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Nontrivial solutions for a class of perturbed fractional differential systems with impulsive effects

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Abstract
In this paper, a class of nonlinear impulsive fractional differential systems including Lipschitz continuous nonlinear terms is studied. Under suitable hypotheses and by using variational methods, some new criteria to guarantee that the fractional differential system has at least two nontrivial and nonnegative solutions are obtained. In addition, an example is presented to illustrate the applicability of the main results.

MSC: 34B15; 26A33; 34K45

Keywords: nontrivial solution; impulsive effects; fractional differential systems; variational methods; critical point theory

1 Introduction

In this paper, we consider the following perturbed fractional differential systems with impulsive effects:

$$\begin{cases} {}_tD_T^{\alpha_i}(a_i(t){}_0^cD_t^{\alpha_i}u_i(t)) = \lambda F_{u_i}(t,u) + h_i(u_i(t)), & 0 < t < T, t \neq t_j, \\ \Delta({}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i))(t_j) = I_{ij}(u_i(t_j)), & j = 1,2,\ldots,m, \\ u_i(0) = u_i(T) = 0, & 1 \leq i \leq N, \end{cases} \tag{P_λ}$$

where $u = (u_1,\ldots,u_N)$, $N \geq 1$, $|u| = \sqrt{\sum_{i=1}^N u_i^2}$, $\lambda > 0$, $0 < \alpha_i \leq 1$ for $1 \leq i \leq N$, $a_i \in L^\infty[0,T]$ with $\overline{a}_i := \operatorname{ess\,inf}_{[0,T]} a_i(t) > 0$ and ${}_tD_T^{\alpha_i}$ denotes the right Riemann-Liouville fractional derivative of order α_i; $0 = t_0 < t_1 < \cdots < t_{m+1} = T$, and $\Delta({}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i))(t_j) = {}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i)(t_j^+) - {}_tD_T^{\alpha_i-1}({}_0^cD_t^{\alpha_i}u_i)(t_j^-)$ where

$${}_tD_T^{\alpha_i-1}\left({}_0^cD_t^{\alpha_i}u_i\right)\left(t_j^+\right) = \lim_{t\to t_j^+} {}_tD_T^{\alpha_i-1}\left({}_0^cD_t^{\alpha_i}u_i\right)(t),$$

$${}_tD_T^{\alpha_i-1}\left({}_0^cD_t^{\alpha_i}u_i\right)\left(t_j^-\right) = \lim_{t\to t_j^-} {}_tD_T^{\alpha_i-1}\left({}_0^cD_t^{\alpha_i}u_i\right)(t),$$

and ${}_0^cD_t^{\alpha_i}$ is the left Caputo fractional derivatives of order α_i. The functions $I_{ij} \in C(\mathbf{R},\mathbf{R})$ are Lipschitz continuous functions with the Lipschitz constants $L_{ij} \geq 0$; *i.e.*,

$$\left|I_{ij}(s_1) - I_{ij}(s_2)\right| \leq L_{ij}|s_1 - s_2| \tag{1.1}$$

for all $s_1, s_2 \in \mathbf{R}$, satisfying $I_{ij}(0) = 0$ for $i = 1, \dots, N$, $j = 1, \dots, m$. $F : [0,T] \times \mathbf{R}^N \to \mathbf{R}$ is measurable with respect to t for every $u \in \mathbf{R}^N$, continuously differentiable in u, for almost every $t \in [0,T]$, and it satisfies the following summability condition:

(F0) $\sup_{|u|\le r_0}(\max\{|F(\cdot,u)|, |F_{u_i}(\cdot,u)|, i = 1, \dots, N\}) \in L^1([0,T])$ for any $r_0 > 0$, and $F(t,0,\dots,0) = 0$ for each $t \in [0,T]$. F_{u_i} denote the partial derivative of F with respect to u_i for $1 \le i \le N$. In addition, the functions $h_i : \mathbf{R} \to \mathbf{R}$ are Lipschitz continuous functions with the Lipschitz constants $L_i \ge 0$; *i.e.*,

$$\left|h_i(x_1) - h_i(x_2)\right| \le L_i|x_1 - x_2| \tag{1.2}$$

for every $x_1, x_2 \in \mathbf{R}$, and $h_i(0) = 0$ for $i = 1, \dots, N$.

Fractional differential equations play a very important role in the modeling of many phenomena in various fields of engineering, chemistry, physics, rheology, and biology. With the help of fractional calculus, the natural phenomena and mathematical models can be more accurately described. Therefore, the theory and application of fractional differential equations have been rapidly developed in recent years. For more details of fractional calculus theory, the reader can see the monographs of Kilbas *et al.* [1], Diethelm [2], and Zhou [3]. Recently, the existence and multiplicity of solutions to boundary value problems for nonlinear fractional differential equations is extremely investigated; see [4–12] and the references therein. Classical approaches to such problems include fixed point theorems, degree theory, the method of upper and lower solutions and so on. In [13] the authors studied a class of fractional boundary value problem by establishing corresponding variational structure and using mountain pass theorem. Since then the variational methods are applied to deal with the existence of solutions for fractional differential equations. The literature on this technique was extended by many authors as [14–22]. More precisely, the authors [22] obtained, by using recent results of Bonanno [23], for the following boundary value problem for fractional order differential equations:

$$\begin{cases} {}_tD_b^{\alpha}({}_a^cD_t^{\alpha}u(t)) + u(t) = \lambda f(t,u), \quad a < t < b, \\ u(a) = u(b) = 0, \end{cases}$$

the existence of at least two nonzero solutions.

On the other hand, impulsive boundary value problems for differential equations have become an important area of investigation in recent year. Such equations appear in describing processes which experience a suddenly changes of their states in chemical technology, physics phenomena, population dynamics, biotechnology, and economics, etc. [24]. Some classical tools of nonlinear analysis as topological methods have been applied to study such problems in the literature. Since very recently, the variational methods and critical point theorems belong to the most promising approaches to integer-order impulsive differential problems, and the literature on this approach has extensively grown; see [25–28] and the references therein.

However, to the best of our knowledge, there are few results on the solutions to impulsive fractional boundary value problems which were studied by the critical point theory and variational methods. Bonanno *et al.* in [15] studied the following impulsive fractional differential equations:

$$\begin{cases} {}_tD_T^{\alpha}({}_0^cD_t^{\alpha}u(t)) + a(t)u(t) = \lambda f(t,u), \quad 0 < t < T, t \neq t_j, \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = \mu I_j(u(t_j)), \quad j = 1,2,\ldots,m, \\ u(0) = u(T) = 0, \end{cases} \tag{1.3}$$

where $\lambda, \mu \in (0,+\infty)$ are two parameters. Under suitable hypotheses and by using the critical point theorem, the existence results of at least one and three solutions for the problem (1.3) are proved. In [16] the authors applying a recent critical point theorem of Bonanno and Marano [29] discussed the existence of at least three distinct weak solutions for the problem (1.3). In [17], by using critical point theory and variational methods, the authors gave some new criteria to guarantee that the problem (1.3) have at least one solution or infinitely many solutions, in the case $\lambda = \mu = 1$.

Motivated by the above work, in the present paper, our main aim is to investigate the multiplicity of nontrivial and nonnegative solutions of the system (P_λ) with Lipschitz continuous impulsive effects. Under some natural assumptions, by employing variational methods, some new results for the existence of at least two nontrivial and nonnegative solutions of the system (P_λ) are obtained. To the best of our knowledge, the investigation of the existence of solutions for impulsive fractional differential systems by employing variational methods has received considerably less attention. Obviously, our results are different from the main results in [15, 22] and extend the second order boundary value problem to the non-integer case in comparison with the papers [25, 27, 28]. The effectiveness of our results is illustrated by an example.

The remainder of this paper is organized as follows. In Section 2, we provide some basic definitions and lemmas that will be useful for our main results. In Section 3 we give the proofs of our main results and an example.

2 Preliminaries

To formulate our main results on the existence of nontrivial solutions for the system (P_λ), we present the following basic notations and lemmas.

Let $C_0^\infty([0,T],\mathbf{R}^N)$ be the set of all functions $x \in C_0^\infty([0,T],\mathbf{R}^N)$ with $x(0) = x(T) = 0$ and the norm

$$\|x\|_\infty = \max_{[0,T]}\left|x(t)\right|. \tag{2.1}$$

Denote the norm of the space $L^p([0,T],\mathbf{R}^N)$ for $1 \le p < \infty$ by

$$\|x\|_{L^p} = \left(\int_0^T \left|x(s)\right|^p ds\right)^{1/p}.$$

The following lemma shows the boundedness of the Riemann-Liouville fractional integral operators from the space $L^p([0,T],\mathbf{R}^N)$ to the space $L^p([0,T],\mathbf{R}^N)$, where $1 \le p < \infty$.

Lemma 2.1 ([13]) *Let $0 < \alpha \le 1$, $1 \le p < \infty$, and $f \in L^p([0,T],\mathbf{R}^N)$. Then*

$$\left\|{}_0D_\xi^{-\alpha}f\right\|_{L^p([0,t])} \le \frac{t^\alpha}{\Gamma(\alpha+1)}\|f\|_{L^p([0,t])}, \quad \text{for } \xi \in [0,t], t \in [0,T],$$

where ${}_0D_t^{-\alpha}$ is left Riemann-Liouville fractional integral of order α.

Definition 2.2 ([3]) Let $\frac{1}{2} < \alpha_i \leq 1$ for $1 \leq i \leq N$. The fractional derivative space $E_0^{\alpha_i}$ is defined by the closure of $C_0^\infty([0,T],\mathbf{R}^N)$, that is,

$$E_0^{\alpha_i} = \overline{C_0^\infty\bigl([0,T],\mathbf{R}^N\bigr)}$$

with respect to the weighted norm

$$\|u_i\|_{\alpha_i} = \left(\int_0^T a_i(t)\bigl|{}_0^cD_t^{\alpha_i}u_i(t)\bigr|^2\,dt + \int_0^T \bigl|u_i(t)\bigr|^2\,dt\right)^{1/2}, \quad \forall u_i \in E_0^{\alpha_i}. \tag{2.2}$$

It is clear that the fractional derivative space $E_0^{\alpha_i}$ is the space of functions $u_i \in L^2(0,T)$ having an α_i-order fractional derivative ${}_0^cD_t^{\alpha_i}u_i \in L^2(0,T)$ and $u_i(0) = u_i(T) = 0$. According to [13], Proposition 3.1, it is well known that the space $E_0^{\alpha_i}$ is a reflexive and separable Banach space. Moreover, from [2, 13] we have

$${}_0^cD_t^{\alpha_i}u_i = {}_0D_t^{\alpha_i}u_i, \qquad {}_t^cD_T^{\alpha_i}u_i = {}_tD_T^{\alpha_i}u_i$$

for any $u_i \in E_0^{\alpha_i}$, $1 \leq i \leq N$.

Lemma 2.3 ([18]) *Let $0 < \alpha_i \leq 1$ for $1 \leq i \leq N$. For every $u_i \in E_0^{\alpha_i}$, one has*

$$\|u_i\|_{L^2} \leq \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)\sqrt{\overline{a}_i}}\left(\int_0^T a_i(t)\bigl|{}_0^cD_t^{\alpha_i}u_i(t)\bigr|^2\,dt\right)^{1/2}. \tag{2.3}$$

Moreover, if $\alpha_i > \frac{1}{2}$, then

$$\|u_i\|_\infty \leq \frac{T^{\alpha_i-\frac{1}{2}}}{\Gamma(\alpha_i)\sqrt{\overline{a}_i(2\alpha_i-1)}}\left(\int_0^T a_i(t)\bigl|{}_0^cD_t^{\alpha_i}u_i(t)\bigr|^2\right)^{1/2}. \tag{2.4}$$

By (2.3), *one can consider $E_0^{\alpha_i}$ with respect to the norm*

$$\|u_i\|_{\alpha_i} = \left(\int_0^T a_i(t)\bigl|{}_0^cD_t^{\alpha_i}u_i(t)\bigr|^2\,dt\right)^{1/2}, \quad \forall u_i \in E_0^{\alpha_i}, \tag{2.5}$$

which is equivalent to (2.2). *Then one has*

$$\sum_{i=1}^N \|u_i\|_{L^2}^2 \leq A_0 \sum_{i=1}^N \|u_i\|_{\alpha_i}^2, \tag{2.6}$$

$$\sum_{i=1}^N \|u_i\|_\infty^2 \leq B_0 \sum_{i=1}^N \|u_i\|_{\alpha_i}^2 \quad \left(\text{if } \alpha_i > \frac{1}{2}\right), \tag{2.7}$$

where

$$A_0 = \max\left\{\frac{T^{2\alpha_i}}{[\Gamma(\alpha_i+1)]^2\overline{a}_i}, 1 \leq i \leq N\right\}, \qquad B_0 = \max\left\{\frac{T^{2\alpha_i-1}}{[\Gamma(\alpha_i)]^2\overline{a}_i(2\alpha_i-1)}, 1 \leq i \leq N\right\}.$$

For the space $E_0^{\alpha_i}$, similarly to the proof of Proposition 3.3 in [13], we have the following results.

Lemma 2.4 *Let $\frac{1}{2} < \alpha_i \le 1$ for $1 \le i \le N$. Assume that the sequence $\{x_n\}$ converges weakly to x in $E_0^{\alpha_i}$, i.e. $x_n \rightharpoonup x$. Then $\{x_n\}$ converges strongly to x in $C([0,T],\mathbf{R})$, i.e. $\|x_n - x\|_\infty \to 0$, as $n \to \infty$.*

In the sequel, X will denote the Cartesian product of N Sobolev spaces $E_0^{\alpha_1}, \ldots, E_0^{\alpha_N}$, i.e., $E_0^{\alpha_1} \times \cdots \times E_0^{\alpha_N}$, which is a reflexive Banach space endowed with the norm

$$\|u\|_X = \big\|(u_1,\ldots,u_N)\big\|_X = \sum_{i=1}^{N} \|u_i\|_{\alpha_i}. \tag{2.8}$$

Obviously, X is compactly embedded in $(C^0([0,T],\mathbf{R}))^N$.

Definition 2.5 By a weak solution of problem (P_λ), one means any $u = (u_1,\ldots,u_N) \in X$ such that

$$\begin{aligned}&\sum_{i=1}^{N}\int_0^T (a_i(t)_0^cD_t^{\alpha_i}u_i(t)_0^cD_t^{\alpha_i}x_i(t)\,dt - \sum_{i=1}^{N}\int_0^T h_i\big(u_i(t)\big)x_i(t)\,dt \\ &\quad + \sum_{i=1}^{N}\sum_{j=1}^{p} a_i(t_j)I_{ij}\big(u_i(t_j)\big)x_i(t_j) - \sum_{i=1}^{N}\lambda\int_0^T F_{u_i}\big(t,u(t)\big)x_i(t)\,dt = 0\end{aligned}$$

for every $x = (x_i,\ldots,x_N) \in X$.

We define

$$H_i(x) = \int_0^x h_i(z)\,dz, \quad i = 1,\ldots,N, \tag{2.9}$$

for every $t \in [0,T]$ and $x \in \mathbf{R}$.

Arguing as in the proof of Theorem 5.1 in [26], we have the following.

Lemma 2.6 *Let $\frac{1}{2} < \alpha_i \le 1$ for $1 \le i \le N$, and $u \in X$. If u is a nontrivial weak solution of problem (P_λ), then u is also a nontrivial solution of problem (P_λ).*

Our analysis is mainly based on Lemmas 2.7 *and* 2.8, *consequences of a local minimum theorem* ([23], *Theorem* 3.1), *which is a more precise result of Ricceri's variational principle* (*see* [30]).

For a given non-empty set Ω and the functionals $\Phi, \Psi : \Omega \to \mathbf{R}$, one defines the following functions:

$$\chi(r_1,r_2) = \inf_{x\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)}\Psi(u) - \Psi(x)}{r_2 - \Phi(x)},$$

$$\rho_1(r_1,r_2) = \sup_{x\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(x) - \sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{\Phi(x) - r_1},$$

for every $r_1, r_2 \in \mathbf{R}$ with $r_1 < r_2$, and

$$\rho(r) = \sup_{x\in\Phi^{-1}(]r,\infty[)} \frac{\Psi(x) - \sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}{\Phi(x) - r},$$

for every $r \in \mathbf{R}$.

Lemma 2.7 ([23], Theorem 5.1) *Let X be a real Banach space; $\Phi : X \to \mathbf{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbf{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there exist $r_1, r_2 \in \mathbf{R}$, $r_1 < r_2$, such that*

$$\chi(r_1, r_2) < \rho_1(r_1, r_2).$$

Then, setting $I_\lambda = \Phi - \lambda\Psi$, for each $\lambda \in \left]\frac{1}{\rho_1(r_1,r_2)}, \frac{1}{\chi(r_1,r_2)}\right[$ there exists $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$, $\forall u \in \Phi^{-1}(]r_1, r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Lemma 2.8 ([23], Theorem 5.3) *Let X be a real Banach space; $\Phi : X \to \mathbf{R}$ be a sequentially weakly lower semicontinuous, coercive, and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbf{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Fix $\inf_X \Phi < r < \sup_X \Phi$ and assume that $\rho(r) > 0$, and for each $\lambda > \frac{1}{\rho(r)}$, the functional $I_\lambda = \Phi - \lambda\Psi$ is coercive. Then for each $\lambda \in \left]\frac{1}{\rho(r)}, +\infty\right[$ there exists $u_{0,\lambda} \in \Phi^{-1}(]r, +\infty[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$, $\forall u \in \Phi^{-1}(]r, +\infty[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.*

3 Main results and proofs

In this section, we shall give and prove our main results. Throughout this paper, we assume that:

(H0) $\frac{1}{2} < \alpha_i \le 1$ for $i = 1, \ldots, N$ and $0 < \beta := pa^*LB_0 + L_0A_0 < 1$, where $a^* = \max\{\operatorname{ess\,sup}_{[0,T]} a_i(t), 1 \le i \le N\}$, $L = \max_{i\in\{1,\ldots,N\}, j\in\{1,\ldots,p\}} L_{ij}$, and $L_0 = \max_{i\in\{1,\ldots,N\}} L_i$.

Proposition 3.1 *Let $T : X \to X^*$ be the operator defined by*

$$T(u)x = \sum_{i=1}^{N} \int_0^T a_i(t){}_0^cD_t^{\alpha_i}u_i(t){}_0^cD_t^{\alpha_i}x_i(t) - h_i\bigl(u_i(t)\bigr)x_i(t)\,dt + \sum_{i=1}^{N}\sum_{j=1}^{p} a_i(t_j)I_{ij}\bigl(u_i(t_j)\bigr)x_i(t_j)$$

for all $u = (u_1, \ldots, u_N)$, $x = (x_i, \ldots, x_N) \in X$. Then T is a continuous inverse on X^*.

Proof From (1.1) and (1.2), we have $-L_i|\xi|^2 \le h_i(\xi)\xi \le L_i|\xi|^2$ $(i = 1, \ldots, N)$ for each $\xi \in \mathbf{R}$, and $-L_{ij}|s|^2 \le I_{ij}(s)s \le L_{ij}|s|^2$ for each $s \in \mathbf{R}$ and $i = 1, \ldots, N$; $j = 1, \ldots, p$. It follows from (2.6) that

$$\begin{aligned} T(u)u &= \sum_{i=1}^{N} \int_0^T a_i(t)\bigl|{}_0^cD_t^{\alpha_i}u(t)\bigr|^2 - h_i\bigl(u_i(t)\bigr)u_i(t)\,dt + \sum_{i=1}^{N}\sum_{j=1}^{p} a_i(t_j)I_{ij}\bigl(u_i(t_j)\bigr)u_i(t_j) \\ &\ge \sum_{i=1}^{N} \|u_i\|_{\alpha_i}^2 - \sum_{i=1}^{N}\sum_{j=1}^{p} a_i(t_j)L_{ij}\|u_i\|_\infty^2 - \sum_{i=1}^{N} L_i\|u_i\|_{L^2}^2 \\ &\ge \sum_{i=1}^{N}\left(1 - \sum_{j=1}^{p} a_i(t_j)L_{ij}B_0 - L_iA_0\right)\|u_i\|_{\alpha_i}^2 \\ &\ge (1-\beta)\sum_{i=1}^{N} \|u_i\|_{\alpha_i}^2. \end{aligned} \tag{3.1}$$

Since $\beta < 1$, the inequality (3.1) shows that T is coercive. For every $u, v \in X$, it is easy to see that

$$\begin{aligned}\langle T(u) - T(v), u - v\rangle &= \sum_{i=1}^{N} \|u_i - v_i\|_{\alpha_i}^2 - \sum_{i=1}^{N} \int_0^T \big[h_i\big(u_i(t)\big) - h_i\big(v_i(t)\big)\big]\big[u_i(t) - v_i(t)\big]\,dt \\ &\quad + \sum_{i=1}^{N}\sum_{j=1}^{p} a_i(t_j)\big[I_{ij}\big(u_i(t_j)\big) - I_{ij}\big(v_i(t_j)\big)\big]\big(u_i(t_j) - v_i(t_j)\big) \\ &\geq (1-\beta)\sum_{i=1}^{N} \|u_i - v_i\|_{\alpha_i}^2,\end{aligned}$$

which implies that T is uniformly monotone. According to Theorem 26.A(d) in [31], the inverse operator T^{-1} of T exists and T^{-1} is continuous on X^*.

For a given nonnegative constant r and a function ω, let

$$\delta_\omega(r) = \frac{\int_0^T \max_{|u| \leq \sqrt{\frac{2B_0}{1-\beta} r}} F(t, u(t))\,dt - \int_0^T F(t, \omega(t))\,dt}{r - \frac{1}{2}(1+\beta)\sum_{i=1}^{N} \|\omega_i\|_{\alpha_i}^2},$$

where $\omega = (\omega_1, \ldots, \omega_N) \in X$. □

Theorem 3.2 *If there exist constants* $r_1 \geq 0$, $r_2 > 0$, *and a function* $\omega = (\omega_1, \ldots, \omega_N) \in X$ *such that*:

(H1) $\frac{2r_1}{1-\beta} < \sum_{i=1}^{N} \|\omega_i\|_{\alpha_i}^2 < \frac{2r_2}{1+\beta}$;

(H2) *there exist* $b_1 \in L^2([0,T], \mathbf{R})$, $b_2 \in L^1([0,T], \mathbf{R})$, *and a positive constant* $\mu < 2$ *such that*

$$\big|F(t,u)\big| \leq b_1(t)|u|^\mu + b_2(t)$$

for almost every $t \in [0,T]$ *and for all* $u \in \mathbf{R}^N$;

(H3) $\delta_\omega(r_2) < \delta_\omega(r_1)$.

Then, for every $\lambda \in]\frac{1}{\delta_\omega(r_1)}, \frac{1}{\delta_\omega(r_2)}[$, *the problem* (P_λ) *has at least two nontrivial solutions* u^*, $u_* \in X$ *such that*

$$r_1 < \frac{1}{2}\sum_{i=1}^{N} \big\|u_i^*\big\|_{\alpha_i}^2 + \sum_{i=1}^{N}\sum_{j=1}^{p} \int_0^{u_i^*(t_j)} I_{ij}(s)\,ds - \sum_{i=1}^{N} \int_0^T H_i\big(u_i^*(t)\big)\,dt < r_2. \tag{3.2}$$

Remark A In Theorem 3.2 and in the results below, by u^*, u_* one means the vectors $u^* = (u_1^*, \ldots, u_N^*)$, $u_* = (u_{*,1}, \ldots, u_{*,N})$, respectively.

Proof of Theorem 3.2 To apply Lemma 2.7 to the problem (P_λ), we define the functional $I_\lambda : X \to \mathbf{R}$ by

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

for all $u = (u_1, \dots, u_N) \in X$, where

$$\Phi(u) = \frac{1}{2}\sum_{i=1}^{N} \|u_i\|_{\alpha_i}^2 + \sum_{i=1}^{N}\sum_{j=1}^{p} \int_0^{u_i(t_j)} I_{ij}(s)\,ds - \sum_{i=1}^{N}\int_0^T H_i\bigl(u_i(t)\bigr)\,dt \tag{3.3}$$

and

$$\Psi(u) = \int_0^T F\bigl(t, u(t)\bigr)\,dt. \tag{3.4}$$

Due to the continuous embedding $X \to (C^0([0,T],\mathbf{R}))^N$ being compact, we know that Ψ is a well-defined Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)x = \sum_{i=1}^{N}\int_0^T F_{u_i}\bigl(t, u(t)\bigr)x_i(t)\,dt$$

for every $x = (x_1, \dots, x_N) \in X$, and Ψ' is a sequentially weakly upper semicontinuous functional on X. Moreover, $\Psi' : X \to X^*$ is a compact operator. In fact, for a given $u \in X$, if $\{u_n = (u_{n,1}, \dots, u_{n,N})\} \subset X$, $u_n \rightharpoonup u$ in X as $n \to +\infty$, then u_n converges uniformly to u on $[0,T]$. Hence, we have $F_{u_i}(t,u_n) \to F_{u_i}(t,u)$ as $n \to +\infty$. So $\Psi'(u_n) \to \Psi'(u)$ as $n \to +\infty$. Therefore, Ψ' is strongly continuous on X, which implies that Ψ' is a compact operator.

It is not difficult to verify that the functional Φ is a continuously differentiable whose differential at the point $u \in X$ is

$$\begin{aligned}\Phi'(u)(x) = {}& \sum_{i=1}^{N}\int_0^T a_i(t)\,{}_0^cD_t^{\alpha_i}u_i(t)\,{}_0^cD_t^{\alpha_i}x_i(t)\,dt - \sum_{i=1}^{N}\int_0^T h_i\bigl(u_i(t)\bigr)x_i(t)\,dt \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{p} a_i(t_j)I_{ij}\bigl(u_i(t_j)\bigr)x_i(t_j) - \lambda\sum_{i=1}^{N}\int_0^T F_{u_i}\bigl(t,u(t)\bigr)x_i(t)\,dt\end{aligned}$$

for every $x \in X$. From Proposition 3.1, it is easy to see that Φ' is a continuous inverse on X^*. Furthermore, Φ admits also sequentially weakly lower semicontinuous on X.

Clearly, the solutions of the equation $I'_\lambda(u) = 0$ are exactly the weak solutions of the problem (P_λ). Similarly to (3.1), we get

$$\frac{1}{2}(1-\beta)\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2 \le \Phi(u) \le \frac{1}{2}(1+\beta)\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2. \tag{3.5}$$

From the condition (H1), we have $r_1 < \Phi(u) < r_2$.

According to (2.1), one has

$$\max_{t\in[0,T]}\sum_{i=1}^{N}\bigl|u_i(t)\bigr|^2 \le B_0\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2.$$

So, for every $r > 0$, from the definition of Φ and by using (3.5) one has

$$\begin{aligned}\Phi^{-1}\bigl(]-\infty,r]\bigr) &:= \bigl\{u \in X : \Phi(u) \le r\bigr\}\\ &\subseteq \left\{u \in X : \frac{1}{2}(1-\beta)\sum_{i=1}^{N}\|u_i\|_{\alpha_i}^2 \le r\right\}\\ &\subseteq \left\{u \in X : \sum_{i=1}^{N}\|u_i\|_{\infty}^2 \le \frac{2B_0}{1-\beta}r\right\}\\ &\subseteq \left\{u \in X : \bigl|u(t)\bigr| \le \sqrt{\frac{2B_0}{1-\beta}r}, \text{ for all } t \in [0,T]\right\}. \qquad (3.6)\end{aligned}$$

Therefore,

$$\begin{aligned}\chi(r_1,r_2) &= \inf_{\omega\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)}\Psi(u)-\Psi(\omega)}{r_2-\Phi(\omega)}\\ &\le \frac{\sup_{u\in\Phi^{-1}((-\infty,r_2[)}\Psi(u)-\Psi(\omega)}{r_2-\Phi(\omega)}\\ &\le \frac{\sup_{u\in\Phi^{-1}((-\infty,r_2[)}\int_0^T F(t,u(t))\,dt-\Psi(\omega)}{r_2-\Phi(\omega)}\\ &\le \frac{\int_0^T \max_{|u|\le\sqrt{\frac{2B_0}{1-\beta}r_2}} F(t,u(t))\,dt-\int_0^T F(t,\omega(t))\,dt}{r_2-\frac{1}{2}(1+\beta)\sum_{i=1}^N\|\omega_i\|_{\alpha_i}^2}\\ &= \delta_\omega(r_2).\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\rho(r_1,r_2) &= \sup_{\omega\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(\omega)-\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{\Phi(\omega)-r_1}\\ &\ge \frac{\Psi(\omega)-\sup_{u\in\Phi^{-1}((-\infty,r_1[)}\Psi(u)}{\Phi(\omega)-r_1}\\ &\ge \frac{\Psi(\omega)-\sup_{u\in\Phi^{-1}((-\infty,r_2[)}\int_0^T F(t,u(t))\,dt}{\Phi(\omega)-r_1}\\ &\ge \frac{\int_0^T \max_{|u|\le\sqrt{\frac{2B_0}{1-\beta}r_1}} F(t,u(t))\,dt-\int_0^T F(t,\omega(t))\,dt}{r_1-\frac{1}{2}(1+\beta)\sum_{i=1}^N\|\omega_i\|_{\alpha_i}^2}\\ &= \delta_\omega(r_1).\end{aligned}$$

According to condition (H3), one has $\rho(r_1,r_2) > \chi(r_1,r_2)$. Hence, applying Lemma 2.7, for every $\lambda \in]\frac{1}{\delta_\omega(r_1)}, \frac{1}{\delta_\omega(r_2)}[$, the functional $I_\lambda(u)$ has at least one critical point $u^* \in X$ such that $r_1 < \Phi(u^*) < r_2$. Obviously, u^* is a nontrivial local minimum for I_λ in X.

Next we show that the existence of a second local minimum of I_λ in X is distinct from the first one. To this aim, we will prove the hypothesis of the mountain pass theorem for the functional I_λ. Obviously, the functional $I_\lambda \in C^1[0,T]$ and $I_\lambda(u) = 0$. From the above proof, we know that $u^* \in X$ is a nontrivial local minimum for I_λ in X. So there exists a

$\zeta > 0$ such that $\inf_{\|u-u^*\|_X=\zeta} I_\lambda(u) > I_\lambda(u^*)$, that is, the condition [[32], I_1, Theorem 2.2] is satisfied. Choosing $u \neq 0$, it follows from (3.3), (3.4), (3.5), (H2), and the Hölder inequality that

$$\begin{aligned} I_\lambda(\tau u) &\geq \frac{\tau^2}{2}\sum_{i=1}^N \|u_i\|_{\alpha_i}^2 - \frac{\tau^2}{2}\sum_{i=1}^N\sum_{j=1}^p \big|u(t_j)\big|^2 - \frac{\tau^2}{2}\sum_{i=1}^N \int_0^T L_i\big|u_i(t)\big|\,dt \\ &\quad - \lambda\tau^\mu \sum_{i=1}^N \int_0^T b_1(t)\big|u_i(t)\big|^\mu\,dt - \lambda\int_0^T b_2(t)\,dt \\ &\geq \frac{\tau^2}{2}(1-\beta)\sum_{i=1}^N \|u_i\|_{\alpha_i}^2 - \lambda\tau^\mu\|b_1\|_{L^2}\sum_{i=1}^N \|u_i\|_{L^2}^\mu - \lambda\|b_2\|_{L^1} \to +\infty \end{aligned}$$

as $\tau \to \infty$, since $\mu < 2$. Hence the condition [[32], I_2, Theorem 2.2] is satisfied.

Furthermore, by standard computations I_λ satisfies (P-S) condition. Therefore, it follows from the classical theorem of Ambrosetti and Rabinowitz that there exists a critical point u_* of $I_\lambda(u)$ such that $I_\lambda(u_*) > I_\lambda(u^*)$. So, the problem (P_λ) has at least two distinct nontrivial weak solutions u^*, u_*, and u^* satisfies (3.2). The proof of Theorem 3.2 is complete. □

For a given constant $\theta \in (0, \frac{1}{2})$, set

$$P_i(\alpha_i,\theta) = \frac{1}{2\theta^2T^2}\left\{\int_0^{\theta T} a_i(t)t^{2(1-\alpha_i)}\,dt + \int_{\theta T}^{(1-\theta)T} a_i(t)p_i^2(t)\,dt + \int_{(1-\theta)T}^{T} a_i(t)q_i^2(t)\,dt\right\},$$

where

$$\begin{aligned} p_i(t) &= t^{1-\alpha_i} - (t-\theta T)^{1-\alpha_i}, \\ q_i(t) &= t^{1-\alpha_i} - (t-\theta T)^{1-\alpha_i} - \big(t-(1-\theta)T\big)^{1-\alpha_i} \end{aligned}$$

and

$$\begin{aligned} K_0 &= \min\big\{P_i(\alpha_i,\theta), 1 \leq i \leq N\big\}, \\ K_1 &= \max\big\{P_i(\alpha_i,\theta), 1 \leq i \leq N\big\}. \end{aligned}$$

For a given nonnegative constant η and a positive constant ξ, let

$$\delta_\xi(\eta) = \frac{\int_0^T \max_{|u|\leq\sqrt{2B_0\eta}} F(t,u(t))\,dt - \int_0^T F(t,\Gamma(2-\alpha_1)\xi,\ldots,\Gamma(2-\alpha_N)\xi)\,dt}{(1-\beta)\eta - N(1+\beta)K_1\xi^2},$$

where $(1-\beta)\eta \neq N(1+\beta)K_1\xi^2$.

Theorem 3.3 *Assume that the condition* (H2) *satisfies. Furthermore, if there exist constants* $c \geq 0$, $b > 0$ *and* ξ *with* $\sqrt{\frac{c}{NK_0}} < \xi < \sqrt{\frac{(1-\beta)b}{NK_1(1+\beta)}}$ *such that*:

(H4) $F(t,u) \geq 0$ *for all* $(t,u) \in ([0,\theta T] \cup [(1-\theta)T, T]) \times [-\xi,\xi]^N$;

(H5) $\delta_\xi(b) < \delta_\xi(c)$.

Then, for every $\lambda \in]\frac{1}{\delta_\xi(c)}, \frac{1}{\delta_\xi(b)}[$, the problem (P_λ) admits at least two nontrivial solutions $u^*, u_* \in X$ such that

$$(1-\beta)c < \frac{1}{2}\sum_{i=1}^{N}\|u_i^*\|_{\alpha_i}^2 + \sum_{i=1}^{N}\sum_{j=1}^{p}\int_0^{u_i^*(t_j)} I_{ij}(s)\,ds - \sum_{i=1}^{N}\int_0^T H_i\bigl(u_i^*(t)\bigr)\,dt < (1-\beta)b. \quad (3.7)$$

Proof We shall show that all the assumptions of Theorem 3.2 are fulfilled by choosing $r_1 = (1-\beta)c$, $r_2 = (1-\beta)b$, and $\omega = (\omega_1(t), \dots, \omega_N(t))$ with

$$\omega_i(t) = \begin{cases} \frac{\Gamma(2-\alpha_i)\xi}{\theta T}t, & t \in [0, \theta T[, \\ \Gamma(2-\alpha_i)\xi, & t \in [\theta T, (1-\theta)T], \\ \frac{\Gamma(2-\alpha_i)\xi}{\theta T}(T-t), & t \in](1-\theta)T, T]. \end{cases} \quad (3.8)$$

Clearly $\omega_i(0) = \omega_i(T) = 0$ and $\omega_i \in L^2[0,T]$ for $i = 1, \dots, N$. By direct calculation we have

$${}_0^cD_t^{\alpha_i}\omega_i(t) = \frac{\xi}{\theta T}\begin{cases} t^{1-\alpha_i}, & t \in [0, \theta T[, \\ p_i(t), & t \in [\theta T, (1-\theta)T], \\ q_i(t), & t \in](1-\theta)T, T]. \end{cases}$$

Furthermore,

$$\begin{aligned} &\int_0^T a_i(t)\bigl|{}_0^cD_t^{\alpha_i}\omega_i(t)\bigr|^2\,dt \\ &\quad = \int_0^{\theta T} + \int_{\theta T}^{(1-\theta)T} + \int_{(1-\theta)T}^{T} a_i(t)\bigl|{}_0D_t^{\alpha_i}\omega_i(t)\bigr|^2\,dt \\ &\quad = \frac{\xi^2}{\theta^2T^2}\left\{\int_0^{\theta T} a_i(t)t^{2(1-\alpha_i)}\,dt + \int_{\theta T}^{(1-\theta)T} a_i(t)p_i^2(t)\,dt + \int_{(1-\theta)T}^{T} a_i(t)q_i^2(t)\,dt\right\} \\ &\quad = 2P_i(\alpha_i, \theta)\xi^2 < +\infty. \end{aligned}$$

Thus, $\omega \in X$, this and (3.5) show that

$$2NK_0\xi^2 \le \sum_{i=1}^{N}\|\omega_i\|_{\alpha_i}^2 \le 2NK_1\xi^2$$

and

$$N(1-\beta)K_0\xi^2 \le \Phi(\omega) \le N(1+\beta)K_1\xi^2.$$

This together with the condition $\sqrt{\frac{c}{NK_0}} < \xi < \sqrt{\frac{(1-\beta)b}{NK_1(1+\beta)}}$ implies (H1) is satisfied.

It follows from (H4) that

$$\begin{aligned} \Psi(\omega) &= \int_0^{\theta T} + \int_{\theta T}^{(1-\theta)T} + \int_{(1-\theta)T}^{T} F(t,\omega)\,dt \ge \int_{\theta T}^{(1-\theta)T} F(t,\omega)\,dt \\ &= \int_{\theta T}^{(1-\theta)T} F\bigl(t, \Gamma(2-\alpha_1)\xi, \dots, \Gamma(2-\alpha_N)\xi\bigr)\,dt. \end{aligned} \quad (3.9)$$

Therefore, one has

$$\begin{aligned}\chi(r_1,r_2) &\le \frac{\sup_{u\in\Phi^{-1}((-\infty,r_2[)}\Psi(u)-\Psi(\omega)}{r_2-\Phi(\omega)}\\ &\le \frac{\int_0^T \max_{|u|\le\sqrt{2B_0b}}F(t,u(t))\,dt-\int_0^T F(t,\Gamma(2-\alpha_1)\xi,\dots,\Gamma(2-\alpha_N)\xi)\,dt}{(1-\beta)b-N(1+\beta)K_1\xi^2}\\ &= \delta_\xi(b)\end{aligned}$$

and

$$\begin{aligned}\rho(r_1,r_2) &\ge \frac{\Psi(\omega)-\sup_{u\in\Phi^{-1}((-\infty,r_1[)}\Psi(u)}{\Phi(\omega)-r_1}\\ &\ge \frac{\int_0^T \max_{|u|\le\sqrt{2B_0c}}F(t,u(t))\,dt-\int_0^T F(t,\Gamma(2-\alpha_1)\xi,\dots,\Gamma(2-\alpha_N)\xi)\,dt}{(1-\beta)c-N(1+\beta)K_1\xi^2}\\ &= \delta_\xi(c),\end{aligned}$$

which implies that (H3) is verified. Therefore, Theorem 3.2 ensures the conclusion. □

Corollary 3.4 *In addition to* (H2), *assume that there exist two constants* $b>0$ *and* ξ *with* $\xi<\sqrt{\frac{(1-\beta)b}{NK_1(1+\beta)}}$ *such that the assumption* (H4) *in Theorem* 3.3 *holds. Furthermore, suppose that*:

$$\text{(H6)}\quad \frac{\int_0^T F(t,\Gamma(2-\alpha_1)\xi,\dots,\Gamma(2-\alpha_N)\xi)\,dt}{N(1+\beta)K_1\xi^2}>\frac{\int_0^T \max_{|u|\le\sqrt{2B_0b}}F(t,u(t))\,dt}{(1-\beta)b}.$$

Then, for every

$$\lambda\in\left]\frac{N(1+\beta)K_1\xi^2}{\int_0^T F(t,\Gamma(2-\alpha_1)\xi,\dots,\Gamma(2-\alpha_N)\xi)\,dt},\frac{(1-\beta)b}{\int_0^T \max_{|u|\le\sqrt{2B_0b}}F(t,u(t))\,dt}\right[,$$

problem (P_λ) *has at least two nontrivial solutions* u^*, $u_*\in X$ *such that*

$$0<\frac{1}{2}\sum_{i=1}^N\|u_i^*\|_{\alpha_i}^2+\sum_{i=1}^N\sum_{j=1}^p\int_0^{u_i^*(t_j)}I_{ij}(s)\,ds-\sum_{i=1}^N\int_0^T H_i\bigl(u_i^*(t)\bigr)\,dt<(1-\beta)b. \tag{3.10}$$

Proof The conclusion follows from Theorem 3.3 by choosing $c=0$. From our assumptions, we have

$$\begin{aligned}\delta_\xi(b) &< \frac{(1-\frac{N(1+\beta)K_1\xi^2}{(1-\beta)b})\int_0^T \max_{|u|\le\sqrt{2B_0b}}F(t,u(t))\,dt}{(1-\beta)b-N(1+\beta)K_1\xi^2}\\ &= \frac{\int_0^T \max_{|u|\le\sqrt{2B_0b}}F(t,u(t))\,dt}{(1-\beta)b}\\ &< \frac{\int_0^T F(t,\Gamma(2-\alpha_1)\xi,\dots,\Gamma(2-\alpha_N)\xi)\,dt}{N(1+\beta)K_1\xi^2}=\delta_\xi(0).\end{aligned}$$

Hence, Theorem 3.3 ensures the conclusion. □

Theorem 3.5 *Assume that there exist a positive constant r and a function $\omega = (\omega_1(t), \dots, \omega_N(t)) \in X$ with $\frac{2r}{1-\beta} < \sum_{i=1}^N \|\omega_i\|_{\alpha_i}^2$ such that:*

$$\text{(H7)} \quad \int_0^T \max_{|u| \leq \sqrt{\frac{2B_0}{1-\beta^2} r}} F(t, u(t))\,dt < \int_0^T F(t, \omega)\,dt;$$

$$\text{(H8)} \quad \liminf_{|u| \to +\infty} \frac{F(t,u)}{|u|^2} \leq 0 \quad \textit{uniformly for } t \in [0, T].$$

Then, for every $\lambda \in]\lambda_0, +\infty)$, where

$$\lambda_0 = \frac{r - \frac{1}{2}(1+\beta)\sum_{i=1}^N \|\omega_i\|_{\alpha_i}^2}{\int_0^T \max_{|u| \leq \sqrt{\frac{2B_0}{1-\beta} r}} F(t, u(t))\,dt - \int_0^T F(t, \omega(t))\,dt},$$

problem (P_λ) has at least one nontrivial solution $u^* \in X$ such that

$$r < \frac{1}{2}\sum_{i=1}^N \|u_i^*\|_{\alpha_i}^2 + \sum_{i=1}^N \sum_{j=1}^p \int_0^{u_i^*(t_j)} I_{ij}(s)\,ds - \sum_{i=1}^N \int_0^T H_i(u_i^*(t))\,dt. \tag{3.11}$$

Proof We may take the functionals Φ and Ψ and the space as in the proof of Theorem 3.2, and choose λ as in the conclusion of the theorem. Obviously, all the regularity assumptions required in Lemma 2.8 are satisfied. According to (H8) there exist a positive constant $\varepsilon > 0$ and a function $m_\varepsilon(t) \in L^1[0, T]$ with $\varepsilon < \frac{1-\beta}{2A_0\lambda}$ such that

$$|F(t,u)| \leq \varepsilon |u(t)|^2 + m_\varepsilon(t) \tag{3.12}$$

for each $t \in [0, T]$.

It follows from (3.3), (3.4), (3.5), and (3.12) that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\sum_{i=1}^N \|u_i\|_{\alpha_i}^2 + \sum_{i=1}^N \sum_{j=1}^p \int_0^{u_i(t_j)} I_{ij}(s)\,ds - \sum_{i=1}^N \int_0^T H_i(u_i(t))\,dt - \lambda \int_0^T F(t, u(t))\,dt \\ &\geq \frac{1}{2}(1-\beta)\sum_{i=1}^N \|u_i\|_{\alpha_i}^2 - \lambda\varepsilon \sum_{i=1}^N \int_0^T |u_i(t)|^2\,dt - \lambda \int_0^T m_\varepsilon(t)\,dt \\ &\geq \frac{1}{2}(1-\beta)\sum_{i=1}^N \|u_i\|_{\alpha_i}^2 - \lambda\varepsilon \sum_{i=1}^N \|u_i\|_{L^2}^2 - \lambda \|m_\varepsilon\|_{L^1} \\ &\geq \frac{1}{2}(1-\beta-2A_0\lambda\varepsilon)\sum_{i=1}^N \|u_i\|_{\alpha_i}^2 - \lambda\|m_\varepsilon\|_{L^1} \end{aligned}$$

and then

$$\lim_{\|u\|_X \to +\infty} I_\lambda(u) = +\infty,$$

which implies that the functional $I_\lambda(u)$ is coercive. Similarly to the proof of Theorem 3.2, it follows from (H7) and (H8) that

$$\rho(r) \geq \frac{\int_0^T \max_{|u|\leq\sqrt{\frac{2B_0}{1-\beta}r}} F(t,u(t))\,dt - \int_0^T F(t,\omega(t))\,dt}{r - \frac{1}{2}(1+\beta)\sum_{i=1}^N \|\omega_i\|_{\alpha_i}^2} > 0.$$

Hence, from Lemma 2.8, the functional I_λ has at least a local minimum $u^* \in X$ such that (3.11) holds. □

Corollary 3.6 *Assume that the condition* (H8) *holds. Furthermore, suppose that there exist positive constants $\bar{c}$ and $\bar{\xi}$ with $\bar{c} < NK_0\bar{\xi}^2$ such that*:

(H9) $F(t,u) \geq 0$ *for all* $(t,u) \in ([0,\theta T] \cup [(1-\theta)T,T]) \times [-\bar{\xi},\bar{\xi}]^N$;

$$\text{(H10)} \quad \int_0^T \max_{|u|\leq\sqrt{2B_0\bar{c}}} F\bigl(t,u(t)\bigr)\,dt < \int_{\theta T}^{(1-\theta)T} F\bigl(t,\Gamma(2-\alpha_1)\bar{\xi},\ldots,\Gamma(2-\alpha_N)\bar{\xi}\bigr)\,dt.$$

Then, for every $\lambda \in \,]\lambda_1,+\infty)$, *where*

$$\lambda_1 = \frac{(1-\beta)\bar{c} - N(1+\beta)K_0\bar{\xi}^2}{\int_0^T \max_{|u|\leq\sqrt{2B_0\bar{c}}} F(t,u(t))\,dt - \int_{\theta T}^{(1-\theta)T} F(t,\Gamma(2-\alpha_1)\bar{\xi},\ldots,\Gamma(2-\alpha_N)\bar{\xi})\,dt},$$

problem (P_λ) *has at least one nontrivial solution* $u^* \in X$ *such that*

$$(1-\beta)\bar{c} < \frac{1}{2}\sum_{i=1}^N \bigl\|u_i^*\bigr\|_{\alpha_i}^2 + \sum_{i=1}^N\sum_{j=1}^p \int_0^{u_i^*(t_j)} I_{ij}(s)\,ds - \sum_{i=1}^N \int_0^T H_i\bigl(u_i^*(t)\bigr)\,dt.$$

Proof The conclusion follows from Theorem 3.5 by choosing $\bar{r} = (1-\beta)\bar{c}$ and taking $\bar{\omega}$ as in (3.8) with ξ replaced by $\bar{\xi}$.

Finally, we present the following example to illustrate the applicability of Theorem 3.2. □

Example 3.7 Consider the following fractional differential systems:

$$\begin{cases} {}_tD_1^{0.75}((1+t^2)\cdot {}_0^cD_t^{0.75}u_1(t)) = \lambda F_{u_1}(t,u_1,u_2) + h_1(u_1), & 0<t<1, \\ {}_tD_1^{0.8}((0.5+t)\cdot {}_0^cD_t^{0.8}u_2(t)) = \lambda F_{u_2}(t,u_1,u_2) + h_2(u_2), & 0<t<1, \\ \Delta({}_tD_T^{-0.25}({}_0^cD_t^{0.75}u_1))(t_1) = I_{11}(u_1(t_1)), \\ \Delta({}_tD_T^{-0.2}({}_0^cD_t^{0.8}u_2))(t_1) = I_{21}(u_2(t_1)), \\ u_1(0) = u_1(1) = 0, \qquad u_2(0) = u_2(1) = 0, \end{cases} \tag{3.13}$$

where $a_1(t) = 1+t^2$, $a_2(t) = 0.5+t$, $F(t,u_1,u_2) = (1+t^2)(u_1^2+u_2^2)^{\frac{3}{4}}$ for $u_1, u_2 \in \mathbf{R}$, $h_1(u_1) = \frac{1}{8}\sin(u_1)$, $h_2(u_2) = \frac{1}{25}\ln(1+u_2^2)$ for $u_1,u_2 \in \mathbf{R}$, $t_1 = \frac{1}{4}$, $I_{i1}(x) = \frac{1}{32}x$ for $x \in \mathbf{R}$ and for $i = 1,2$.

Obviously, $h_1,h_2 : \mathbf{R} \to \mathbf{R}$ are two Lipschitz continuous functions with Lipschitz constants $L_1 = \frac{1}{8}$, $L_2 = \frac{1}{25}$ and $h_1(0) = h_2(0) = 0$; $I_{i1} : \mathbf{R} \to \mathbf{R}$ $(i = 1,2)$ are also Lipschitz continuous functions with Lipschitz constants $L_{11} = L_{21} = \frac{1}{32}$. $F(t,0,0) = 0$ for all $t \in [0,1]$ and by taking $\mu = \frac{3}{2}$, $b_1(t) = 1+t^2$, $b_2(t) = t$, then the condition (H2) holds. By simple calculations, we see that $a^* = 2$, $L = \frac{1}{32}$, $L_0 = \frac{1}{8}$, $A_0 \approx 2.3052$, $B_0 \approx 2.8637$, and $\beta \approx 0.4672$.

By choosing, for instance, $\omega = (\omega_1, \omega_2)$, where $\omega_1(t) = \Gamma(1.25)t(1-t)$, $\omega_2(t) = \Gamma(1.2)t(1-t)$, and $r_1 = \frac{1}{500}$, $r_2 = 30$, then the conditions (H1) and (H3) are verified. In fact, $\omega_i(0) = \omega_i(1) = 0$, $i = 1, 2$, and $\|\omega_1\|_{0.75}^2 \approx 0.1582$, $\|\omega_2\|_{0.8}^2 \approx 0.1389$. It is easy to show that the condition (H1) holds and

$$\delta_\omega(r_1) = \frac{\int_0^1 \max_{|u|\leq 0.14663} F(t, u(t))\, dt - \int_0^1 F(t, \omega(t))\, dt}{r_1 - 0.7336(\|\omega_1\|_{0.75}^2 + \|\omega_2\|_{0.8}^2)} \approx 0.3977,$$

$$\delta_\omega(r_2) = \frac{\int_0^1 \max_{|u|\leq 17.958} F(t, u(t))\, dt - \int_0^1 F(t, \omega(t))\, dt}{r_2 - 0.7336(\|\omega_1\|_{0.75}^2 + \|\omega_2\|_{0.8}^2)} \approx 0.3770,$$

which imply that the condition (H3) is satisfied. Therefore, according to Theorem 3.2, for every $\lambda \in]2.5146, 2.6525[$ the problem (3.13) has at least two nontrivial solutions in $E_0^{0.75} \times E_0^{0.8}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References
1. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
2. Diethelm, K: The Analysis of Fractional Differential Equations. Springer, Berlin (2010)
3. Zhou, Y: Basic Theory of Fractional Differential Equation. World Scientific, Singapore (2014)
4. Agarwal, RP, Benchohra, M, Hamani, S: A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl. Math. **109**, 973-1033 (2010)
5. Ahmad, B, Sivasundaram, S: On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order. Appl. Math. Comput. **217**, 480-487 (2010)
6. Su, X: Boundary value problem for a coupled system of nonlinear fractional differential equations. Appl. Math. Lett. **22**, 64-69 (2009)
7. Zhao, Y, Chen, H, Huang, L: Existence of positive solutions for nonlinear fractional functional differential equation. Comput. Math. Appl. **64**, 3456-3467 (2012)
8. Fečkan, M, Zhou, Y, Wang, J: On the concept and existence of solution for impulsive fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **17**, 3050-3060 (2012)
9. Guo, L, Zhang, X: Existence of positive solutions for the singular fractional differential equations. J. Appl. Math. Comput. **44**, 215-228 (2014)
10. Cabada, A, Hamdi, Z: Nonlinear fractional differential equations with integral boundary value conditions. Appl. Math. Comput. **228**, 251-257 (2014)
11. Jia, M, Liu, X: Multiplicity of solutions for integral boundary value problems of fractional differential equations with upper and lower solutions. Appl. Math. Comput. **232**, 313-323 (2014)
12. Wang, G, Ahmad, B, Zhang, L, Nieto, JJ: Comments on the concept of existence of solution for impulsive fractional differential equations. Commun. Nonlinear Sci. Numer. Simul. **19**, 401-403 (2014)
13. Jiao, F, Zhou, Y: Existence of solutions for a class of fractional boundary value problems via critical point theory. Comput. Math. Appl. **62**, 1181-1199 (2011)
14. Zhao, Y, Chen, H, Zhang, Q: Infinitely many solutions for fractional differential system via variational method. J. Appl. Math. Comput. **50**, 589-609 (2016)
15. Bonanno, G, Rodríguez-López, R, Tersian, S: Existence of solutions to boundary value problem for impulsive fractional differential equations. Fract. Calc. Appl. Anal. **17**(3), 717-744 (2014)
16. Rodríguez-López, R, Tersian, S: Multiple solutions to boundary value problem for impulsive fractional differential equations. Fract. Calc. Appl. Anal. **17**(4), 1016-1038 (2014)
17. Nyamoradia, N, Rodríguez-López, R: On boundary value problems for impulsive fractional differential equations. Appl. Math. Comput. **271**, 874-892 (2015)

18. Zhao, Y, Chen, H, Qin, B: Multiple solutions for a coupled system of nonlinear fractional differential equations via variational methods. Appl. Math. Comput. **257**, 417-427 (2015)
19. Bai, C: Infinitely many solutions for a perturbed nonlinear fractional boundary-value problem. Electron. J. Differ. Equ. **2013**, 136 (2013)
20. Sun, H, Zhang, Q: Existence of solutions for a fractional boundary value problem via the mountain pass method and an iterative technique. Comput. Math. Appl. **64**, 3436-3443 (2012)
21. Klimek, M, Odzijewicz, T, Malinowska, AB: Variational methods for the fractional Sturm-Liouville problem. J. Math. Anal. Appl. **416**, 402-426 (2014)
22. Averna, D, Tersian, S, Tornatore, E: On the existence and multiplicity of solutions for Dirichlet's problem for fractional differential equations. Fract. Calc. Appl. Anal. **19**, 253-266 (2016)
23. Bonanno, G: A critical point theorem via the Ekeland variational principle. Nonlinear Anal. **75**, 2992-3007 (2012)
24. Benchohra, M, Henderson, J, Ntouyas, S: Theory of Impulsive Differential Equations, Contemporary Mathematics and Its Applications. Hindawi Publishing Corporation, New York (2006)
25. Xiao, J, Nieto, JJ: Variational approach to some damped Dirichlet nonlinear impulsive differential equations. J. Franklin Inst. **348**, 369-377 (2011)
26. Tian, Y, Ge, W: Multiple solutions of impulsive Sturm-Liouville boundary value problem via lower and upper solutions and variational methods. J. Math. Anal. Appl. **387**, 475-489 (2012)
27. Zhao, Y, Huang, L, Zhang, Q: Existence results for an impulsive Sturm-Liouville boundary value problems with mixed double parameters. Bound. Value Probl. **2015**, 150 (2015)
28. D'Aguì, G, Di Bella, B, Tersian, S: Multiplicity results for superlinear boundary value problems with impulsive effects. Math. Methods Appl. Sci. **39**, 1060-1068 (2016)
29. Bonanno, G, Marano, SA: On the structure of the critical set of non-differentiable functions with a weak compactness condition. Appl. Anal. **89**, 1-10 (2010)
30. Ricceri, B: A general variational principle and some of its applications. J. Comput. Appl. Math. **113**, 401-410 (2000)
31. Zeidler, E: Nonlinear Functional Analysis and Its Applications, Vol. II. Springer, Berlin (1985)
32. Rabinowitz, PH: Minimax Methods in Critical Point Theory with Applications to Differential Equations. CBMS Reg. Conf. Ser. Math., vol. 65. Am. Mat. Soc., Providence (1986)