measure of E. By (A_1), for any $\varepsilon > 0$, there is $C_{14}(\varepsilon) > 0$ such that

$$\big|f(x,t)t\big| \le C_{14}(\varepsilon) + \frac{\varepsilon}{2M'}|t|^{p^*(s)}, \quad (x,t) \in \overline{\Omega} \times \mathbb{R}.$$

Let $\delta = \frac{\varepsilon}{2C_{14}(\varepsilon)} > 0$, if $E \subset \Omega$, $\operatorname{meas} E < \delta$, we have

$$\begin{aligned}\left|\int_E f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+)\,dx\right| &\le \int_E \left|f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+)\right|dx \\ &\le \int_E C_{14}(\varepsilon)\,dx + \frac{\varepsilon}{2M'}\int_E \left|u_\lambda + v_n^+\right|^{p^*(s)}dx \\ &\le C_{14}(\varepsilon)\operatorname{meas} E + \frac{\varepsilon}{2} < \varepsilon.\end{aligned}$$

By the Vitali theorem, we have

$$\int_\Omega f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+)\,dx \to \int_\Omega f(x, u_\lambda + v_0^+)(u_\lambda + v_0^+)\,dx \quad \text{as } n \to \infty.$$

Hence,

$$\begin{aligned}\int_\Omega f(x, u_\lambda + v_n^+)(u_\lambda + v_n)\,dx &= \int_\Omega f(x, u_\lambda + v_n^+)(u_\lambda + v_n^+)\,dx - \int_\Omega f(x, u_\lambda)(v_n^-)\,dx \\ &\to \int_\Omega f(x, u_\lambda + v_0^+)(u_\lambda + v_0)\,dx \quad \text{as } n \to \infty. \end{aligned}\tag{3.11}$$

By the same method, we get

$$\begin{aligned}&\int_\Omega f(x, u_\lambda + v_n^+)\omega\,dx \to \int_\Omega f(x, u_\lambda + v_0^+)\omega\,dx, \\ &\int_\Omega F(x, u_\lambda + v_n^+)\,dx \to \int_\Omega F(x, u_\lambda + v_0^+)\,dx\end{aligned}\tag{3.12}$$

as $n \to \infty$ for $\omega \in W_0^{1,p}(\Omega)$. Similar to the proof of Theorem 1.1, we have

$$0 = \lim_{n\to\infty}\langle J'(v_n), \omega\rangle = \langle J'(v_0), \omega\rangle$$

for $\omega \in W_0^{1,p}(\Omega)$, which implies that $J'(v_0) = 0$. Therefore, v_0 is a critical point of J in $W_0^{1,p}(\Omega)$. By the assumption that $v = 0$ is the only critical point of J, we have $v_0=0$. Now, we want to prove $v_0 \to 0$ strongly in $W_0^{1,p}(\Omega)$. By (3.10), (3.12), and the Brezis-Leib Lemma (see [25]), we have

$$J(v_n) = \frac{1}{p}\left\|v_n^-\right\|^p + I(u_\lambda + v_n^+) - I(u_\lambda) = \frac{1}{p}\|v_n\|^p - \frac{1}{p^*(s)}\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx + o(1).$$

Therefore,

$$\langle J'(v_n), v_n\rangle = \|v_n\|^p - \int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx + o(1) \to 0.$$

In fact, $\|v_n\|^p \to 0$ as $n \to \infty$. If not, then there is a subsequence (still denoted by v_n) such that

$$\lim_{n\to\infty}\|v_n\|^p = k, \qquad \lim_{n\to\infty}\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx = k, \quad k > 0.$$

By (1.2), we get

$$\|v_n\|^p \geq A_{\mu,s}\left(\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx\right)^{\frac{p}{p^*(s)}}, \quad \text{for all } n \in \mathbb{N}.$$

Then $k \geq A_{\mu,s}k^{\frac{p}{p^*(s)}}$, *i.e.*, $k \geq A_{\mu,s}^{\frac{N-s}{p-s}}$. Thus, we have

$$\begin{aligned} c &= o(1) + J(v_n) = \frac{1}{p}\|v_n\|^p - \frac{1}{p^*(s)}\int_\Omega \frac{(v_n^+)^{p^*(s)}}{|x|^s}\,dx + o(1) \\ &= \frac{p-s}{p(N-s)}k + o(1) \\ &\geq \frac{p-s}{p(N-s)}A_{\mu,s}^{\frac{N-s}{p-s}}. \end{aligned}$$

It is a contradiction. So $v_n \to 0$ strongly in $W_0^{1,p}(\Omega)$ as $n \to \infty$. □

Lemma 3.3 [21] *If* $1 < p < N$, $0 \leq s < p$ *and* $0 \leq \mu < \mu_1$, *then the limiting problem*

$$\begin{cases} -\triangle_p u - \mu\frac{u^{p-1}}{|x|^p} = \frac{u^{p^*(s)-1}}{|x|^s}, & \text{in } \mathbb{R}^N \setminus \{0\}, \\ u > 0, & \text{in } \mathbb{R}^N \setminus \{0\}, \\ u \in D^{1,p}(\mathbb{R}^N), \end{cases} \tag{P}$$

has radially symmetric ground states,

$$\widetilde{V}_\varepsilon(x) := \varepsilon^{-\frac{N-p}{p}}\widetilde{U}_{p,\mu}\left(\frac{x}{\varepsilon}\right) = \varepsilon^{-\frac{N-p}{p}}\widetilde{U}_{p,\mu}\left(\frac{|x|}{\varepsilon}\right), \quad \forall\varepsilon > 0,$$

and it satisfies

$$\int_{\mathbb{R}^N}\left(\left|\nabla\widetilde{V}_\varepsilon(x)\right|^p - \mu\frac{|\widetilde{V}_\varepsilon(x)|^p}{|x|^p}\right)dx = \int_{\mathbb{R}^N}\frac{|\widetilde{V}_\varepsilon(x)|^{p^*(s)}}{|x|^s}\,dx = A_{\mu,s}^{\frac{N-s}{p-s}},$$

where $\widetilde{U}_{p,\mu}(x) = \widetilde{U}_{p,\mu}(|x|)$ *is the unique radial solution of* (P), *satisfying*

$$\widetilde{U}_{p,\mu}(1) = \left(\frac{(N-s)(\mu_1-\mu)}{N-p}\right)^{\frac{1}{p^*(s)-p}}.$$

Moreover, $\widetilde{U}_{p,\mu}$ *has the following properties*:

$$\begin{aligned} &\lim_{r\to 0} r^{a(\mu)}\widetilde{U}_{p,\mu}(r) = c_1 > 0, &&\lim_{r\to+\infty} r^{b(\mu)}\widetilde{U}_{p,\mu}(r) = c_2 > 0, \\ &\lim_{r\to 0} r^{a(\mu)+1}\widetilde{U}'_{p,\mu}(r) = c_1 a(\mu) \geq 0, &&\lim_{r\to+\infty} r^{b(\mu)+1}\widetilde{U}'_{p,\mu}(r) = c_2 b(\mu) > 0, \end{aligned}$$

where c_1 *and* c_2 *are positive constants depending on* p *and* N; $a(\mu)$ *and* $b(\mu)$ *are zeros of the function*

$$f(t) = (p-1)t^p - (N-p)t^{p-1} + \mu, \quad t \geq 0, 0 \leq \mu < \mu_1,$$

satisfying $0 \leq a(\mu) < \frac{N-p}{p} < b(\mu) < \frac{N-p}{p-1}$.

Since $u_\lambda > 0$ is a solution of (1.1), similar to the proof of Theorem 1.1 in [26], there are constants $R > 0$ and $r_0 > 0$ such that $B_{2R}(0) \subset \Omega$ and

$$0 < r_0 \le u_\lambda(x), \quad \forall x \in B_{2R}(0) \setminus \{0\}. \tag{3.13}$$

Let $\varphi \in C_0^\infty(\Omega)$ such that $0 \le \varphi(x) \le 1$ and

$$\varphi(x) := \begin{cases} 1, & |x| \le R, \\ 0, & |x| \ge 2R, \end{cases}$$

where $B_{2R}(0) \subset \Omega$. Set $v_\varepsilon(x) = \varphi(x)\widetilde{V}_\varepsilon(x)$, $\varepsilon > 0$, where $\widetilde{V}_\varepsilon(x)$ is defined in Lemma 3.3. Then we can get the following results by the method used in [27]:

$$\|v_\varepsilon\|^p = A_{\mu,s}^{\frac{N-s}{p-s}} + O\bigl(\varepsilon^{b(\mu)p+p-N}\bigr), \tag{3.14}$$

$$\int_\Omega \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}\,dx = A_{\mu,s}^{\frac{N-s}{p-s}} + O\bigl(\varepsilon^{b(\mu)p^*(s)+s-N}\bigr), \tag{3.15}$$

$$\int_\Omega \frac{|v_\varepsilon|^r}{|x|^s}\,dx = O\bigl(\varepsilon^{p-s}\bigr), \quad \frac{N-s}{b(\mu)} < r < p^*(s). \tag{3.16}$$

Lemma 3.4 *For $\gamma \ge 2$, $1 \le t \le \gamma - 1$, $\forall a, b > 0$, there exists a positive constant C such that*

$$(a+b)^\gamma \ge a^\gamma + b^\gamma + Ca^{\gamma-t}b^t.$$

Proof To prove this lemma, we only need to prove

$$(1+x)^\gamma \ge 1 + x^\gamma + Cx^t, \quad 0 < x < \infty.$$

Let $\gamma = k + \theta$, $t = m + \eta$, where $k \ge 2$, $1 \le m \le k-1$ are integral numbers and $0 \le \eta \le \theta < 1$ are real numbers. Clearly,

$$\begin{aligned}(1+x)^\gamma &= (1+x)^{k+\theta} = (1+x)^k(1+x)^\theta \ge \bigl(1 + x^k + Cx^m\bigr)(1+x)^\theta \\ &\ge 1 + x^{k+\theta} + Cx^m(1+x)^\theta \\ &\ge 1 + x^{k+\theta} + Cx^m x^\eta = 1 + x^\gamma + Cx^t. \end{aligned}$$

□

Lemma 3.5 *If $N \ge 3$, $0 \le s < p$, $0 \le \mu < \mu_1$, $1 < p < N$, (A_1), (A_2), (A_3) (or (A_3')), and $f(x,0) \equiv 0$ hold, then there is $v_* \in W_0^{1,p}(\Omega)$, $v_* \not\equiv 0$, such that*

$$\sup_{t\ge 0} J(tv_*) < \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}}.$$

Proof By (3.2), (A_2), and Lemma 3.4, we have

$$g(x,l) \ge \frac{l^{p^*(s)-1}}{|x|^s} + C\frac{l^{p-1}u_\lambda^{p^*(s)-p}}{|x|^s}.$$

By (A_3) or (A_3'), we have $p \geq 2$ and $s \leq N - \frac{(N-p)(1+p)}{p}$, which imply $p^*(s) - 1 \geq 2$ and $1 \leq p - 1 \leq (p^*(s) - 1) - 1$. Therefore,

$$G(x, tv_\varepsilon) \geq \frac{t^{p^*(s)}}{p^*(s)} \frac{v_\varepsilon^{p^*(s)}}{|x|^s} + \frac{Ct^p}{p} \frac{v_\varepsilon^p u_\lambda^{p^*(s)-p}}{|x|^s}.$$

From (A_3) (or (A_3')), we have $s > N - Pb(\mu)$, which implies $p > \frac{N-s}{b(\mu)}$, so (3.16) holds. So by (3.13)-(3.16), we have

$$\begin{aligned} J(tv_\varepsilon) &= \frac{t^p}{p} \|v_\varepsilon\|^p - \int_\Omega G(x, tv_\varepsilon)\,dx \\ &\leq \frac{t^p}{p} \|v_\varepsilon\|^p - \frac{t^{p^*(s)}}{p^*(s)} \int_\Omega \frac{|v_\varepsilon|^{p^*(s)}}{|x|^s}\,dx - C_{15} t^p \int_\Omega \frac{v_\varepsilon^p}{|x|^s}\,dx \\ &= \frac{t^p}{p}\big(A_{\mu,s}^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p+p-N}\big)\big) - \frac{t^{p^*(s)}}{p^*(s)}\big(A_{\mu,s}^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p^*(s)+s-N}\big)\big) - C_{15} t^p O\big(\varepsilon^{p-s}\big), \end{aligned}$$

where $C_{15} = \frac{Cr_0^{p^*(s)-p}}{p}$. Let

$$Q(t) := \frac{t^p}{p}\big(A_{\mu,s}^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p+p-N}\big)\big) - \frac{t^{p^*(s)}}{p^*(s)}\big(A_{\mu,s}^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p^*(s)+s-N}\big)\big) - C_{15} t^p O\big(\varepsilon^{p-s}\big).$$

Clearly, the following equation:

$$0 = Q'(t) = t^{p-1}\big[A_{\mu,s}^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p+p-N}\big) - O\big(\varepsilon^{p-s}\big)\big] - t^{p^*(s)-1}\big[A_{\mu,s}^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p^*(s)+s-N}\big)\big]$$

has only a positive root

$$t_\varepsilon = \left(\frac{A_{\mu,s}^{\frac{N-s}{p-s}} + O(\varepsilon^{b(\mu)p+p-N}) - O(\varepsilon^{p-s})}{A_{\mu,s}^{\frac{N-s}{p-s}} + O(\varepsilon^{b(\mu)p^*(s)+s-N})}\right)^{\frac{1}{p^*(s)-p}}.$$

We have

$$\begin{aligned} Q(t_\varepsilon) &= \frac{t_\varepsilon^p}{p}\big(A_{\mu,s}^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p+p-N}\big) - O\big(\varepsilon^{p-s}\big)\big) - \frac{t_\varepsilon^{p^*(s)}}{p^*(s)}\big(A_{\mu,s}^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p^*(s)+s-N}\big)\big) \\ &= \left(\frac{1}{p} - \frac{1}{p^*(s)}\right)\big[A_{\mu,s}^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p^*(s)+s-N}\big)\big]\left[\frac{A_{\mu,s}^{\frac{N-s}{p-s}} + O(\varepsilon^{b(\mu)p+p-N}) - O(\varepsilon^{p-s})}{A_{\mu,s}^{\frac{N-s}{p-s}} + O(\varepsilon^{b(\mu)p^*(s)+s-N})}\right]^{\frac{p^*(s)}{p^*(s)-p}} \\ &= \frac{p-s}{p(N-s)}\big[A_{\mu,s}^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p^*(s)+s-N}\big)\big]^{-\frac{N-p}{p-s}}\big[A_{\mu,s}^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p+p-N}\big) - O\big(\varepsilon^{p-s}\big)\big]^{\frac{N-s}{p-s}} \\ &= \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}} + O\big(\varepsilon^{b(\mu)p^*(s)+s-N}\big) + O\big(\varepsilon^{b(\mu)p+p-N}\big) - O\big(\varepsilon^{p-s}\big). \end{aligned}$$

By $s > N - pb(\mu)$ (see (A_3) or (A_3')), we have

$$b(\mu)p + p - N > p - s.$$

Since $b(\mu) > \frac{N-p}{p}$ implies $b(\mu)p^*(s) + s - N > b(\mu)p + p - N$, we have

$$b(\mu)p^*(s) + s - N > p - s.$$

Since $Q(0) = 0$ and $\lim_{t\to+\infty} Q(t) = -\infty$, we have

$$\sup_{t\geq 0} Q(t) = Q(t_\varepsilon) < \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}}$$

for $\varepsilon > 0$ sufficiently small. So we get

$$\sup_{t\geq 0} J(tv_\varepsilon) \leq \sup_{t\geq 0} Q(t) < \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}},$$

for $\varepsilon > 0$ sufficiently small. It completes the proof if we let $v_* = v_\varepsilon$ with $\varepsilon > 0$ being sufficiently small. □

Proof of Theorem 1.2 If $v = 0$ is the only critical point of J in $W_0^{1,p}(\Omega)$. By Lemma 3.1, we know there is $\alpha > 0$ such that $J(v) > \alpha$, $\forall v \in \partial B_\rho = \{v \in W_0^{1,p}(\Omega), \|v\| = \rho\}$, where $\rho > 0$ is small enough. Lemma 3.5 implies that there is $v_* \in W_0^{1,p}(\Omega)$ and $v_* \not\equiv 0$ such that

$$\sup_{t\geq 0} J(tv_*) < \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}}.$$

By (3.8), we get $\lim_{t\to\infty} J(tv_*) \to -\infty$. Hence, we can choose $t_0 > 0$ such that $\|t_0 v_*\| > \rho$ and $J(t_0 v_*) < 0$. By the mountain pass lemma in [22], there is a sequence $\{v_n\} \subset W_0^{1,p}(\Omega)$ satisfying

$$J(v_n) \to c \geq \alpha \quad \text{and} \quad J'(v_n) \to 0,$$

where

$$c = \inf_{h\in\Gamma} \max_{t\in[0,1]} J\bigl(h(t)\bigr),$$

$$\Gamma = \bigl\{h \in C\bigl([0,1], X\bigr) \mid h(0) = 0, h(1) = t_0 v_*\bigr\}.$$

We have

$$0 < \alpha \leq c = \inf_{h\in\Gamma} \max_{t\in[0,1]} J\bigl(h(t)\bigr) \leq \max_{t\in[0,1]} J(tt_0 v_*) \leq \sup_{t\geq 0} J(tv_*) < \frac{p-s}{p(N-s)} A_{\mu,s}^{\frac{N-s}{p-s}},$$

and this together with Lemma 3.2 implies that $v_n \to 0$ strongly in $W_0^{1,p}(\Omega)$ as $n \to \infty$. Hence, we have $0 = J(0) = \lim_{n\to\infty} J(v_n) = c \geq \alpha > 0$, a contradiction. So, Theorem 1.2 holds. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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