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Existence and continuity of positive solutions on a parameter for second-order impulsive differential equations

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Abstract

Applying the eigenvalue theory and theory of α-concave operator, we establish some new sufficient conditions to guarantee the existence and continuity of positive solutions on a parameter for a second-order impulsive differential equation. Furthermore, two nonexistence results of positive solutions are also given. In particular, we prove that the unique solution $u_\lambda(t)$ of the problem is strongly increasing and depends continuously on the parameter λ.

Keywords: continuity on a parameter; impulsive differential equations; transformation technique; L^p-integrable; eigenvalue

1 Introduction

We consider the second-order impulsive differential equation

$$\begin{cases} u''(t) + \lambda\omega(t)f(u(t)) = 0, & t \in (0,1), t \neq t_k, \\ u(t_k^+) - u(t_k) = c_k u(t_k), & k = 1,2,\ldots,n, \\ u'(0) = 0, & au(1) + bu'(1) = \int_0^1 g(t)u(t)\,dt, \end{cases} \tag{1.1}$$

where $\lambda > 0, \omega \in L^p[0,1]$ for some $1 \leq p \leq +\infty$, $f \in C(R^+, R^+), R^+ = [0,+\infty], t_k$ $(k = 1,2,\ldots,n)$ are fixed points with $0 < t_1 < t_2 < \cdots < t_k < \cdots < t_n < 1, a, b > 0$, $\{c_k\}$ is a real sequence with $c_k > -1, k = 1,2,\ldots,n$, $x(t_k^+)$ $(k = 1,2,\ldots,n)$ denotes the right-hand limit of $x(t)$ at $t = t_k$, and $g \in C[0,1]$ is a nonnegative function. In addition, we assume that ω, f, c_k, and g satisfy

(H_1) $\omega \in L^p[0,1]$ for some $1 \leq p \leq +\infty$, and there exists $\xi > 0$ such that $\omega(t) \geq \xi$ a.e. on J;

(H_2) $f \in C([0,+\infty),[0,+\infty))$ with $f(0) = 0$ and $f(u) > 0$ for $u > 0$, $\{c_k\}$ is a real sequence with $c_k > -1$, $k = 1,2,\ldots,n$, and $c(t) := \Pi_{0<t_k<t}(1 + c_k)$;

(H_3) $g \in C[0,1]$ is nonnegative with

$$\mu := \int_0^1 g(t)c(t)\,dt \in \bigl[0, ac(1)\bigr). \tag{1.2}$$

Remark 1.1 We always assume that the product $c(t) := \Pi_{0<t_k<t}(1+c_k)$ equals unity if the number of factors is equal to zero, and let

$$c_M = \max_{t\in J} c(t), \qquad c_m = \min_{t\in J} c(t), \qquad c^{-1}(t) = \Pi_{0<t_k<t}(1+c_k)^{-1}.$$

Remark 1.2 Combining (H_2) and the definition of $c(t)$, we know that $c(t)$ is a step function, which is bounded on J, and

$$c(t) > 0, \quad \forall t \in J, \qquad c(t) = 1, \quad \forall t \in [0, t_1].$$

Such problems were first studied by Zhang and Feng [1]. By using transformation technique to deal with impulse term of second-order impulsive differential equations, the authors obtained existence results of positive solutions by using fixed point theorems in a cone. However, they only considered the case $\omega(t) \equiv 1$ on $t \in [0,1]$. The other related results can be found in [2–14]. However, there are almost no papers on second-order boundary value problems, especially second-order boundary value problems with impulsive effects, using the eigenvalue theory. In this paper, we solve this problem.

The first goal of this paper is to establish several criteria for the optimal intervals of the parameter λ so as to ensure the existence of positive solutions for problem (1.1). Our method is based on transformation technique, Hölder's inequality, and the eigenvalue theory and is completely different from those used in [1–14].

Another contribution of this paper is to study the expression and properties of Green's function associated with problem (1.1). It is interesting to point out that the Green's function associated with problem (1.1) is positive, which is different from that of [15].

Moreover, we give two nonexistence results. The arguments that we present here are based on geometric properties of the super-sublinearity of f at zero and infinity, which was first used by Sánchez in [16] (see Properties 1.1-1.2).

For convenience, we introduce the following notations:

$$f_0 = \lim_{u\to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to +\infty} \frac{f(u)}{u}.$$

The following geometric Properties 1.1-1.2 will be very important in our arguments.

Property 1.1 If $f_0 = +\infty$ and $f_\infty = +\infty$, then there exists $R > 0$ such that

$$\frac{f(R)}{R} = \min_{t>0} \frac{f(t)}{t}. \tag{1.3}$$

Let $\bar{R}$ be a point where f attains its maximum on the interval $(0, R]$.

Property 1.2 If $f_0 = 0$ and $f_\infty = 0$, then there exists $R > 0$ such that

$$\frac{f(R)}{R} = \max_{u>0} \frac{f(u)}{u}. \tag{1.4}$$

Finally, we are able to obtain the uniqueness results of problem (1.1) by using theory of α-concave operators. We also obtain the following analytical properties: the unique solution $u_\lambda(t)$ of the above problem is strongly increasing and depends continuously on the parameter λ.

The rest of this paper is organized as follows. In Section 2, we provide some necessary background. In particular, we introduce some lemmas and definitions associated with the eigenvalue theory and theory of α-concave (or $-\alpha$-convex) operators. Several technical lemmas are given in Section 3. In Section 4, we establish the existence and nonexistence of positive solutions for problem (1.1). In Section 5, we prove the uniqueness of a positive solution for problem (1.1) and its continuity on a parameter . In Section 6, we offer some remarks and comments on the associated problem (1.1). Finally, in Section 7, two examples are also included to illustrate the main results.

2 Preliminaries

In this section, we collect some known results, which can be found in the book by Guo and Lakshmikantham [17].

Definition 2.1 Let E be a real Banach space over $\mathbf{R}$. A nonempty closed set $P \subset E$ is said to be a cone if

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0, b \geq 0$, and
(ii) $u, -u \in P$ implies $u = 0$.

Definition 2.2 A cone P of a real Banach space E is a solid cone if P° is not empty, where P° is the interior of P.

Every cone $P \subset E$ induces a semiorder in E given by "$\leq$". That is, $x \leq y$ if and only if $y - x \in P$. If a cone P is solid and $y - x \in P^\circ$, then we write $x \ll y$.

Definition 2.3 A cone P is said to be normal if there exists a positive constant δ such that

$$\|x + y\| \geq \delta, \quad \forall x, y \in P, \|x\| = 1, \|y\| = 1.$$

Geometrically, normality means that the angle between two positive unit vectors is bounded away from π. In other words, a normal cone cannot be too large.

Lemma 2.1 *Let P be a cone in E. Then the following assertions are equivalent*:

(i) *P is normal*;
(ii) *There exists a constant $\gamma > 0$ such that*

$$\|x + y\| \geq \gamma \max\{\|x\|, \|y\|\}, \quad \forall x, y \in P;$$

(iii) *There exists a constant $\eta > 0$ such that $0 \leq x \leq y$ implies that $\|x\| \leq \eta\|y\|$, that is, the norm $\|\cdot\|$ is semimonotone*;
(iv) *There exists an equivalent norm $\|\cdot\|_1$ on E such that $0 \leq x \leq y$ implies that $\|x\|_1 \leq \|y\|_1$, that is, the norm $\|\cdot\|_1$ is semimonotone*;
(v) *$x_n \leq z_n \leq y_n$ $(n = 1, 2, 3, \ldots)$ and $\|x_n - x\| \to 0, \|y_n - x\| \to 0$ imply that $\|z_n - x\| \to 0$*;
(vi) *The set $(B + P) \cap (B - P)$ is bounded, where*

$$B = \{x \in E : \|x\| \leq 1\};$$

(vii) *Every order interval* $[x,y] = \{z \in E : x \leq z \leq y\}$ *is bounded.*

Remark 2.1 Some authors use assertion (iii) as the definition of normality of a cone P and call the smallest number η the normal constant of P.

Definition 2.4 Let P be a solid cone of a real Banach space E. An operator $A : P^\circ \to P^\circ$ is called an α-concave operator ($-\alpha$-convex operator) if

$$A(tx) \geq t^\alpha Ax\bigl(A(tx) \leq t^{-\alpha} Ax\bigr), \quad \forall x \in P^\circ, 0 < t < 1,$$

where $0 \leq \alpha < 1$. The operator A is increasing (decreasing) if $x_1, x_2 \in P^\circ$ and $x_1 \leq x_2$ imply $Ax_1 \leq Ax_2$ ($Ax_1 \geq Ax_2$), and further, the operator A is strongly increasing (decreasing) if $x_1, x_2 \in P^\circ$ and $x_1 < x_2$ imply $Ax_2 - Ax_1 \in P^\circ$ ($Ax_1 - Ax_2 \in P^\circ$). Let x_λ be a proper element of an eigenvalue λ of A, that is, $Ax_\lambda = \lambda x_\lambda$. Then x_λ is called strongly increasing (decreasing) if $\lambda_1 > \lambda_2$ implies that $x_{\lambda_1} - x_{\lambda_2} \in P^\circ$ ($x_{\lambda_2} - x_{\lambda_1} \in P^\circ$), which is denoted by $x_{\lambda_1} \gg x_{\lambda_2}$ ($x_{\lambda_2} \gg x_{\lambda_1}$).

Definition 2.5 An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Lemma 2.2 (Arzelà-Ascoli) *A set* $M \subset C(J,R)$ *is said to be a precompact set if the following two conditions are satisfied*:

(i) *All the functions in the set* M *are uniformly bounded, which means that there exists a constant* $r > 0$ *such that* $|u(t)| \leq r, \forall t \in J, u \in M$;

(ii) *All the functions in the set* M *are equicontinuous, which means that for every* $\varepsilon > 0$, *there is* $\delta = \delta(\varepsilon) > 0$, *which is independent of the function* $u \in M$, *such that*

$$\bigl|u(t_1) - u(t_2)\bigr| < \varepsilon$$

whenever $|t_1 - t_2| < \delta, t_1, t_2 \in J$.

Lemma 2.3 *Suppose* D *is an open subset of an infinite-dimensional real Banach space* E, $\theta \in D$, *and* P *is a cone of* E. *If the operator* $\Gamma : P \cap D \to P$ *is completely continuous with* $\Gamma\theta = \theta$ *and satisfies*

$$\inf_{x \in P \cap \partial D} \Gamma x > 0,$$

then Γ *has a proper element on* $P \cap \partial D$ *associated with a positive eigenvalue. That is, there exist* $x_0 \in P \cap \partial D$ *and* μ_0 *such that* $\Gamma x_0 = \mu_0 x_0$.

Lemma 2.4 *Suppose that* P *is a normal cone of a real Banach space and* $A : P^\circ \to P^\circ$ *is an* α-*concave increasing* (*or* $-\alpha$-*convex decreasing*) *operator. Then* A *has exactly one fixed point in* P°.

3 Some lemmas

Let $J = [0,1]$. A function $u(t)$ is said to be a solution of problem (1.1) on J if:

(i) $u(t)$ is absolutely continuous on each interval $(0, t_1]$ and $(t_k, t_{k+1}]$, $k = 1, 2, \ldots, n$;

(ii) for any $k = 1, 2, \ldots, n$, $u(t_k^+)$ and $u(t_k^-)$ exist, and $u(t_k^-) = u(t_k)$;

(iii) $u(t)$ satisfies (1.1).

We shall reduce problem (1.1) to a system without impulse. To this goal, firstly, by means of the transformation

$$u(t) = c(t)y(t) \tag{3.1}$$

we convert problem (1.3) into

$$\begin{cases} -y''(t) = \lambda\omega(t)c^{-1}(t)f(c(t)y(t)), & t \in J, \\ y'(0) = 0, \quad ac(1)y(1) + bc(1)y'(1) = \int_0^1 g(s)c(s)y(s)\,ds. \end{cases} \tag{3.2}$$

The following lemmas will be used in the proof of our main results.

Lemma 3.1 *Assume that* (H_1)-(H_3) *hold. Then*

(i) *If* $y(t)$ *is a solution of problem* (3.2) *on* J, *then* $u(t) = c(t)y(t)$ *is a solution of problem* (1.1) *on* J;

(ii) *If* $u(t)$ *is a solution of problem* (1.1) *on* J, *then* $y(t) = c^{-1}(t)u(t)$ *is a solution of problem* (3.2) *on* J.

Proof (i) Let $y(t)$ be a solution of (3.2) on J. It is easy to see that $u(t) = c(t)y(t)$ is absolutely continuous on each interval $(t_k, t_{k+1}]$, $k = 1, 2, \dots, n$. By the definition of $c(t)$ we have $c'(t) = 0$ for $t \neq t_k$. Then, for $t \neq t_k$, we have

$$u'(t) = c'(t)y(t) + c(t)y'(t) = c(t)y'(t),$$
$$u''(t) = c'(t)y'(t) + c(t)y''(t) = c(t)y''(t).$$

It follows that

$$-u''(t) = -c(t)y''(t) = \lambda\omega(t)f\bigl(c(t)y(t)\bigr) = \lambda\omega(t)f\bigl(u(t)\bigr).$$

For $t = t_k$, we have

$$u\bigl(t_k^+\bigr) = \lim_{t\to t_k^+} c(t)y(t) = \Pi_{0\le t_i\le t_k}(1 + c_i)y(t_k),$$
$$u\bigl(t_k^-\bigr) = \lim_{t\to t_k^-} c(t)y(t) = \Pi_{0\le t_i\le t_{k-1}}(1 + c_i)y(t_k).$$

By (ii) of Definition 2.2, $u(t_k^-) = u(t_k)$, so we have

$$u\bigl(t_k^+\bigr) - u(t_k) = u\bigl(t_k^+\bigr) - u\bigl(t_k^-\bigr) = \Pi_{0\le t_i\le t_{k-1}}(1 + c_i)c_k y(t_k) = c_k c(t_k)y(t_k) = c_k u(t_k).$$

Thus, $u(t_k^+) - u(t_k) = c_k u(t_k)$.

It is obvious that $u(t)$ satisfies the boundary conditions.

Then $u(t)$ is a solution of problem (1.1) on J.

(ii) It is easy to see that, for $t \in J$,

$$-c(t)y''(t) = \lambda\omega(t)f\bigl(c(t)y(t)\bigr).$$

For $t = t_k$,

$$\begin{aligned} y(t_k^+) - y(t_k^-) &= c^{-1}(t_k^+)u(t_k^+) - c^{-1}(t_k^-)u(t_k^-) \\ &= c^{-1}(t_k^+)\big(u(t_k) + c_k u(t_k)\big) - c^{-1}(t_k^-)u(t_k^-) \\ &= c^{-1}(t_k^-)u(t_k^-) - c^{-1}(t_k^-)u(t_k^-) \\ &= 0. \end{aligned}$$

Then $y(t)$ is continuous on J. It is easy to prove that $y(t)$ is absolutely continuous on J and satisfies the boundary conditions.

Then $y(t)$ is a solution of problem (3.2) on J. □

Lemma 3.2 *If* (H_1)-(H_3) *hold, then problem* (3.2) *has a solution* y, *and* y *can be expressed in the form*

$$y(t) = \lambda \int_0^1 H(t,s)\omega(s)c^{-1}(s)f\big(c(s)y(s)\big)\,ds, \tag{3.3}$$

where

$$H(t,s) = G(t,s) + \frac{1}{ac(1) - \mu}\int_0^1 G(\tau,s)g(\tau)c(\tau)\,d\tau + \frac{bc(1)}{ac(1) - \mu}, \tag{3.4}$$

$$G(t,s) = \begin{cases} 1 - t, & 0 \le s \le t \le 1, \\ 1 - s, & 0 \le t \le s \le 1. \end{cases} \tag{3.5}$$

Proof First, suppose that y is a solution of problem (3.2). Integrating problem (3.2) from 0 to t, by the boundary conditions we obtain that

$$y'(t) = -\int_0^t z(s)\,ds, \tag{3.6}$$

where $z(s) = \lambda\omega(s)c^{-1}(s)f(c(s)y(s))$.

Integrating (3.6) from 0 to t, we have

$$y(t) = y(0) - \int_0^t (t-s)z(s)\,ds. \tag{3.7}$$

Letting $t = 1$ in (3.6) and (3.7), we find

$$y(1) = y(0) - \int_0^1 (1-s)z(s)\,ds, \qquad y'(1) = -\int_0^1 z(s)\,ds.$$

Combining these equalities with (3.7) and the boundary conditions $ac(1)y(1) + bc(1)y'(1) = \int_0^1 g(t)c(t)y(t)\,dt$, we obtain

$$\begin{aligned} y(t) &= \int_0^1 (1-s)z(s)\,ds + \frac{1}{ac(1)}\int_0^1 g(s)c(s)y(s)\,ds - \int_0^t (t-s)z(s)\,ds + \frac{b}{a}\int_0^1 z(s)\,ds \\ &= \int_0^1 G(t,s)z(s)\,ds + \frac{1}{ac(1)}\int_0^1 g(s)c(s)y(s)\,ds + \frac{b}{a}\int_0^1 z(s)\,ds, \end{aligned} \tag{3.8}$$

and further

$$\begin{aligned}
&\int_0^1 g(s)c(s)y(s)\,ds \\
&\quad= \int_0^1 g(s)c(s)\left[\int_0^1 G(s,\tau)z(\tau)\,d\tau + \frac{1}{ac(1)}\int_0^1 g(\tau)c(\tau)y(\tau)\,d\tau + \frac{b}{a}\int_0^1 z(\tau)\,d\tau\right]ds \\
&\quad= \frac{1}{ac(1)}\int_0^1 g(s)c(s)\,ds\int_0^1 g(\tau)c(\tau)y(\tau)\,d\tau + \int_0^1 g(s)c(s)\left[\int_0^1 G(s,\tau)z(\tau)\,d\tau\right]ds \\
&\qquad+ \frac{b}{a}\int_0^1 g(s)c(s)\,ds\int_0^1 z(\tau)\,d\tau.
\end{aligned} \tag{3.9}$$

Therefore, we have

$$\begin{aligned}
\int_0^1 g(s)c(s)y(s)\,ds = \frac{ac(1)}{ac(1)-\mu}\biggl\{&\int_0^1 g(s)c(s)\left[\int_0^1 G(s,\tau)z(\tau)\,d\tau\right]ds \\
&+ \frac{b}{a}\int_0^1 g(s)c(s)\,ds\int_0^1 z(\tau)\,d\tau\biggr\}.
\end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.8), we obtain

$$\begin{aligned}
y(t) &= \int_0^1 G(t,s)z(s)\,ds + \frac{1}{ac(1)-\mu}\int_0^1 g(s)c(s)\left[\int_0^1 G(s,\tau)z(\tau)\,d\tau\right]ds \\
&\quad+ \frac{b}{a}\int_0^1 g(s)c(s)\,ds\int_0^1 z(\tau)\,d\tau + \frac{b}{a}\int_0^1 z(s)\,ds \\
&= \int_0^1 H(t,s)z(s)\,ds.
\end{aligned} \tag{3.11}$$

Then

$$y(t) = \lambda\int_0^1 H(t,s)\omega(s)c^{-1}(s)f\bigl(c(s)y(s)\bigr)\,ds, \tag{3.12}$$

and the proof of sufficiency is complete.

Conversely, from (3.3) it is easy to obtain

$$\begin{aligned}
&-y''(t) = \lambda\omega(t)c^{-1}(t)f\bigl(c(t)y(t)\bigr), \\
&y'(0) = 0, \qquad ac(1)y(1) + bc(1)y'(1) = \int_0^1 g(t)c(t)y(t)\,dt.
\end{aligned}$$

Lemma 3.2 is proved. □

Lemma 3.3 *Let* $\mu \in [0, ac(1))$, G, *and* H *be given as in Lemma* 3.2. *Then we have the following results*:

$$H(t,s) > 0, \qquad G(t,s) \geq 0, \quad \forall t,s \in J, \tag{3.13}$$

$$e(t)e(s) \leq G(t,s) \leq e(s), \quad \forall t,s \in J, \tag{3.14}$$

where $0 \leq e(t) = 1 - t \leq 1$, *and*

$$0 < \alpha^* \leq H(t,s) \leq \beta' h(s) \leq \beta^*, \quad \forall t,s \in J, \tag{3.15}$$

where

$$\alpha^* = \frac{bc(1)}{ac(1) - \mu}, \qquad \beta' = \frac{c(1)}{ac(1) - \mu}, \qquad h(s) = ae(s) + b, \qquad \beta^* = \frac{(a+b)c(1)}{ac(1) - \mu}. \tag{3.16}$$

Proof Relation (3.13) is simple to prove. For $0 \leq s \leq t \leq 1$, we have

$$e(t)e(s) \leq e(t) = G(t,s) = 1 - t \leq 1 - s = e(s).$$

For $0 \leq t \leq s \leq 1$, we have

$$e(t)e(s) \leq e(s) = G(t,s) = 1 - s.$$

Then,

$$e(t)e(s) \leq G(t,s) \leq e(s).$$

This gives the proof of (3.14).

For any $t,s \in J$, by (3.13), (3.14), and (3.16) we have

$$\begin{aligned} H(t,s) &= G(t,s) + \frac{1}{ac(1) - \mu} \int_0^1 G(\tau,s) g(\tau) c(\tau)\, d\tau + \frac{bc(1)}{ac(1) - \mu} \\ &\geq \frac{bc(1)}{ac(1) - \mu} \\ &= \alpha^*. \end{aligned}$$

On the other hand,

$$\begin{aligned} H(t,s) &= G(t,s) + \frac{1}{ac(1) - \mu} \int_0^1 G(\tau,s) g(\tau) c(\tau)\, d\tau + \frac{bc(1)}{ac(1) - \mu} \\ &\leq e(s) + \frac{1}{ac(1) - \mu} \int_0^1 e(s) g(\tau) c(\tau)\, d\tau + \frac{bc(1)}{ac(1) - \mu} \\ &= e(s)\left[1 + \frac{1}{ac(1) - \mu} \int_0^1 g(\tau) c(\tau)\, d\tau\right] + \frac{bc(1)}{ac(1) - \mu} \\ &= e(s)\frac{ac(1)}{ac(1) - \mu} + \frac{bc(1)}{ac(1) - \mu} \\ &= \beta' \big[ae(s) + b\big] \\ &\leq \beta' h(s) \leq \beta^*. \end{aligned}$$

Therefore, the proof of (3.15) is complete. □

To obtain some of the norm inequalities in our main results, we employ Hölder's inequality.

Lemma 3.4 (Hölder) *Let $e \in L^p[a,b]$ with $p > 1$, $h \in L^q[a,b]$ with $q > 1$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then $eh \in L^1[a,b]$ and*

$$\|eh\|_1 \le \|e\|_p \|h\|_q.$$

Let $e \in L^1[a,b]$, $h \in L^\infty[a,b]$. Then $eh \in L^1[a,b]$, and

$$\|eh\|_1 \le \|e\|_1 \|h\|_\infty.$$

Let $E = C[0,1]$. Then E is a real Banach space with the norm $\|\cdot\|$ defined by

$$\|x\| = \max_{t \in J} |x(t)|, \quad x \in E.$$

Define two cones K and K_1 in E by

$$K = \{y \in E : y(t) \ge 0, y(t) \ge \delta \|y\|, t \in J\} \tag{3.17}$$

and

$$K_1 = \{y \in E : y(t) \ge 0, t \in J\},$$

where $\delta = \frac{\alpha^*}{\beta^*} = \frac{b}{a+b}$. It is easy to see that K and K_1 are two solid normal cones and

$$K^0 = \{y \in E : y(t) > 0, y(t) \ge \delta \|y\|, t \in J\},$$
$$K_1^0 = \{y \in E : y(t) > 0, t \in J\}.$$

For $r > 0$, define Ω_r by

$$\Omega_r = \{y \in K : \|y\| < r\},$$
$$\partial\Omega_r = \{y \in K : \|y\| = r\}.$$

Define $T: K \to K$ by

$$(Ty)(t) = \int_0^1 H(t,s)\omega(s)c^{-1}(s)f(c(s)y(s))\,ds, \quad t \in J. \tag{3.18}$$

Lemma 3.5 *Assume that* (H_1)-(H_3) *hold. Then $T(K) \subset K$, and $T : K \to K$ is completely continuous.*

Proof For $y \in K$, it follows from (3.7) and (3.12) that

$$\begin{aligned}(Ty)(t) &= \int_0^1 H(t,s)\omega(s)c^{-1}(s)f(c(s)y(s))\,ds \\ &\ge \alpha^* \int_0^1 \omega(s)c^{-1}(s)f(c(s)y(s))\,ds\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\alpha^*}{\beta^*} \max_{t\in J} \int_0^1 H(t,s)\omega(s)c^{-1}(s)f\big(c(s)y(s)\big)\,ds \\
&= \delta\|Ty\|.
\end{aligned}
$$

Thus, $T(K) \subset K$.

Next, we prove that the operator $T : K \to K$ is completely continuous by standard methods and the Arzelà-Ascoli theorem.

Let $B_r = \{y \in E \mid \|y\| \leq r\}$ be a bounded set. Then, for all $y \in B_r$, we have

$$
\|Ty\| = \max_{t\in J} \int_0^1 H(t,s)\omega(s)c^{-1}(s)f\big(c(s)y(s)\big)\,ds \leq c_m^{-1}\beta'\|h\|_q\|\omega\|_p L,
$$

where $L = \max_{\|c(s)y(s)\|\leq c_M r} f(c(s)y(s))$. Therefore, $T(B_r)$ is uniformly bounded.

On the other hand, noticing that $H(t,s)$ is uniformly continuous on $J \times J$, we have that, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that if $|t_1 - t_2| < \delta_1$, then

$$
\big|H(t_1,s) - H(t_2,s)\big| < \frac{\varepsilon}{\|\omega\|_1 c_m^{-1} L}.
$$

Then, for any $y \in B_r$, taking $|t_1 - t_2| < \delta_1$, we get

$$
\begin{aligned}
\big|(Ty)(t_1) - (Ty)(t_2)\big| &= \bigg|\int_0^1 H(t_1,s)\omega(s)c^{-1}(s)f\big(c(s)y(s)\big)\,ds \\
&\quad - \int_0^1 H(t_2,s)\omega(s)c^{-1}(s)f\big(c(s)y(s)\big)\,ds\bigg| \\
&= \int_0^1 \big|H(t_1,s) - H(t_2,s)\big|\omega(s)c^{-1}(s)f\big(c(s)y(s)\big)\,ds \\
&\leq \frac{\varepsilon}{\|\omega\|_1 c_m^{-1} L} \int_0^1 \omega(s)c^{-1}(s)f\big(c(s)y(s)\big)\,ds \\
&\leq \|\omega\|_1 c_m^{-1} L \frac{\varepsilon}{\|\omega\|_1 c_m^{-1} L} \\
&= \varepsilon.
\end{aligned}
$$

Thus, the set $\{T : y \in B_r\}$ is equicontinuous. The Arzelà-Ascoli theorem implies that T is completely continuous, and Lemma 3.5 is proved. □

4 Existence and nonexistence of positive solutions on a parameter

In this section, we establish some sufficient conditions for the existence and nonexistence of positive solutions of problem (1.1). We consider the following three cases for $\omega \in L^p[0,1] : p > 1, p = 1$, and $p = \infty$. The case $p > 1$ is treated in the following theorem.

Theorem 4.1 *Assume that* (H_1)-(H_3) *hold. If* $0 < f_\infty < +\infty$, *then there exists* $R_0 > 0$ *such that for any* $r > R_0$, *problem* (1.1) *has a positive solution* $u_r(t)$ *satisfying* $\|u_r(t)\| = c_M r$ *for any*

$$
\lambda = \lambda_r \in [\lambda_1, \lambda_2], \tag{4.1}
$$

where λ_1 *and* λ_2 *are two positive finite numbers.*

Proof By (3.3) and (3.18) problem (1.1) has a positive solution $u_r(t)$ associated with $\lambda > 0$ if and only if the operator T has a proper element y_r associated with the eigenvalue $\frac{1}{\lambda} > 0$.

Since $0 < f_\infty < +\infty$, there exist $l_2 > l_1 > 0$ and $\eta > 0$ such that

$$l_1 y < f(y) < l_2 y, \quad \forall y \geq \eta. \tag{4.2}$$

Now, we prove that $R_0 = \frac{\eta}{c_m \delta}$ is required. Thus, for all $r > R_0$, if $y \in K \cap \partial\Omega_r$, we have

$$y(t) \geq \delta \|y\| = \delta r, \quad t \in J.$$

Noticing $r > R_0$, we have

$$c(t)y(t) \geq c_m \delta \|y\| = c_m \delta r > c_m \delta R_0 = \eta, \quad t \in J.$$

Together with Lemma 3.5, we have that $T : K \cap \bar{\Omega}_r \to K$ is completely continuous with $T\theta = \theta$. In addition,

$$\begin{aligned}(Ty)(t) &= \int_0^1 H(t,s)\omega(s)c^{-1}(s)f\bigl(c(s)y(s)\bigr)\,ds \\ &\geq \alpha^* \xi c_M^{-1} \int_0^1 l_1 c(s)y(s)\,ds \\ &\geq \alpha^* \xi c_M^{-1} l_1 c_m \delta \|y\| \\ &= \alpha^* \xi c_M^{-1} l_1 c_m \delta r > 0.\end{aligned}$$

Therefore, for any $r > R_0$ and $y \in K \cap \partial\Omega_r$, we have

$$\inf_{y \in K \cap \partial\Omega_r} \|Ty\| \geq \alpha^* \xi c_M^{-1} l_1 c_m \delta r > 0.$$

By Lemma 2.3, for any $r > R_0$, the operator T has a proper element $y_r \in K$ associated with the eigenvalue $\gamma > 0$; further, y_r satisfies $\|y_r\| = r$. Let $\lambda = \frac{1}{\gamma}$. Then problem (3.2) has a positive solution $y_r(t)$ associated with λ.

Hence, it follows from Lemma 3.1 that problem (1.1) has a positive solution $u_r(t)$ associated with λ and satisfying $\|u_r\| = c_M r$.

From the proof above, for any $r > R_0$, there exists a positive solution $y_r \in K \cap \partial\Omega_r$ associated with $\lambda > 0$, that is,

$$y_r(t) = \lambda \int_0^1 H(t,s)\omega(s)c^{-1}(s)f\bigl(c(s)y(s)\bigr)\,ds$$

with $\|y_r\| = r$.

On the one hand,

$$y_r(t) \leq \lambda c_m^{-1} \beta' \|h\|_q \|\omega\|_p l_2 c_M r,$$

and, further,

$$\|y_r\| = r \leq \lambda c_m^{-1} \beta' \|h\|_q \|\omega\|_p l_2 c_M r,$$

which means that

$$\lambda \geq \frac{1}{l_2 c_m^{-1} c_M \beta' \|h\|_q \|\omega\|_p} = \lambda_1.$$

On the other hand,

$$y_r(t) \geq \lambda \alpha^* \xi c_M^{-1} l_1 c_m \delta r, \tag{4.3}$$

and thus

$$\|y_r\| = r \geq \lambda \alpha^* \xi c_M^{-1} l_1 c_m \delta r,$$

which leads to

$$\lambda \leq \frac{1}{l_1 \delta \alpha^* \xi c_M^{-1} c_m} = \lambda_2.$$

It is easy to see by calculating that $\lambda_1 < \lambda_2$.

In conclusion, $\lambda \in [\lambda_1, \lambda_2]$. The proof is complete. □

The following Corollary 4.1 deals with the case $p = \infty$.

Corollary 4.1 *Assume that* (H_1)-(H_3) *hold. If* $0 < f_\infty < +\infty$, *then there exists* $R_1 > 0$ *such that for any* $r > R_1$, *problem* (1.1) *has a positive solution* $u_r(t)$ *satisfying* $\|u_r(t)\| = c_M r$ *for any*

$$\lambda \in \left[\lambda_1', \lambda_2\right],$$

where

$$\lambda_1' = \frac{1}{l_2 c_m^{-1} c_M \beta' \|h\|_1 \|\omega\|_\infty}.$$

Proof Replacing $\|h\|_q \|\omega\|_p$ by $\|h\|_1 \|\omega\|_\infty$ and repeating the argument above, we get the corollary. □

Finally, we consider the case of $p = 1$.

Corollary 4.2 *Assume that* (H_1)-(H_3) *hold. If* $0 < f_\infty < +\infty$, *then there exists* $R_2 > 0$ *such that for any* $r > R_2$, *problem* (1.1) *has a positive solution* $u_r(t)$ *satisfying* $\|u_r(t)\| = c_M r$ *for any*

$$\lambda \in \left[\lambda_1'', \lambda_2\right],$$

where

$$\lambda_1'' = \frac{1}{l_2 c_m^{-1} c_M \beta^* \|\omega\|_1}.$$

Proof Replacing $\beta' \|h\|_q \|\omega\|_p$ by $\beta^* \|\omega\|_1$ and repeating the argument above, we get the corollary. □

In the following theorems, we only consider the case $1 < p < +\infty$.

Theorem 4.2 *Assume that* (H_1)-(H_3) *hold. If* $f_\infty = +\infty$, *then there exists* $R_3 > 0$ *such that for any* $r > R_3$, *problem* (1.1) *has a positive solution* $u_r(t)$ *satisfying* $\|u_r\| = c_M r$ *for any*

$$\lambda = \lambda_r \in (0, \lambda_3],$$

where λ_3 *is a positive finite number.*

Proof Similarly to the proof of Theorem 4.1, it is easy to see from (4.2) and (4.3) that Theorem 4.2 is also true. □

Theorem 4.3 *Assume that* (H_1)-(H_3) *hold. If* $0 < f_0 < +\infty$, *then there exists* $r_0 > 0$ *such that for any* $0 < r < r_0$, *problem* (1.1) *has a positive solution* $u_r(t)$ *satisfying* $\|u_r(t)\| = c_M r$ *for any*

$$\lambda \in [\hat{\lambda}_1, \hat{\lambda}_2],$$

where $\hat{\lambda}_1$ *and* $\hat{\lambda}_2$ *are two positive finite numbers.*

Proof By (3.3) and (3.18) problem (1.1) has a positive solution $u_r(t)$ associated with $\lambda > 0$ if and only if the operator T has a proper element y_r associated with the eigenvalue $\frac{1}{\lambda} > 0$.

Since $0 < f_0 < +\infty$, there exist $\eta' > 0$ and constants $c_2 > c_1 > 0$ such that

$$c_1 u < f(u) < c_2 u, \quad \forall 0 < u < \eta'.$$

Set

$$U_r = \{y \in E : \|y\| < r\},$$

where $0 < r < r_0$.

Then U_r is a bounded open subset of the Banach space E, and $\theta \in U_r$.

Now, we prove that $r_0 = \frac{\eta'}{c_M}$ is required.

Thus, for $y \in K \cap \partial U_r$, noticing $0 < r < r_0$, we have

$$y(t) \geq \delta \|y\| = \delta r, \quad t \in J,$$

and

$$0 < c(t)y(t) \leq c_M \|y\| = c_M r < c_M r_0 = \eta', \quad t \in J.$$

Together with Lemma 3.5, we note that $T : K \cap \bar{U}_r \to K$ is completely continuous with $T\theta = \theta$ and that

$$\begin{aligned}(Ty)(t) &= \int_0^1 H(t,s)\omega(s)c^{-1}(s)f\big(c(s)y(s)\big)\,ds \\ &\geq \alpha^*\xi c_M^{-1}\int_0^1 c_1 c(s)y(s)\,ds \\ &\geq \alpha^*\xi c_M^{-1}c_1 c_m\delta\|y\| \\ &= \alpha^*\xi c_M^{-1}c_1 c_m\delta r > 0.\end{aligned}$$

So, for any $0 < r < r_0$ and $y \in K \cap \partial U_r$, we have

$$\inf_{y\in K\cap\partial U_r}\|Ty\| \geq \alpha^*\xi c_M^{-1}c_1 c_m\delta r > 0.$$

By Lemma 2.3, for any $0 < r < r_0$, the operator T has a proper element $y_r \in K$ associated with the eigenvalue $\gamma > 0$; further, y_r satisfies $\|y_r\| = r$. Letting $\lambda = \frac{1}{\gamma}$ and following the proof of Theorem 4.1, we complete the proof of Theorem 4.3. □

Theorem 4.4 *Assume that* (H_1)-(H_3) *hold. If* $f_0 = +\infty$, *then there exists* $r_1 > 0$ *such that for any* $0 < r < r_1$, *problem* (1.1) *has a positive solution* $u_r(t)$ *satisfying* $\|u_r\| = c_M r$ *for any*

$$\lambda = \lambda_r \in (0, \hat{\lambda}_3],$$

where $\hat{\lambda}_3$ *is a positive finite number.*

Proof The proof is similar to that of Theorem 4.3, so we omit it here. □

Theorem 4.5 *Assume that* (H_1)-(H_3) *hold. If* $f_0 = f_\infty = +\infty$, *then there exists* $\bar{\lambda} > 0$ *such that problem* (1.1) *has no positive solutions for all* $\lambda \in [\bar{\lambda}, +\infty)$.

Proof We argue by contradiction. Suppose that there exists a sequence $\{\lambda_n\}$ with $\lambda_n > n$ such that for each n, problem (3.2) has a positive solution $y_n \in K$. Let $\mu_n = \frac{1}{\lambda_n}$. Since $(Ty_n)(t) = \mu_n y_n(t)$ for $t \in J$ and $f(u) \geq Nu$ for all $u > 0$, where $N = \frac{f(R)}{R}$, we have

$$\begin{aligned}\|y_n\| &= \max_{t\in J}\big|\lambda_n Ty_n(t)\big| \\ &\geq \lambda_n\int_0^1 H(t,s)\omega(s)c^{-1}(s)f\big(c(s)y_n(s)\big)\,ds \\ &\geq \lambda_n\alpha^*\xi c_M^{-1}\int_0^1 Nc_m y_n(s)\,ds \\ &\geq \lambda_n\alpha^*\xi c_M^{-1}Nc_m\delta\|y_n\| \\ &> n\alpha^*\xi c_M^{-1}Nc_m\delta\|y_n\|,\end{aligned}$$

which implies that $1 > n\alpha^*\xi c_M^{-1}Nc_m\delta$.

Since n may be arbitrarily large, we obtain a contradiction.

Therefore, by Lemma 3.1 problem (1.1) has no positive solutions for all $\lambda \geq \bar{\lambda}$. This gives the proof of Theorem 4.5. □

Theorem 4.6 *Assume that* (H_1)-(H_3) *hold. If* $f_0 = f_\infty = 0$, *then there exists* $\underline{\lambda} > 0$ *such that problem* (1.1) *has no positive solutions for* $\lambda \in (0, \underline{\lambda})$.

Proof It follows from $f_0 = f_\infty = 0$ and (1.4) that there exists $\bar{\nu}_0 > 0$ such that

$$\frac{f(\bar{\nu}_0)}{\bar{\nu}_0} = \max_{\nu > 0} \frac{f(\nu)}{\nu}.$$

Let

$$\mathbf{M} = \frac{f(\bar{\nu}_0)}{\bar{\nu}_0} + 1.$$

Then $\mathbf{M} > 0$ and

$$f(\nu) \leq \mathbf{M}\nu, \quad \forall \nu > 0. \tag{4.4}$$

Let $y(t)$ be a positive solution of problem (3.2). We will show that this leads to a contradiction for $\lambda < \underline{\lambda}$, where $\underline{\lambda} = (\beta' \|h\|_q \|\omega\|_p c_m^{-1} c_M \mathbf{M})^{-1}$. Let $\mu = \frac{1}{\lambda}$. Since $(Ty)(t) = \mu y(t)$ for $t \in J$, it follows from (3.18) that

$$\begin{aligned} y(t) &= \lambda \int_0^1 H(t,s)\omega(s)c^{-1}(s) f\bigl(c(s)y(s)\bigr)\, ds \\ &\leq \lambda \beta' \|h\|_q \|\omega\|_p c_m^{-1} \int_0^1 \mathbf{M} c(s) y(s)\, ds \\ &\leq \lambda \beta' \|h\|_q \|\omega\|_p c_m^{-1} c_M \mathbf{M} \|y\|, \end{aligned}$$

which shows that

$$\begin{aligned} \|y\| &\leq \lambda \beta' \|h\|_q \|\omega\|_p c_m^{-1} c_M \mathbf{M} \|y\| \\ &< \underline{\lambda} \beta' \|h\|_q \|\omega\|_p c_m^{-1} c_M \mathbf{M} \|y\| \\ &= \|y\|, \end{aligned}$$

which is a contradiction. This finishes the proof. □

Remark 4.1 The method to study the existence and nonexistence results of positive solutions is completely different from those of Zhang and Feng [18].

5 Uniqueness and continuity of positive solution on a parameter

In the previous section, we have established some existence and nonexistence criteria of positive solutions for problem (1.1). Next, we consider the uniqueness and continuity of positive solutions on a parameter for problem (1.1).

Theorem 5.1 *Suppose that $f(u):[0,+\infty)\to[0,+\infty)$ is a nondecreasing function with $f(u)>0$ for $u>0$ and satisfies $f(\rho u)\geq\rho^{\alpha}f(u)$ for any $0<\rho<1$, where $0\leq\alpha<1$. Then, for any $\lambda\in(0,\infty)$, problem* (1.1) *has a unique positive solution $u_\lambda(t)$. Furthermore, such a solution $u_\lambda(t)$ satisfies the following properties*:

(i) *$u_\lambda(t)$ is strongly increasing in λ, that is, $\lambda_1>\lambda_2>0$ implies $u_{\lambda_1}(t)\gg u_{\lambda_2}(t)$ for $t\in J$.*
(ii) $\lim_{\lambda\to0^+}\|u_\lambda\|=0, \lim_{\lambda\to+\infty}\|u_\lambda\|=+\infty$.
(iii) *$u_\lambda(t)$ is continuous with respect to λ, that is, $\lambda\to\lambda_0>0$ implies $\|u_\lambda-u_{\lambda_0}\|\to0$.*

Proof Set $\Psi=\lambda T$, where T is the same as in (3.18). Similarly to Lemma 3.5, the operator Ψ maps K_1 into K_1. In view of $H(t,s)>0, \omega(s)>0, c^{-1}(s)>0$, and $f(u)>0$ for $u>0$, it is easy to see that $\Psi:K_1^0\to K_1^0$. We assert that $\Psi:K_1^0\to K_1^0$ is an α-concave increasing operator. Indeed,

$$\begin{aligned}\Psi(\rho y)&=\lambda\int_0^1H(t,s)\omega(s)c^{-1}(s)f\bigl(c(s)\rho y(s)\bigr)\,ds\\&\geq\rho^{\alpha}\lambda\int_0^1H(t,s)\omega(s)c^{-1}(s)f\bigl(c(s)y(s)\bigr)\,ds\\&=\rho^{\alpha}\Psi(y),\quad\forall0<\rho<1,\end{aligned}$$

where $0\leq\alpha<1$. Since $f(u)$ is nondecreasing, we have

$$\begin{aligned}(\Psi y_*)(t)&=\lambda\int_0^1H(t,s)\omega(s)c^{-1}(s)f\bigl(c(s)y_*(s)\bigr)\,ds\\&\leq\lambda\int_0^1H(t,s)\omega(s)c^{-1}(s)f\bigl(c(s)y_{**}(s)\bigr)\,ds\\&=(\Psi y_{**})(t)\quad\text{for }y_*\leq y_{**},y_*,y_{**}\in X.\end{aligned}$$

In view of Lemma 2.4, Ψ has a unique fixed point $y_\lambda\in K_1^0$. This shows that problem (3.2) has a unique positive solution $y_\lambda(t)$. It follows from Lemma 3.1 that problem (1.1) has a unique positive solution $u_\lambda(t)$.

Next, we give a proof for (i)-(iii). Let $\gamma=\frac{1}{\lambda}$ and denote $\lambda Ty_\lambda=y_\lambda$ by $Ty_\gamma=\gamma y_\gamma$. Assume that $0<\gamma_1<\gamma_2$. Then $y_{\gamma_1}\geq y_{\gamma_2}$. Indeed, set

$$\bar{\eta}=\sup\{\eta:y_{\gamma_1}\geq\eta y_{\gamma_2}\}.\tag{5.1}$$

We assert $\bar{\eta}\geq1$. If this is not true, then $0<\bar{\eta}<1$, and further

$$\gamma_1y_{\gamma_1}=Ty_{\gamma_1}\geq T(\bar{\eta}y_{\gamma_2})\geq\bar{\eta}^{\alpha}Ty_{\gamma_2}=\bar{\eta}^{\alpha}\gamma_2y_{\gamma_2},$$

which implies

$$y_{\gamma_1}\geq\bar{\eta}^{\alpha}\frac{\gamma_2}{\gamma_1}y_{\gamma_2}\gg\bar{\eta}^{\alpha}y_{\gamma_2}\gg\bar{\eta}y_{\gamma_2}.$$

This is a contradiction to (5.1).

In view of the discussion above, we have

$$y_{\gamma_1} = \frac{1}{\gamma_1} T y_{\gamma_1} \geq \frac{1}{\gamma_1} T y_{\gamma_2} = \frac{\gamma_2}{\gamma_1} y_{\gamma_2} \gg y_{\gamma_2}. \tag{5.2}$$

Hence, $y_\gamma(t)$ is strongly decreasing in γ. Namely, $y_\lambda(t)$ is strongly increasing in λ. By Lemma 3.1, (i) is proved.

Setting $\gamma_2 = \gamma$ and fixing γ_1 in (5.2), we have $y_{\gamma_1} \geq \frac{\gamma}{\gamma_1} y_\gamma$ for $\gamma > \gamma_1$. Further,

$$\|y_\gamma\| \leq \frac{\gamma_1 N_1}{\gamma} \|y_{\gamma_1}\|, \tag{5.3}$$

where $N_1 > 0$ is a normal constant. Noting that $\gamma = \frac{1}{\lambda}$, we have $\lim_{\lambda \to 0^+} \|y_\lambda(t)\| = 0$. Then it follows from Lemma 3.1 that $\lim_{\lambda \to 0^+} \|u_\lambda(t)\| = 0$.

Similarly, letting $\gamma_1 = \gamma$ and fixing γ_2, again by (5.2) and the normality of K_1 we have $\lim_{\lambda \to +\infty} \|y_\lambda(t)\| = +\infty$. Then, it follows from Lemma 3.1 that $\lim_{\lambda \to +\infty} \|u_\lambda(t)\| = +\infty$.

This gives the proof of (ii).

Next, we show the continuity of $u_\gamma(t)$. For given $\gamma_0 > 0$, by (i),

$$y_\gamma \ll y_{\gamma_0} \quad \text{for any } \gamma > \gamma_0. \tag{5.4}$$

Let $l_\gamma = \sup\{\nu > 0 \mid y_\gamma \geq \nu y_{\gamma_0}, \gamma > \gamma_0\}$. Obviously, $0 < l_\gamma < 1$ and $y_\gamma \geq l_\gamma y_{\gamma_0}$. So, we have

$$\gamma y_\gamma = T y_\gamma \geq T(l_\gamma y_{\gamma_0}) \geq l_\gamma^\alpha T y_{\gamma_0} = l_\gamma^\alpha \gamma_0 y_{\gamma_0},$$

and further

$$y_\gamma \geq \frac{\gamma_0}{\gamma} l_\gamma^\alpha y_{\gamma_0}.$$

By the definition of l_γ,

$$\frac{\gamma_0}{\gamma} l_\gamma^\alpha \leq l_\gamma \quad \text{or} \quad l_\gamma \geq \left(\frac{\gamma_0}{\gamma}\right)^{\frac{1}{1-\alpha}}.$$

Again by the definition of l_γ, we have

$$y_\gamma \geq \left(\frac{\gamma_0}{\gamma}\right)^{\frac{1}{1-\alpha}} y_{\gamma_0} \quad \text{for any } \gamma > \gamma_0. \tag{5.5}$$

Noticing that K_1 is a normal cone, in view of (5.4) and (5.5), we obtain

$$\|y_{\gamma_0} - y_\gamma\| \leq N_2 \left[1 - \left(\frac{\gamma_0}{\gamma}\right)^{\frac{1}{1-\alpha}}\right] \|y_{\gamma_0}\| \to 0, \quad \gamma \to \gamma_0 + 0.$$

In the same way,

$$\|y_\gamma - y_{\gamma_0}\| \to 0, \qquad \gamma \to \gamma_0 - 0,$$

where $N_2 > 0$ is a normal constant.

Therefore, by Lemma 3.1 we have

$$\|u_{\gamma_0}-u_\gamma\|\le c_M\|y_{\gamma_0}-y_\gamma\|\le c_M N_2\left[1-\left(\frac{\gamma_0}{\gamma}\right)^{\frac{1}{1-\alpha}}\right]\|y_{\gamma_0}\|\to 0,\quad \gamma\to\gamma_0+0.$$
$$\|u_\gamma-u_{\gamma_0}\|\le c_M\|y_\gamma-y_{\gamma_0}\|\to 0,\quad \gamma\to\gamma_0-0.$$

Consequently, (iii) holds. The proof is complete. □

6 Remarks and comments

In this section, we offer some remarks and comments on the associated problem (1.1).

Remark 6.1 Some ideas of the proof of Theorem 5.1 come from Theorem 2.2.7 in [17] and Theorem 6 in [19], but there are almost no papers considering the uniqueness of positive solution for second impulsive differential equations, especially in the case where $\omega(t)$ is L^p-integrable.

Remark 6.2 Generally, it is difficult to study the uniqueness of a positive solution for nonlinear second-order differential equations with or without impulsive effects (see, e.g., [4, 5, 20] and references therein).

For example, we consider the following problem:

$$\begin{cases} u''(t)+\lambda\omega(t)f(t,u(t))=0, & t\in J, t\neq t_k,\\ u(t_k^+)-u(t_k)=c_k u(t_k), & k=1,2,\ldots,n,\\ u(0)=u(1)=\int_0^1 h(s)u(t)\,dt, \end{cases}\tag{6.1}$$

where $\lambda>0$ is a positive parameter, $J=[0,1]$, $\omega\in L^p[0,1]$ for some $1\le p\le+\infty$, $f\in C(J\times R^+,R^+)$, $R^+=[0,+\infty)$, t_k ($k=1,2,\ldots,n$) are fixed points with $0<t_1<t_2<\cdots<t_k<\cdots<t_n<1$, $\{c_k\}$ is a real sequence with $c_k>-1$, $k=1,2,\ldots,n$, $x(t_k^+)$ ($k=1,2,\ldots,n$) is the right-hand limit of $x(t)$ at t_k, and $h\in C[0,1]$ is nonnegative.

By means of transformation (3.1) we can convert problem (6.1) into

$$\begin{cases} -y''(t)=\lambda c^{-1}(t)\omega(t)f(t,c(t)y(t)), & t\in J,\\ y(0)=c(1)y(1)=\int_0^1 h(s)c(s)y(s)\,ds. \end{cases}\tag{6.2}$$

Using a proof similar to that of Lemma 3.2, we can obtain the following results.

Lemma 6.1 *If* (H_1)-(H_3) *hold, then problem* (6.2) *has a solution* y, *and* y *can be expressed in the form*

$$y(t)=\lambda\int_0^1 H^*(t,s)\omega(s)c^{-1}(s)f\bigl(c(s)y(s)\bigr)\,ds,\tag{6.3}$$

where

$$H^*(t,s)=G^*(t,s)+\frac{1}{1-\nu}\int_0^1 G^*(s,\tau)h(\tau)\,d\tau,\tag{6.4}$$

$$G^*(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{6.5}$$

It is not difficult to prove that $H^*(t,s)$ and $G^*(t,s)$ have similar properties to those of $H(t,s)$ and $G(t,s)$. However, we cannot guarantee that $H^*(t,s) > 0$ for any $t,s \in J$. This implies that we cannot apply Lemma 2.4 *to study the uniqueness of a positive solution for problem* (6.1).

Remark 6.3 In Theorem 5.1, even though we do not assume that T is completely continuous or even continuous, we can assert that u_λ depends continuously on λ.

Remark 6.4 If we replace K_1, K_1^0 by K, K^0, respectively, then Theorem 5.1 also holds.

7 Examples

To illustrate how our main results can be used in practice, we present two examples.

Example 7.1 Let $n = 1, t_1 = \frac{1}{2}, p = 3$. It follows from $p = 3$ that $q = \frac{3}{2}$. Consider the following boundary value problem:

$$\begin{cases} u''(t) + \lambda(\frac{1}{|t-\frac{1}{3}|^{\frac{1}{4}}})(6u + \arctan u) = 0, & t \in J, t \neq t_k, \\ u(\frac{1}{2}^+) - u(\frac{1}{2}) = \frac{1}{2}u(\frac{1}{2}), \\ u'(0) = 0, \qquad u(1) + u'(1) = \int_0^1 u(t)\,dt, \end{cases} \tag{7.1}$$

Conclusion Problem (7.1) has at least one positive solution for any $\lambda \in [0.0056, 0.09]$.

Proof Problem (7.1) can be regarded as a problem of the form (1.1), where

$$\omega(t) = \frac{1}{|t - \frac{1}{3}|^{\frac{1}{4}}} \in L^3[0,1], \qquad f\bigl(u(t)\bigr) = 6u + \arctan u,$$

and

$$a = b = 1, \qquad g(t) = 1.$$

We convert problem (7.1) into

$$\begin{cases} -y''(t) = \lambda(\frac{1}{|t-\frac{1}{3}|^{\frac{1}{4}}})c^{-1}(t)(6c(t)y(t) + \arctan c(t)y(t)), & t \in J, \\ y'(0) = 0, \qquad \frac{3}{2}y(1) + \frac{3}{2}y'(1) = \int_0^1 c(s)y(s)\,ds, \end{cases} \tag{7.2}$$

where

$$c(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{2}, \\ \frac{3}{2}, & \frac{1}{2} < t \le 1. \end{cases}$$

From $\omega(t) = \frac{1}{|t-\frac{1}{3}|^{\frac{1}{4}}}, t \in J$, choosing $p = 3, q = \frac{3}{2}$, it follows that

$$\|\omega\|_p = \|\omega\|_3 = \left(\int_0^1 \left(\frac{1}{|t - \frac{1}{3}|^{\frac{1}{4}}}\right)^3 dt\right)^{\frac{1}{3}} = \frac{(4 + 4 \times 2^{\frac{1}{4}})^{\frac{1}{3}}}{3^{\frac{1}{12}}} \approx 1.879,$$

$$\|h\|_q = \|h\|_{\frac{3}{2}} = \left(\int_0^1 (2-t)^{\frac{3}{2}}\,dt\right)^{\frac{2}{3}} = \left(\frac{2}{5}\times 2^{\frac{5}{2}} - \frac{2}{5}\right)^{\frac{2}{3}} \approx 1.513.$$

Thus, it is easy to see by calculating that $\omega(t) \geq \xi = \sqrt[4]{\frac{3}{2}}$ for a.e. $t \in J$ and that

$$\mu = \int_0^1 g(t)c(t)\,dt = \frac{5}{4}, \qquad c_M = \frac{3}{2}, \qquad c_m = 1,$$

and

$$\alpha^* = \frac{bc(1)}{ac(1)-\mu} = 6, \qquad \beta' = \frac{c(1)}{ac(1)-\mu} = 6,$$
$$\beta^* = \frac{(a+b)c(1)}{ac(1)-\mu} = 12, \qquad \delta = \frac{\alpha^*}{\beta^*} = \frac{1}{2}.$$

Therefore, it follows from the definitions $\omega(t)$,f, and g that (H_1)-(H_3) hold and

$$f_\infty = \lim_{u\to+\infty}\frac{f(u)}{u} = \lim_{u\to+\infty}\left(6+\frac{\arctan u}{u}\right) = 6 + \lim_{u\to+\infty}\frac{\arctan u}{u} = 6,$$

so $0 < f_\infty = 6 < +\infty$.

Thus, we have

$$5 < f_\infty < 7.$$

Set $l_1 = 5$ and $l_2 = 7$. Then

$$\lambda_1 = \frac{1}{l_2 c_m^{-1} c_M \beta' \|h\|_{\frac{3}{2}} \|\omega\|_3} \approx 0.0056,$$
$$\lambda_2 = \frac{1}{l_1 \delta \alpha^* \xi c_M^{-1} c_m} \approx 0.09.$$

Hence, by Theorem 4.1 the conclusion follows, and the proof is complete. □

Example 7.2 Let $n = 1, t_1 = \frac{1}{2}, p = 1$. It follows from $p = 1$ that $q = \infty$. Consider the following boundary value problem:

$$\begin{cases} u''(t) + \lambda(2t+3)(6u+\arctan u) = 0, & t \in J, t \neq t_1, \\ u(\frac{1}{2}^+) - u(\frac{1}{2}) = \frac{1}{2}u(\frac{1}{2}), \\ u'(0) = 0, \qquad u(1) + u'(1) = \int_0^1 u(t)\,dt. \end{cases} \tag{7.3}$$

Conclusion Problem (7.3) has at least one positive solution for any $\lambda \in [\frac{1}{504}, \frac{1}{30}]$.

Proof Problem (7.3) can be regarded as a problem of the form (1.1), where

$$\omega(t) = 2t + 3 \in L^1[0,1], \qquad f(u) = 6u + \arctan u,$$

and

$$a = b = 1, \qquad g(t) = 1.$$

We convert problem (7.3) into

$$\begin{cases} -y''(t) = \lambda(2t+3)c^{-1}(t)(6c(t)y(t) + \arctan c(t)y(t)), & t \in J, \\ y'(0) = 0, \qquad \frac{3}{2}y(1) + \frac{3}{2}y'(1) = \int_0^1 c(s)y(s)\,ds, \end{cases} \tag{7.4}$$

where

$$c(t) = \begin{cases} 1, & 0 \le t \le \frac{1}{2}, \\ \frac{3}{2}, & \frac{1}{2} < t \le 1. \end{cases}$$

Thus, it is easy to see by calculating that $\omega(t) \ge \xi = 3$ for a.e. $t \in J$ and that

$$\mu = \int_0^1 g(t)c(t)\,dt = \frac{5}{4}, \qquad c_M = \frac{3}{2}, \qquad c_m = 1,$$

and

$$\alpha^* = \frac{bc(1)}{ac(1) - \mu} = 6, \qquad \beta^* = \frac{(a+b)c(1)}{ac(1) - \mu} = 12, \qquad \delta = \frac{\alpha^*}{\beta^*} = \frac{1}{2}.$$

Therefore, it follows from the definitions $\omega(t)$, f, and g that (H_1)-(H_3) hold.
On the other hand, it follows from $\omega(t) = 2t + 3$ that

$$\|\omega\|_1 = \int_0^1 (2t+3)\,dt = 4.$$

Thus, we have

$$5 < f_\infty < 7.$$

Set $l_1 = 5$ and $l_2 = 7$. Then

$$\lambda_1'' = \frac{1}{l_2 c_m^{-1} c_M \beta^* \|\omega\|_1} = \frac{1}{504},$$
$$\lambda_2 = \frac{1}{l_1 \delta \alpha^* \xi c_M^{-1} c_m} = \frac{1}{30}.$$

Hence, by Corollary 4.2 the conclusion follows, and the proof is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All results belong to YT and XZ. Both authors read and approved the final manuscript.

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