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A reaction-diffusion-advection logistic model with a free boundary in heterogeneous environment

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Abstract

The aim of this paper is to investigate the dynamics of the solution for a class of reaction-diffusion-advection logistic model with a free boundary in heterogeneous environment. The species undergoes diffusion and advection in a one dimensional heterogeneous environment, and it invades the environment with a spreading front evolving as the free boundary. To understand the effects of the advection rate α and the expansion capacity μ on the dynamics of this model, we derive a spreading-vanishing dichotomy and obtain the sharp criterion for the spreading and vanishing by choosing α and μ as variable parameters. That is, the invasion species can unconditionally survive for a slow advection rate, while, for a fast advection rate, whether it can survive or not depends on the expansion capacity and initial values of the invasion species.

Keywords: reaction-diffusion-advection; free boundary condition; heterogeneous environment; spreading; vanishing

1 Introduction

Due to the serious threat of invasive species to bio-diversity conservation and the global economy, mathematical modeling has become an important tool in analyzing the prediction and prevention of biological invasions [1, 2]. Recently, a great deal of attention has been paid to developing more realistic mathematical models for the invasion dynamics. There have been a number of works on modeling the invasion of a species described by a reaction-diffusion system; see [3–7] and the references cited therein for further details.

There is growing interest in modeling and understanding spatial species dynamics in advection environments, *i.e.*, environments where individuals are exposed to unidirectional flow or biased dispersal [8]. For example, water flow imposes directional bias (advection) buoyancy and turbulence leads to unbiased movement (diffusion) [9]. The West Nile virus appeared for the first time in New York city in the summer of 1999. In the second year the wave front traveled 187 km to the North and 1,100 km to the South [10]. It should be noticed that the term 'advection' was used in a series of works, see [3, 11–15], to show movement toward a higher quality habitat. In [16], Lou and Lutscher established the existence of a critical advection speed for the persistence of a single species as follows:

$$\begin{cases} u_t = du_{xx} - \alpha u_x + u(r(x) - u), & t > 0, 0 < x < L, \\ \alpha u(0,t) - du_x(0,t) = 0, \quad u(L,t) = 0, & t > 0, \end{cases} \tag{1.1}$$

where u denotes the density of the invasion species at time t and location x in the bounded interval $[0, L]$, d and α are the diffusion and advection rates, respectively. The function $r(x)$ stands for the quality of the habitat, the population can grow if $r(x) > 0$ and will decline if $r(x) < 0$. They showed that if the species cannot persist for $\alpha = 0$, then it cannot persist for any advection, otherwise it can persist for some α in the interval $[0, \alpha^*]$, where α^* is the critical advection speed and can be obtained by considering the following eigenvalue problem:

$$\begin{cases} d\phi_{xx} - \alpha\phi_x + \phi r(x) = \lambda\phi, & 0 < x < L, \\ \alpha\phi(0) - d\phi_x(0) = 0, & \phi(L) = 0. \end{cases}$$

Moreover, it is shown that $\alpha^* < c_0$, where $c_0 = 2\sqrt{d\overline{r}}$ is the minimal speed of the traveling waves of (1.1) and $\overline{r} = \max_{x\in[0,L]} r(x)$.

However, it is still considered that model (1.1) is not very realistic for describing the invasion dynamics because of the lack of information as regards the precise invasion dynamics. Thus, it is necessary to consider the impact of the free boundary on the dynamics of the new invasion species. That is, the species evolves according to the free boundary condition

$$h'(t) = -\mu u_x\big(h(t), t\big), \tag{1.2}$$

where μ is a given positive constant and denotes the expansion capacity, $x = h(t)$ denotes the spreading front, *i.e.*, the free boundary that needs to be determined. The equation (1.2) implies that the spreading front expands at a speed that is proportional to the species gradient at the front. Note that (1.2) is a special case of the well-known Stefen condition. Systems with the free boundary condition have been used in describing ecological models over bounded spatial dynamics in several earlier papers; see [17–19]. The results of free boundary problems have been used in the modeling many practical problems, such as wound healing [20], the combustion process [21], the American option pricing problem [22], the chemical vapor deposition on a hot wall reactor [23], image processing [24], tumor growth [25, 26], and so on. For example, Zhou and Xiao [27] introduced a free boundary to describe the spreading of an invasion species as follows:

$$\begin{cases} u_t - du_{xx} = u(r(x) - u), & t > 0, 0 < x < h(t), \\ u_x(0,t) = 0, \quad u(h(t),t) = 0, & t > 0, \\ h'(t) = -\mu u_x(h(t),t), & t > 0, \\ u(x,0) = u_0(x) \geq 0, \quad h(0) = h_0 > 0, & 0 \leq x \leq h_0. \end{cases}$$

The authors divided the heterogeneous environment into a strong heterogeneous environment and a weak heterogeneous environment. They derived sufficient conditions for species spreading (resp. vanishing) in the strong heterogeneous environment, furthermore, they obtained sharp criteria for the spreading and vanishing in a weak heterogeneous environment by varying the parameters d and μ. A large number of related works about diffusive problem with a free boundary in homogeneous and heterogeneous environments can be found; see [28–35].

Inspired by the previous work, in this paper, we consider the following free boundary condition problem based on the reaction-diffusion-advection logistic model (1.1):

$$\begin{cases} u_t - u_{xx} + \alpha u_x = u(r(x) - u), & t > 0, 0 < x < h(t), \\ \alpha u(0,t) - u_x(0,t) = 0, \quad u(h(t),t) = 0, & t > 0, \\ h'(t) = -\mu u_x(h(t),t), & t > 0, \\ h(0) = h_0, \quad u(x,0) = u_0(x), & 0 \le x \le h_0, \end{cases} \tag{1.3}$$

and the initial function $u_0(x)$ satisfies

$$\begin{cases} u_0 \in C^2([0,h_0]), \\ \alpha u_0(0) - u_0'(0) = 0, \quad u_0(h_0) = 0, \\ u_0'(h_0) < 0, \quad u_0(x) > 0, \quad 0 \le x \le h_0. \end{cases} \tag{1.4}$$

The biological meanings of all parameters are the same as model (1.1) and equation (1.2), and h_0 is a positive constant. Furthermore, we assume that $r(x)$ satisfies

$$(\mathrm{H}): \quad r(x) \in C^1\big([0,\infty)\big) \text{ and } 0 < m_1 \le r(x) \le m_2 < \infty \quad \text{for } x \in [0,\infty),$$

where m_1 and m_2 are two positive constants. Throughout this paper, we always assume that (H) holds true. For simplicity, in the following we always denote $c_0 = 2\sqrt{d\bar{r}}$, which is the minimal speed of the traveling waves of (1.1). The aim of this paper is to study the effects of advection and the free boundary condition on the dynamics of model (1.3) in the heterogeneous environment. To the best of our knowledge, no work has been done for model (1.3).

The rest of our paper is arranged as follows. In the next section, we present some preliminaries including the global existence and uniqueness of the solution of model (1.3) and the comparison principle in the moving domain. In order to show the results of the spreading-vanishing dichotomy, the eigenvalue problems associated with model (1.3) are given in Section 3, and a sharp criterion for spreading and vanishing is established in Section 4. In Section 5, the asymptotic spreading speed of the free boundary is estimated if spreading of the invasion species occurs. In Section 6, we give a brief discussion.

2 Preliminaries

In this section, we first present the local existence and uniqueness of the solution of model (1.3) and then use suitable estimates to show that the solution is global.

First, we show that the local existence and uniqueness of the solution of model (1.3). Similar to those presented in [20] and [32], the proof can be shown via minor modifications. Thus, we omit it here.

Theorem 2.1 *For any given $u_0(x)$ satisfying* (1.4) *and any $\eta \in (0,1)$, there exists a $T > 0$ such that model* (1.3) *admits a unique solution*

$$(u,h) \in C^{1+\eta,\frac{1+\eta}{2}}\big(\big[0,h(t) \times [0,T]\big]\big) \times C^{1+\frac{\eta}{2}}\big([0,T]\big).$$

Moreover,

$$\|u\|_{C^{1+\eta,\frac{1+\eta}{2}}([0,h(t)]\times[0,T])} + \|h\|_{C^{1+\frac{\eta}{2}}([0,T])} \le C,$$

where C and T depend on h_0, η, and $\|u_0\|_{C^2([0,h_0])}$.

Next, we show the global existence of the solution of model (1.3) gained in Theorem 2.1. The following lemma is needed.

Lemma 2.2 *Let (u,h) be a solution of model* (1.3) *defined for $t \in (0,T)$ for some $T > 0$, then there exist constants M_1, M_2 independent of T such that*

$$0 < u(x,t) \le M_1, \quad \text{for } 0 < t < T, 0 \le x < h(t),$$
$$0 < h'(t) \le M_2, \quad \text{for } 0 < t < T.$$

Proof The strong maximum principle implies that $u(x,t) > 0$ and $u_x(h(t),t) < 0$ in $[0,h(t)] \times [0,T]$. By using the free boundary condition of model (1.3), we have $h'(t) > 0$ for $t \in (0,T)$. Note that $u(x,t)$ satisfies

$$\begin{cases} u_t - u_{xx} + \alpha u_x = u(r(x) - u), & t > 0, 0 < x < h(t), \\ \alpha u(0,t) - u_x(0,t) = 0, \qquad u(h(t),t) = 0, & t > 0, \\ u(x,0) = u_0(x), \quad 0 \le x \le h_0. \end{cases}$$

Thus, we get

$$u(x,t) \le M_1 \triangleq \max\big\{\|u_0\|_{L^\infty([0,h_0])}, \big\|r(x)\big\|_{L^\infty([0,\infty])}\big\}$$

for $0 < t < T$ and $0 \le x < h(t)$.

Now, we show that $h'(t)$ is bounded from above. To do this, we construct an auxiliary function

$$\omega(x,t) = M_1\big[2M\big(h(t) - x\big) - M^2\big(h(t) - x\big)^2\big], \quad \text{for } (x,t) \in \Omega_M,$$

where $\Omega_M = \{(x,t) : h(t) - M^{-1} < x < h(t), 0 < t < T\}$. In the following, we will choose M ($> h_0^{-1}$) such that $u(x,t) \ge \omega(x,t)$ holds over Ω_M.

Direct calculations show that for $(x,t) \in \Omega_M$, one has

$$\begin{cases} \omega_t - \omega_{xx} + \alpha\omega_x \ge 2M_1M(M - \alpha) \ge u(r(x) - u), \\ \omega(h(t) - M^{-1},t) = M_1 \ge u(h(t) - M^{-1},t), \\ \omega(h(t),t) = u(h(t),t) = 0, \end{cases}$$

provided $M \ge \alpha/2 + \sqrt{\alpha^2 + 2\|r(x)\|_{L^\infty[0,\infty)}}/2$.

It remains to show that $u_0(x) \leq w(x,0)$ for $x \in [h_0 - M^{-1}, h_0]$. To do this, the interval $[h_0 - M^{-1}, h_0]$ is divided into two subsets: $[h_0 - M^{-1}, h_0 - (2M)^{-1}]$ and $[h_0 - (2M)^{-1}, h_0]$. For $[h_0 - (2M)^{-1}, h_0]$, if $M \geq 4\|u_0\|_{C^1([0,h_0])}/(3M_1)$, then we have

$$\omega_x(x,0) = -2M_1M\bigl(1 - M(h_0 - x)\bigr) \leq -M_1M \leq u_0'(x),$$

which, together with $\omega(h_0,0) = u_0(h_0,0) = 0$, shows that $u_0(x) \leq w(x,0)$ for $[h_0 - (2M)^{-1}, h_0]$. For $[h_0 - M^{-1}, h_0 - (2M)^{-1}]$, if $M \geq 4\|u_0\|_{C^1([0,h_0])}/(3M_1)$, we have

$$w(x,0) \geq \frac{3M_1M}{4} \geq \|u_0\|_{C^1([0,h_0])}M^{-1} \geq u_0(x).$$

Therefore, we choose

$$M = \max\left\{\frac{\alpha}{2} + \frac{\sqrt{\alpha^2 + 2\|r(x)\|_{L^\infty[0,\infty)}}}{2}, \frac{4\|u_0\|_{C^1([0,h_0])}}{3M_1}, \frac{1}{h_0}\right\}.$$

Applying the maximum principle to $w - u$ over Ω_M, one finds $u(x,t) \leq w(x,t)$ for $(x,t) \in \Omega_M$. Furthermore, one has

$$u_x\bigl(h(t),t\bigr) \geq \omega_x\bigl(h(t),t\bigr) = -2MM_1$$

and

$$h'(t) = -\mu u_x\bigl(h(t),t\bigr) \leq 2MM_1\mu \triangleq M_2.$$

This completes the proof. □

Following the standard proof in [36], we obtain the following theorem for the global existence of the solution of model (1.3).

Theorem 2.3 *The solution of model* (1.3) *exists and is unique for* $t \in (0,\infty)$.

Finally, we introduce the comparison principle for model (1.3).

Lemma 2.4 (The comparison principle [37]) *Suppose that* $T \in (0,\infty)$. *Let* (u,h) *be the solution of model* (1.3), *suppose that* $(\overline{u},\overline{h}) \in C^{2,1}(D_T) \cap C(\overline{D}_T)$ *with* $D_T = \{(x,t) \in \mathbb{R}^2 : 0 < t < T, 0 < x < \overline{h}(t)\}$ *satisfies*

$$\begin{cases} \overline{u}_t - \overline{u}_{xx} + \alpha\overline{u}_x \geq \overline{u}(r(x) - \overline{u}), & 0 < t < T, 0 < x < \overline{h}(t), \\ \alpha\overline{u}(0,t) - \overline{u}_x(0,t) \geq 0, \qquad \overline{u}(\overline{h}(t),t) = 0, & 0 < t < T, \\ \overline{h}'(t) \geq -\mu\overline{u}_x(\overline{h}(t),t), & 0 < t < T. \end{cases}$$

If $\overline{h}(0) \geq h_0$ *and* $\overline{u}(x,0) \geq u_0(x)$ *for all* $0 \leq x < h_0$, *then*

$$u(x,t) \leq \overline{u}(x,t), \quad \text{for } 0 < t < T, 0 < x < \overline{h}(t),$$

$$h(t) \leq \overline{h}(t), \quad \text{for } 0 < t < T.$$

Here, we refer to the pair $(\overline{u}, \overline{h})$ as the upper solution of model (1.3). Similarly, the lower solution can be defined by reversing all the inequalities.

To investigate the dependence of the solution of model (1.3) on the expanding capability μ, we rewrite the solution (u, h) of model (1.3) as (u_μ, h_μ). As a direct consequence of Lemma 2.4, we have the following corollary.

Corollary 2.5 *Let* (u_μ, h_μ) *be the solution of model* (1.3). *For fixed* $u_0, \alpha, h_0, r(x)$, *if* $\mu_1 \leq \mu_2$, *then* $u_{\mu_1}(x,t) \leq u_{\mu_2}(x,t)$ *for* $(x,t) \in [0, h_{\mu_1}(t)] \times (0,\infty)$, *and* $h_{\mu_1}(t) \leq h_{\mu_2}(t)$ *for* $t \in (0,\infty)$.

3 An eigenvalue problem

In this section, we mainly study the principal eigenvalue problem and the properties of its principal eigenvalue. These results are significant for later sections.

Consider the following eigenvalue problem:

$$\begin{cases} \phi_{xx} - \alpha\phi_x + r(x)\phi = \tau\phi, & 0 < x < h_0, \\ \alpha\phi(0) - \phi_x(0) = 0, & \phi(h_0) = 0. \end{cases} \tag{3.1}$$

Let τ_1 and ϕ_1 denote the principal eigenvalue and the corresponding eigenfunction of problem (3.1), respectively, and τ_1 uniquely exists [3]. Using the transformation $\phi = e^{\alpha x}\psi$, where $\psi > 0$ is uniquely determined by the normalization $\int_0^{h_0} \psi^2\,dx = 1$, then (3.1) becomes

$$\begin{cases} \psi_{xx} + \alpha\psi_x + r\psi = \tau\psi, & 0 < x < h_0, \\ \psi_x(0) = 0, & \psi(h_0) = 0. \end{cases} \tag{3.2}$$

By (3.2), it follows the variational method that τ_1 has the following form (refer to [16] for the detailed derivation):

$$-\tau_1 = \inf_{\varphi \in S(h_0)} \left\{ \frac{\int_0^{h_0} \exp(\alpha x)(\varphi_x^2 - r\varphi^2)\,dx}{\int_0^{h_0} \exp(\alpha x)\varphi^2\,dx} \right\}, \tag{3.3}$$

where

$$S(h_0) = \left\{\varphi \in W^{1,2}\big([0,h_0]\big) : \varphi \not\equiv 0, \varphi(h_0) = 0\right\}.$$

If we set $\psi = e^{\frac{\alpha x}{2}}\varphi$, then (3.3) is equivalent to

$$\begin{aligned} -\tau_1 &= \inf_{\psi \in S(h_0)} \left\{ \frac{\int_0^{h_0} [(\psi_x - \frac{\alpha}{2}\psi)^2 - r\psi^2]\,dx}{\int_0^{h_0} \psi^2\,dx} \right\} \\ &= \inf_{\psi \in S(h_0)} \left\{ \frac{\int_0^{h_0} (\psi_x^2 - r\psi^2)\,dx + \frac{\alpha}{2}\psi^2(0)}{\int_0^{h_0} \psi^2\,dx} + \frac{\alpha^2}{4} \right\}. \end{aligned} \tag{3.4}$$

First, we fix $r(x)$ and α. Note that α is bounded. Then we have the following theorem.

Theorem 3.1 *Assume that* $\alpha < c_0$. *Let* $\tau_1(h_0)$ *denote the principal eigenvalue of* (3.1), *then*

(i) $\tau_1(h_0)$ *is strictly increasing function of* h_0;

(ii) $\lim_{h_0\to 0}\tau_1(h_0) = -\infty$;
(iii) $\lim_{h_0\to\infty}\tau_1(h_0) = \max_{x\in[0,h_0]} r(x) - \frac{\alpha^2}{4} > 0$.

Proof For the proof of part (i), it is similar to [27], Theorem 3.2, with some minor modifications. Here, we omit it.

Now we show the proof of part (ii) and (iii). In fact, according to (3.4), we have

$$-\tau_1 = \inf_{\psi\in S(h_0)}\left\{\frac{\int_0^{h_0}(\psi_x^2 - r\psi^2)\,dx + \frac{\alpha}{2}\psi^2(0)}{\int_0^{h_0}\psi^2\,dx} + \frac{\alpha^2}{4}\right\} \ge \frac{\pi^2}{4h_0^2} + \frac{\alpha^2}{4} - \max_{x\in[0,h_0]} r(x),$$

which implies that the conclusions of (ii) and (iii) hold true. This completes the proof. □

As a consequence of the above theorem, we have the following corollary.

Corollary 3.2 *Assume that $\alpha < c_0$. There exists $h^* = h^*(r,\alpha) > 0$ such that*

$$\tau_1(h_0) < 0 \quad \text{if } 0 < h_0 < h^*; \qquad \tau_1(h_0) = 0 \quad \text{if } h_0 = h^*; \qquad \tau_1(h_0) > 0 \quad \text{if } h_0 > h^*.$$

Next, we fix $r(x)$ and h_0. To consider the effects of advection on dynamics of model (1.3), we introduce the following theorem, which is the counterpart of Theorem 3.1, and we refer to [16], Lemmas 4.5, 4.6 and Theorem 4.1, for a detailed proof.

Theorem 3.3 *For fixed h_0 and $r(x) > 0$, let $\tau_1(\alpha)$ denote the principal eigenvalue of problem* (3.1), *then*

(i) *$\tau_1(\alpha)$ is a strictly decreasing function of α*;
(ii) $\lim_{\alpha\to\infty}\tau_1(\alpha) = -\infty$;
(iii) $\lim_{\alpha\to 0}\tau_1(\alpha) > 0$.

The above theorem implies the following corollary.

Corollary 3.4 *For fixed h_0 and $r(x) > 0$, there exists $\alpha^* = \alpha^*(r,h_0) > 0$ such that*

$$\tau_1(\alpha) > 0 \quad \text{if } 0 < \alpha < \alpha^*; \qquad \tau_1(\alpha) = 0 \quad \text{if } \alpha = \alpha^*; \qquad \tau_1(\alpha) < 0 \quad \text{if } \alpha > \alpha^*.$$

Note that the domain of model (1.3) changes with t, we replace h_0 of τ_1 defined in (3.4) by $h(t)$, *i.e.*,

$$-\tau_1\bigl(h(t)\bigr) = \inf_{\psi\in S(h(t))}\left\{\frac{\int_0^{h(t)}(\psi_x^2 - r\psi^2)\,dx + \frac{\alpha}{2}\psi^2(0)}{\int_0^{h(t)}\psi^2\,dx} + \frac{\alpha^2}{4}\right\},$$

where

$$S\bigl(h(t)\bigr) = \bigl\{\psi \in W^{1,2}\bigl(\bigl[0,h(t)\bigr]\bigr) : \psi \not\equiv 0, \psi_x(0) = \psi\bigl(h(t)\bigr) = 0\bigr\}.$$

Thus, Lemma 2.2 implies that the following theorem holds true.

Theorem 3.5 *$\tau_1(h(t))$ is a strictly monotone increasing function of t, it is equivalent to $\tau_1(h(t))$ being a strictly monotone increasing function of $h(t)$.*

4 Spreading and vanishing of an invasive species

This section is devoted to the proofs of the spreading-vanishing dichotomy, and the sharp criteria for spreading and vanishing.

4.1 Spreading-vanishing dichotomy

It follows from Lemma 2.2 that $x = h(t)$ is a monotone increasing function of t. There exists $h_\infty \in (h_0, \infty]$ such that $\lim_{t\to\infty} h(t) = h_\infty$. The spreading-vanishing dichotomy is a consequence of the following three theorems.

First, we show that the species will die out if the species cannot spread to the whole domain. Thus, the following theorem holds true.

Theorem 4.1 *Assume that* $\alpha < c_0$. *If* $h_\infty < \infty$, *then* $h_\infty \le h^*$ *and* $\lim_{t\to\infty} \|u(x,t)\|_{C([0,h(t)])} = 0$.

Proof Theorem 4.1 can be proved by the following two steps.

Step 1. Proof of $h_\infty \le h^*$ if $h_\infty < \infty$. By contradiction, we assume that there exists $T_0 = T_0(\varepsilon)$ for small enough $\varepsilon > 0$ such that $h_\infty > h(t) > h_\infty - \varepsilon > h^*$ for all $t \ge T_0$. Consider the following auxiliary problem:

$$\begin{cases} w_t - w_{xx} + \alpha w_x = w(r(x) - w), & t > T_0, 0 < x < h_\infty - \varepsilon, \\ \alpha w(0,t) - w_x(0,t) = 0, \qquad w(h_\infty - \varepsilon, t) = 0, & t > T_0, \\ w(x, T_0) = u(x, T_0), & 0 < x < h_\infty - \varepsilon. \end{cases} \tag{4.1}$$

Following Corollary 3.2 and $h_\infty - \varepsilon > h^*$, we know problem (4.1) is a logistic problem with $\tau_1 = \tau_1(\alpha, h_\infty - \varepsilon) > 0$. It follows from [3], Proposition 3.3, that problem (4.1) admits a unique positive solution $\underline{w} = \underline{w}_\varepsilon(x,t)$ satisfying

$$\underline{w}_\varepsilon(x,t) \to v_{h_\infty - \varepsilon}(x), \quad \text{in } C^2\big([0, h_\infty - \varepsilon]\big) \text{ as } t \to \infty,$$

where $v_{h_\infty-\varepsilon}(x)$ is the unique positive solution of the following problem:

$$\begin{cases} -v_{xx} + \alpha v_x = v(r(x) - v), & 0 < x < h_\infty - \varepsilon, \\ \alpha v(0) - v_x(0) = 0, \qquad v(h_\infty - \varepsilon) = 0. \end{cases}$$

By the comparison principle, one has

$$u(x,t) \ge \underline{w}_\varepsilon(x,t) \ge v_{h_\infty-\varepsilon}(x), \quad \text{for } 0 \le x \le h_\infty - \varepsilon \text{ and } t > T_0,$$

which implies

$$\varliminf_{t\to\infty} u(x,t) \ge v_{h_\infty-\varepsilon}(x), \quad \text{for } 0 \le x \le h_\infty - \varepsilon \text{ and } t > T_0. \tag{4.2}$$

Furthermore, consider the following problem:

$$\begin{cases} w_t - w_{xx} + \alpha w_x = w(r(x) - w), & t > T_0, 0 \le x \le h_\infty, \\ \alpha w(0,t) - w_x(0,t) = 0, \qquad w(h_\infty, t) = 0, & t > T_0, \\ w(x, T_0) = \overline{u}(x, T_0), & 0 \le x \le h_\infty, \end{cases} \tag{4.3}$$

where

$$\overline{u}(x,T_0)=\begin{cases}u(x,t), & 0\le x\le h(T_0),\\ 0, & h(T_0)<x<h_\infty.\end{cases}$$

Obviously, (4.3) admits a unique positive solution $\overline{w}=\overline{w}(x,t)$ which satisfies

$$\overline{w}(x,t)\to v_{h_\infty}(x),\quad \text{in } C^2\big([0,h_\infty]\big) \text{ as } t\to\infty,$$

where $v_{h_\infty}(x)$ is the unique positive solution of the problem

$$\begin{cases}-v_{xx}+\alpha v_x=v(r(x)-v), & 0\le x\le h_\infty,\\ \alpha v(0)-v_x(0)=0, & v(h_\infty)=0.\end{cases}$$

The comparison principle leads to

$$u(x,t)\le\overline{w}(x,t),\quad \text{for } 0\le x\le h_\infty \text{ and } t>T_0.$$

Thus, we have

$$\overline{\lim_{t\to\infty}}\,u(x,t)\le v_{h_\infty}(x),\quad \text{for } 0\le x\le h_\infty \text{ and } t>T_0. \tag{4.4}$$

By a standard compactness and uniqueness argument, we can easily show

$$v_{h_\infty-\varepsilon}\to v_{h_\infty},\quad \text{in } C^2_{\mathrm{loc}}\big([0,h_\infty]\big) \text{ as } \varepsilon\to 0^+.$$

Combining (4.2), (4.4), and the arbitrariness of ε yields

$$\lim_{t\to\infty}u(x,t)=v_{h_\infty}(x),\quad \text{for } 0\le x\le h_\infty, \tag{4.5}$$

which implies

$$\big\|u(x,t)-v_{h_\infty}(x)\big\|_{C^2([0,h(t)])}\to 0\quad \text{as } t\to\infty.$$

Thus, one has

$$u_x\big(h(t),t\big)\to v'_{h_\infty}(h_\infty)<0\quad \text{as } t\to\infty.$$

It follows from the boundary condition that

$$h'(t)=-\mu u_x\big(h(t),t\big)\to -\mu v'_{h_\infty}(h_\infty)>0\quad \text{as } t\to\infty,$$

which implies $h_\infty=\infty$, a contradiction with $h_\infty<\infty$. Therefore we have $h_\infty\le h^*$.

Step 2. Proof of $\lim_{t\to\infty}\|u(x,t)\|_{C([0,h(t)])}=0$. Let $\overline{u}(x,t)$ be the unique solution of the problem

$$\begin{cases}\overline{u}_t-\overline{u}_{xx}+\alpha\overline{u}_x=\overline{u}(r(x)-\overline{u}), & t>0, 0<x<h_\infty,\\ \alpha\overline{u}(0,t)-\overline{u}_x(0,t)=0, \quad \overline{u}(h(t),t)=0, & t>0,\\ \overline{u}(x,0)=\overline{u}_0(x), & 0<x<h_\infty,\end{cases}\tag{4.6}$$

with

$$\overline{u}_0(x)=\begin{cases}u_0(x), & 0\le x\le h_0,\\ 0, & h_0<x<h_\infty.\end{cases}$$

From the comparison principle, one has $0\le u(x,t)\le\overline{u}(x,t)$, for $t>0$ and $0\le x\le h(t)$. Since $h_\infty\le h^*$, we have $\tau_1(\alpha,r,h_\infty)\le 0$ from Corollary 3.2. Following [3], Corollary 3.4, we obtain $\overline{u}(x,t)\to 0$ uniformly for $x\in[0,h_\infty]$ as $t\to\infty$. Thus $\lim_{t\to\infty}\|u(x,t)\|_{C([0,h(t)])}=0$. □

Next, we show the case that the species can spread to the whole domain. Thus, we have the following theorem.

Theorem 4.2 *If* $0<\alpha\le\alpha^*$, *then* $\lim_{t\to\infty}\|u(x,t)\|_{C([0,h(t)])}>0$ *and* $h_\infty=\infty$, *i.e., spreading occurs.*

Proof Here, we still use τ_1 and ϕ_1 to denote the principal eigenvalue and the corresponding eigenfunction of problem (3.1), respectively.

Consider the case $0<\alpha<\alpha^*$. It follows from Corollary 3.4 that we have $\tau_1>0$. Define

$$\underline{u}(x,t)=\begin{cases}\delta\phi_1(x), & t>0, 0\le x\le h_0,\\ 0, & t>0, x>h_0,\end{cases}$$

where δ is sufficient small such that $\delta\phi_1\le\min\{\tau_1,u_0(x)\}$ for $x\in[0,h_0]$. Direct computations yield

$$\underline{u}_t-\underline{u}_{xx}+\alpha\underline{u}_x-\underline{u}\big(r(x)-\underline{u}\big)=\delta\phi_1(x)(-\tau_1+\delta\phi_1)\le 0.$$

Furthermore, one has

$$\begin{cases}\underline{u}_t-\underline{u}_{xx}+\alpha\underline{u}_x\le\underline{u}(r(x)-\underline{u}), & t>0, 0\le x\le h_0,\\ \underline{u}_x(0,t)-\alpha\underline{u}(0,t=\underline{u}(h_0,t)=0, & t>0,\\ 0=h_0'\le-\mu\underline{u}_x(h_0,t)=-\mu\delta\phi_1'(h_0), & t>0,\\ \underline{u}(x,0)\le u_0(x), & 0\le x\le h_0.\end{cases}$$

Following Lemma 2.4, we conclude that $u(x,t)\ge\underline{u}(x,t)$ in $[0,h_0]\times(0,\infty)$. Then $\lim_{t\to\infty}\|u(x,t)\|_{C([0,h(t)])}\ge\delta\phi_1(0)>0$ and $h_\infty=\infty$ hold from Theorem 4.1.

Consider the case $\alpha=\alpha^*$. By Corollary 3.4, one has $\tau_1=0$. The monotonicity of $h(t)$ shows that $h(t_0)>h_0$ for small $t_0>0$. It follows from Theorem 3.1 that we have

$\tau_1(\alpha^*, r, h(t_0)) > \tau_1(\alpha^*, r, h_0)$. Hence, replacing h_0 by $h(t_0)$, we can repeat the same process as above to construct a lower solution over $[0, h(t_0)] \times [t_0, \infty)$. And so, the desired result follows. The proof is finished. □

Finally, we study the long time behavior of the spreading species. The proof is similar with those in [38], Theorem 2.3, and [39], Propositions 3, 4. Here, we omit the proof for brevity.

Theorem 4.3 *If $\tau_1 > 0$ or $h_\infty = \infty$, then the solution of model* (1.3) *satisfies* $\lim_{t\to\infty} u(x,t) = u^*(x)$ *uniformly in any bounded subset of* $(0,\infty)$, *where u^* is the unique positive equilibrium of the stationary problem*

$$-u^*_{xx} + \alpha u^*_x = \bigl(r(x) - u^*\bigr)u^*, \quad \textit{for } 0 < x < \infty. \tag{4.7}$$

Combining Theorems 4.1, 4.2 and 4.3, we obtain the following spreading-vanishing dichotomy theorem.

Theorem 4.4 *Let* $(u(x,t),h(t))$ *be the solution of model* (1.3). *The following alternatives hold.*

(i) *Spreading*: $h_\infty = \infty$ *and* $\lim_{t\to\infty} u(x,t) = u^*(x)$ *uniformly in any bounded subset of* $(0,\infty)$;

(ii) *Vanishing*: $h_\infty < \infty$ *and* $\lim_{t\to\infty} \|u(x,t)\|_{C([0,h(t)])} = 0$.

4.2 Sharp criteria for spreading and vanishing

From Theorem 4.2, we see that vanishing is possible only when $\alpha > \alpha^*$. The following theorem reveals the sharp criteria for spreading and vanishing, which provides some sufficient conditions for vanishing.

Theorem 4.5 *If $\alpha > \alpha^*$ and $\|u_0\|_{C([0,h_0])}$ is sufficiently small, then $h_\infty < \infty$ and*

$$\lim_{t\to\infty} \bigl\|u(x,t)\bigr\|_{C([0,h(t)])} = 0.$$

Proof For $t > 0$, we define

$$\sigma(t) = h_0\left(1 + \delta - \frac{\delta}{2}e^{-\delta t}\right), \qquad w(x,t) = \varepsilon e^{-\delta t} e^{-\frac{\alpha}{2}(\sigma(t)-x)} \cos\left(\frac{\pi x}{2\sigma(t)}\right), \quad 0 \le x \le \sigma(t),$$

where $0 < \delta < 1$ and ε are positive constants to be determined later. It is easy to observe that for $t > 0$, one has

$$\left(1 + \frac{\delta}{2}\right)h_0 \le \sigma(t) \le (1+\delta)h_0 \quad \text{and} \quad 0 \le \sigma'(t) = \frac{h_0\delta^2 e^{-\delta t}}{2} \le \frac{h_0\delta^2}{2}.$$

By Corollary 3.4, we have $\tau_1 < 0$ if $\alpha > \alpha^*$. It follows from (3.4) that we may choose sufficiently small $\delta > 0$ such that

$$-\delta - \frac{\alpha\delta^2 h_0}{4} + \frac{\alpha^2}{4} + \frac{\pi^2}{4h_0^2(1+\delta)^2} - \hat{r} > 0,$$

where $\hat{r} = \max_{x\in[0,\sigma(t)]} r(x)$. For $t > 0$ and $0 \le x \le \sigma(t)$, direct calculations give

$$\begin{aligned} & w_t - w_{xx} + \alpha w_x - r(x)w \\ &= w\left[-\delta - \frac{\alpha\sigma'(t)\phi}{2} + \frac{\alpha^2}{4} + \frac{\pi^2}{4\sigma^2(t)} - r(x)\right] + \frac{\pi x\sigma'(t)}{2\sigma^2(t)}\epsilon e^{-\delta t} e^{-\frac{\alpha}{2}(\sigma(t)-x)} \sin\frac{\pi x}{2\sigma(t)} \\ &\ge w\left[-\delta - \frac{\alpha\delta^2 h_0}{4} + \frac{\alpha^2}{4} + \frac{\pi^2}{4h_0^2(1+\delta)^2} - \hat{r}\right] > 0. \end{aligned}$$

Obviously,

$$\alpha w(0,t) - w_x(0,t) \ge 0, \quad \text{and} \quad w\big(\sigma(t),t\big) = 0.$$

Thus, we choose $0 < \varepsilon \le \frac{\delta^2 h_0^2}{\mu\pi}(1 + \frac{\delta}{2})$ such that $\sigma'(t) \ge -\mu w_x(\sigma(t))$. Furthermore, let u_0 be small enough such that

$$\|u_0\|_{C([0,h_0])} \le \varepsilon e^{\frac{\alpha\delta h_0}{4}} \cos\frac{\pi}{2+\delta} = w(h_0,t).$$

Therefore, $(w(x,t),\sigma(t))$ is an upper solution of model (1.3). Applying Lemma 2.4, we show that $h(t) \le \sigma(t)$ and $u(x,t) \le w(x,t)$ for $t > 0$ and $0 \le x \le h(t)$. Clearly,

$$\lim_{t\to\infty} \sigma(t) = h_0(1+\delta) < \infty.$$

Thus, $h_\infty < \infty$. Applying Theorem 4.1, we have $\lim_{t\to\infty} \|u(x,t)\|_{C([0,h(t)])} = 0$. □

As a direct consequence of Theorem 4.5, we are able to show the following theorem.

Theorem 4.6 *If $\alpha > \alpha^*$ and μ is sufficiently small, then $h_\infty < \infty$ and*

$$\lim_{t\to\infty} \big\|u(x,t)\big\|_{C([0,h(t)])} = 0.$$

The next result implies that spreading occurs for large expansion capacity and the proof will be leaved out, since it is similar to that in [32], Lemma 3.7, or [33], Lemma 2.8.

Theorem 4.7 *For $\alpha > \alpha^*$ and any given u_0 satisfying model* (1.3), *if μ is sufficiently large then $h_\infty = \infty$.*

Combining Theorems 4.5-4.7, we can derive the sharp criteria for spreading-vanishing for the invasion species.

Theorem 4.8 (Sharp criteria) *For given h_0, α, and u_0 satisfying model* (1.3), *there exists $\mu^* \in [0,\infty)$ (depending on u_0 and α) such that spreading occurs if $\mu > \mu^*$; and vanishing occurs if $0 < \mu \le \mu^*$. Furthermore, $\mu^* = 0$ if $0 < \alpha \le \alpha^*$, while $\mu^* > 0$ if $\alpha > \alpha^*$.*

Proof First, for the case $0 < \alpha \le \alpha^\star$, it follows from Theorem 4.2 that spreading always occurs if $0 < \alpha \le \alpha^*$. Therefore, we can choose $\mu^* = 0$.

Next, for the case $\alpha > \alpha^\star$, we define

$$\mu^* = \sup\{\sigma_0 : h_\infty(\mu) \le h^*, \text{ for } \mu \in (0, \sigma_0]\}.$$

Applying Theorems 4.6-4.7, it follows that $\mu^* \in [0, \infty)$. Following Corollary 2.5, we observe that species spreads if $\mu > \mu^*$ and species vanishes if $0 < \mu < \mu^*$.

Finally, we show that vanishing happens if $\mu = \mu^*$. Otherwise, $h_\infty = \infty$ for $\mu = \mu^*$. It follows from Corollary 3.2 that there exists $T_0 > 0$ such that $h(T_0) > h^*$. Now, the symbol (u_μ, h_μ) is used to emphasize the dependence of solution (u, h) of model (1.3) on μ. Hence, $h_\mu(T_0) > h^*$ follows. Since (u_μ, h_μ) has a continuous dependence on μ, we see that there exists small $\varepsilon > 0$ such that $\mu = \mu^* - \varepsilon$, which implies that the species spreads, contradicting the definition of μ^*. Hence, vanishing occurs if $\mu = \mu^*$. The proof is completed. □

5 Asymptotic spreading speed

This section is devoted to rough estimates of the asymptotic spreading speed. Following [40], Proposition 2.1, we have the following proposition.

Proposition 5.1 ([40]) *Let (u, h) be the unique solution of model* (1.3) *with $\alpha = 0$, if $h_\infty = +\infty$, then $\lim_{t\to\infty} h(t)/t = k_0$, where $(k_0, q(z))$ is the unique solution of the problem*

$$\begin{cases} -q'' + kq' = q(\overline{r} - q), & z > 0, k \ge 0, \\ q(0) = 0, \quad q(\infty) = \overline{r}, \quad q(z) > 0, & z > 0, \\ \mu q'(0) = k_0. \end{cases}$$

Furthermore, we introduce the following problem:

$$\begin{cases} q'' - (k - \alpha)q' + q(\overline{r} - q) = 0, & z > 0, k \ge 0, \\ q(0) = 0, \quad q(\infty) = \overline{r}, \quad q(z) > 0, & z > 0. \end{cases} \tag{5.1}$$

The following proposition is useful for the proof of the main result.

Proposition 5.2 ([41]) *Consider problem* (5.1), *the following statements hold*:

- (i) *Problem* (5.1) *has exactly one solution (k^*, q^*) such that $\mu(q^*)'(0) = k^*$. Moreover, $k^* = k^*(\alpha, r) \in (0, 2\sqrt{\overline{r}} + \alpha)$;*
- (ii) $0 < \overline{k}^* < k^*$, *where $\overline{k}^*$ is the speed of problem* (5.1) *with $\alpha = 0$;*
- (iii) *k^* is strictly increasing of parameter r, i.e., for any $r_1 > r_2 > 0$, we have*

$$k^*(\alpha, r_1) > k^*(\alpha, r_2), \qquad \lim_{\varepsilon \to 0} k^*(\alpha, r + \varepsilon) = k^*(\alpha, r).$$

Now, we have the following estimates of the asymptotic spreading speed for model (1.3) when spreading happens.

Theorem 5.3 *Assume that $\tilde{r} = \lim_{x\to\infty} r(x)$ exists. If $h_\infty = \infty$, then*

$$\lim_{t\to\infty} \frac{h(t)}{t} = k^*(\alpha, \tilde{r}).$$

Proof By assumption (H) and Theorem 4.3, the unique positive solution $u^*(x)$ of the stationary problem (4.7) satisfies

$$0 < m_1 \le \liminf_{x\to\infty} u^*(x) \le \limsup_{x\to\infty} u^*(x) \le m_2.$$

For any $\varepsilon > 0$, there exists $L = L(\varepsilon) > 0$ such that for all $x \ge L$

$$m_1 - \varepsilon \le r(x) \le m_2 + \varepsilon \quad \text{and} \quad m_1 - \frac{\varepsilon}{2} < u^*(x) < m_2 + \frac{\varepsilon}{2}.$$

It follows from the comparison principle and [33], Theorem 3.6 that we can derive

$$\liminf_{t\to\infty} \frac{h(t)}{t} \ge k^*(\alpha, m_1 - \varepsilon), \qquad \limsup_{t\to\infty} \frac{h(t)}{t} \le k^*(\alpha, m_2 + \varepsilon).$$

Due to the arbitrariness of ε, we have

$$\liminf_{t\to\infty} \frac{h(t)}{t} \ge k^*(\alpha, m_1), \qquad \limsup_{t\to\infty} \frac{h(t)}{t} \le k^*(\alpha, m_2).$$

Furthermore, we suppose that $m_1 = m_2 = \tilde{r}$, then it follows from (iii) of Proposition 5.2 that we have

$$\lim_{t\to\infty} \frac{h(t)}{t} = k^*(\alpha, \tilde{r}).$$ □

6 Discussion

In this paper, we incorporated the free boundary and heterogenous environment into the reaction-diffusion-advection logistic model, which is more realistic for describing the invasion dynamics of a new species, investigated the influence of the advection term and spatial heterogeneous environment features on the dynamics of the invading species, and gave a rough estimates of the asymptotic spreading speed.

According to the analysis, we determined the spreading-vanishing dichotomy and the sharp criteria for the spreading and vanishing by choosing the advection rate α and the expansion capacity μ as variable parameters. Therefore, whether the invasion species will spread or vanish depends on the advection rate α, the expansion capacity μ, and the initial function $u_0(x)$. More specifically, we found a positive threshold α^* such that the invasive species always spreads if $0 < \alpha \le \alpha^*$, which is consistent with the result of [16]. However, for $\alpha > \alpha^*$, there exists a critical criterion, $\mu^* > 0$, such that species spreads if $\mu > \mu^*$ and species vanishes if $0 < \mu \le \mu^*$, which is distinct from that in [16]. Biologically, these results mean that the invasive species spreads either under a small advection rate, or under a big advection rate when it is favored by its expansion capacity and initial values. Furthermore, species with a large expansion capacity will benefit to survive.

Now, we make some comparisons with the previous work. Compared with the work of [32, 33], we considered the influence of the advection term on the dynamics of the invading species. Compared with the work of [37], we considered the influence of spatial heterogeneous environment features on the dynamics of the invasive population. Compared with the work of [9, 16], our work implied that spreading or vanishing of invasive population not only depends on advection term and no-flux boundary condition but it also relates to free boundary condition. It should be noticed that we found two thresholds, α^* and μ^*, which are different from the previous work. Therefore, our work and the corresponding conclusions are more general. It should be noticed that, due to the assumption of the heterogeneous environment, we only obtain the weaker result on the asymptotic behavior of the invasive species.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this paper. All authors read and approved the final manuscript.

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