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New applications of Calvert and Gupta's results to hyperbolic differential equation with mixed boundaries

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Abstract

Calvert and Gupta's results concerning the perturbations on the ranges of *m*-accretive mappings have been employed widely in the discussion of the existence of solutions of nonlinear *elliptic* differential equation with *Neumann* boundary. In this paper, we shall focus our attention on certain *hyperbolic* differential equation with *mixed* boundaries. By defining some suitable nonlinear mappings, we shall demonstrate that Calvert and Gupta's results can be applied to hyperbolic equations, in addition to its wide usage in elliptic equations. Due to the differences between hyperbolic and elliptic equations, some new techniques have been developed in this paper, which can be regarded as the complement and extension of the previous work.

MSC: Primary 47H05; 47H09

Keywords: accretive mapping; duality mapping; subdifferential; hyperbolic equation; mixed boundaries

1 Introduction and preliminaries

1.1 Introduction

Nonlinear boundary value problems involving the generalized p-Laplacian operator arise from many physical phenomena, such as reaction-diffusion problems, petroleum extraction, flow through porous media, and non-Newtonian fluids, just to name a few. Thus, the study of such problems and their generalizations have attracted much attention in recent years. Many methods have been employed to tackle the existence of solutions of boundary value problems and one important method is to apply theories of the perturbations on ranges of nonlinear operators. Indeed, we recall that Calvert and Gupta [1] have used such a perturbation result (namely Theorem 1.1, which is stated in Section 1.2) to provide sufficient conditions so that some nonlinear boundary value problems with Neumann boundaries involving the Laplacian operator have solutions in $L^p(\Omega)$.

Inspired by Calvert and Gupta's perturbation result of Theorem 1.1, the p-Laplacian boundary value problems and their general forms have been extensively studied in the work of [2–7]. For example, the following problem that involves the generalized

p-Laplacian operator with Neumann boundaries has been discussed in [7]:

$$\begin{cases} -\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u]+\varepsilon|u|^{q-2}u+g(x,u(x))=f(x), & \text{a.e. in } \Omega, \\ -\langle\vartheta,(C(x)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle\in\beta_x(u(x)), & \text{a.e. on } \Gamma. \end{cases} \tag{1.1}$$

It is shown that (1.1) has solutions in $L^s(\Omega)$ under some conditions, where $\frac{2N}{N+1}<p\leq s<+\infty$, $1\leq q<+\infty$ if $p\geq N$, and $1\leq q\leq\frac{Np}{N-p}$ if $p<N$, for $N\geq 1$. The study of (1.1) in [7] can be regarded as the summary of the work done in [2–5].

A more general version of (1.1) has recently been tackled in [8]. Here, by using Calvert and Gupta's perturbation result (Theorem 1.1) again, Wei, Agarwal, and Wong [8] tackled the following elliptic p-Laplacian-like equation with Neumann boundary conditions, which is more general than (1.1) and includes (1.1):

$$\begin{cases} -\operatorname{div}[(C(x)+|\nabla u|^2)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u]+\varepsilon|u|^{q-2}u+g(x,u(x))=f(x), & \text{a.e. in } \Omega, \\ -\langle\vartheta,(C(x)+|\nabla u|^2)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u\rangle\in\beta_x(u), & \text{a.e. on } \Gamma. \end{cases} \tag{1.2}$$

It is shown that (1.2) has solutions in $L^p(\Omega)$ under some conditions, where $\frac{2N}{N+1}<p<+\infty$, $1\leq q<+\infty$ if $p\geq N$, and $1\leq q\leq\frac{Np}{N-p}$ if $p<N$, for $N\geq 1$.

On the other hand, a type of integro-differential equation with generalized p-Laplacian operator has also been studied in [9],

$$\begin{cases} \frac{\partial u}{\partial t}-\operatorname{div}[(C(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u] & \\ \quad+\varepsilon|u|^{q-2}u+a\frac{\partial}{\partial t}\int_\Omega u(x,t)\,dx=f(x,t), & (x,t)\in\Omega\times(0,T), \\ -\langle\vartheta,(C(x,t)+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\rangle\in\beta_x(u(x,t)), & (x,t)\in\Gamma\times(0,T), \\ u(x,0)=u(x,T), & x\in\Omega. \end{cases} \tag{1.3}$$

By using some results on the ranges of bounded pseudo-monotone operator and maximal monotone operator presented in [10–12], it is shown that (1.3) has solutions in $L^p(0,T;W^{1,p}(\Omega))$, for $1<q\leq p<+\infty$.

Motivated by the above research on elliptic and integro-differential equations, in this paper we focus our attention on *hyperbolic* differential equations. We shall explore the applicability of Calvert and Gupta's perturbation result (Theorem 1.1) to the existence of solution of certain hyperbolic differential equation. To be specific, we shall consider the following *hyperbolic* p-Laplacian-like problem with mixed boundaries:

$$\begin{cases} -\frac{\partial}{\partial t}(\alpha(\frac{\partial u}{\partial t}))-\operatorname{div}[(C(x,t)+|\nabla u|^2)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u] & \\ \quad+\varepsilon|u|^{q-2}u+g(x,u(x,t))=f(x,t), & (x,t)\in\Omega\times(0,T), \\ -\langle\vartheta,(C(x,t)+|\nabla u|^2)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u\rangle\in\beta_x(u(x,t)), & (x,t)\in\Gamma\times(0,T), \\ \alpha(\frac{\partial u}{\partial t}(x,0))=\alpha(\frac{\partial u}{\partial t}(x,T)), & x\in\Omega, \\ u(x,0)=u(x,T), & x\in\Omega. \end{cases} \tag{1.4}$$

In (1.4), α is the subdifferential of j, *i.e.*, $\alpha=\partial j$, where $j:\mathbb{R}\to\mathbb{R}$ is a proper, convex, and lower-semi continuous function, and β_x is the subdifferential of φ_x, *i.e.*, $\beta_x\equiv\partial\varphi_x$, where $\varphi_x=\varphi(x,\cdot):\mathbb{R}\to\mathbb{R}$ is a proper, convex, and lower-semicontinuous function. More details of (1.4) will be presented in Section 2. We shall investigate the existence of a solution of (1.4) in $L^2(0,T;L^2(\Omega))$. In the subsequent development, we shall demonstrate new applications of Calvert and Gupta's perturbation result.

Compared to most of the previous studies on hyperbolic differential equations, the main term $-\frac{\partial^2 u}{\partial t^2}$ in the previous work is replaced by $-\frac{\partial}{\partial t}(\alpha(\frac{\partial u}{\partial t}))$ in (1.4), which leads to the differences in the proofs of the main result. Moreover, the existence of solution of (1.4) will be discussed in $L^2(0,T;L^2(\Omega))$, which does not change while p is varying from $\frac{2N}{N+1}$ to $+\infty$ for $N \geq 1$.

1.2 Preliminaries

Let X be a real Banach space with a strictly convex dual space X^*. We shall use $(\cdot,\cdot)$ to denote the *generalized duality pairing* between X and X^*. We shall use '$\to$' and 'w-lim' to denote *strong* and *weak convergence*, respectively. Let '$X \hookrightarrow Y$' denote the space X embedded *continuously* in space Y. Let '$X \hookrightarrow\hookrightarrow Y$' denote the space X embedded *compactly* in space Y. For any subset G of X, we denote by $\operatorname{int} G$ its *interior* and $\overline{G}$ its *closure*, respectively. For two subsets G_1 and G_2 in X, if $\overline{G_1} = \overline{G_2}$ and $\operatorname{int} G_1 = \operatorname{int} G_2$, then we say G_1 is *almost equal* to G_2, which is denoted by $G_1 \simeq G_2$. A mapping $T : X \to X^*$ is said to be *hemi-continuous* on X [10, 11] if $w - \lim_{t\to 0} T(x+ty) = Tx$ for any $x, y \in X$.

A function Φ is called a *proper convex function* on X [10, 11] if Φ is defined from X to $(-\infty,+\infty]$, not identically $+\infty$, such that $\Phi((1-\lambda)x+\lambda y) \leq (1-\lambda)\Phi(x)+\lambda\Phi(y)$, whenever $x, y \in X$ and $0 \leq \lambda \leq 1$.

A function $\Phi : X \to (-\infty,+\infty]$ is said to be *lower-semicontinuous* on X [10, 11] if $\liminf_{y\to x} \Phi(y) \geq \Phi(x)$, for any $x \in X$.

Given a proper convex function Φ on X and a point $x \in X$, we denote by $\partial\Phi(x)$ the set of all $x^* \in X^*$ such that $\Phi(x) \leq \Phi(y) + (x-y, x^*)$, for every $y \in X$. Such an element x^* is called the *subgradient* of Φ at x, and $\partial\Phi(x)$ is called the *subdifferential* of Φ at x [10].

Let J_r denote the *duality mapping* from X into 2^{X^*}, which is defined by

$$J_r(x) = \{f \in X^* : (x,f) = \|x\|^r, \|f\| = \|x\|^{r-1}\}, \quad \forall x \in X,$$

where $r > 1$ is a constant. If $r = 2$, then J_2 is called *normalized duality mapping*, which is denoted by J in our paper. If X^* is strictly convex, then J is a single-valued mapping [1]. If, X is reduced to the Hilbert space, then J is the identity mapping.

A multi-valued mapping $A : X \to 2^X$ is said to be *accretive* [1] if $(v_1 - v_2, J_r(u_1 - u_2)) \geq 0$, for any $u_i \in D(A)$ and $v_i \in Au_i$, $i = 1, 2$. The accretive mapping A is said to be *m-accretive* if $R(I + \lambda A) = X$ for some $\lambda > 0$. We say that a mapping $A : X \to 2^X$ is *boundedly inversely compact* [1] if, for any pair of bounded subsets G and G' of X, the subset $G \cap A^{-1}(G')$ is relatively compact in X.

A multi-valued operator $B : X \to 2^{X^*}$ is said to be *monotone* [11] if its graph $G(B)$ is a monotone subset of $X \times X^*$ in the sense that $(u_1 - u_2, w_1 - w_2) \geq 0$, for any $[u_i, w_i] \in G(B)$, $i = 1, 2$. Further, B is called *strictly monotone* if $(u_1 - u_2, w_1 - w_2) \geq 0$ and the equality holds if and only if $u_1 = u_2$. The monotone operator B is said to be *maximal monotone* if $G(B)$ is maximal among all monotone subsets of $X \times X^*$ in the sense of inclusion. Also, B is maximal monotone if and only if $R(B + \lambda J) = X^*$, for any $\lambda > 0$. The mapping B is said to be *coercive* [11] if $\lim_{n\to+\infty} (x_n, x_n^*)/\|x_n\| = +\infty$ for all $[x_n, x_n^*] \in G(B)$ such that $\lim_{n\to+\infty} \|x_n\| = +\infty$.

Let $1 < p < +\infty$, then $L^p(0,T;X)$ denotes the space of all X-valued strongly measurable functions $x(t)$ defined a.e. on $(0,T)$ such that $\|x(t)\|_X^p$ is Lebesgue integrable over $(0,T)$. It is well known that $L^p(0,T;X)$ is a Banach space with the norm defined by

$$\|x\|_{L^p(0,T;X)} = \left(\int_0^T \|x(t)\|_X^p \,dt\right)^{\frac{1}{p}}.$$

If X is reflexive, then $L^p(0,T;X)$ is reflexive, and its dual space coincides with $L^{p'}(0,T;X^*)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, $L^p(0,T;X)$ is reflexive in the case when X is reflexive, and $L^p(0,T;X)$ is strictly (uniformly) convex in the case when X is strictly (uniformly) convex.

For $1 \leq r < p < +\infty$, if $X \hookrightarrow Y$, then $L^p(0,T;X) \hookrightarrow L^r(0,T;Y)$.

Lemma 1.1 ([11]) *If $A : X \to 2^{X^*}$ is a everywhere defined, monotone and hemi-continuous mapping, then A is maximal monotone. If, moreover, A is coercive, then $R(A) = X^*$.*

Lemma 1.2 ([11]) *If $\Phi : X \to (-\infty, +\infty]$ is a proper convex and lower-semicontinuous function, then $\partial\Phi$ is maximal monotone from X to X^*.*

Lemma 1.3 ([11]) *If A_1 and A_2 are two maximal monotone operators in X such that $(\operatorname{int} D(A_1)) \cap D(A_2) \neq \emptyset$, then $A_1 + A_2$ is maximal monotone.*

Definition 1.1 ([1]) The duality mapping $J_r : X \to X^*$ is said to satisfy *Condition* (I) if there exists a function $\eta : X \to [0, +\infty)$ such that for $u, v \in X$,

$$\|J_r u - J_r v\| \leq \eta(u - v). \tag{I}$$

Lemma 1.4 ([1]) *Let Ω be a bounded domain in $\mathbb{R}^N$ and let $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ denote the duality mapping, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then J_p satisfies Condition* (I). *Moreover, for $2 \leq p < +\infty$, $J_p u = |u|^{p-1} \operatorname{sgn} u \|u\|_p^{2-p}$, for any $u \in L^p(\Omega)$; for $1 < p \leq 2$, $J_p u = |u|^{p-1} \operatorname{sgn} u$, for any $u \in L^p(\Omega)$.*

Definition 1.2 ([1]) Let $A : X \to 2^X$ be an accretive mapping and $J_r : X \to X^*$ be a duality mapping. We say that A satisfies *Condition* $(*)$ if, for any $f \in R(A)$ and $a \in D(A)$, there exists a constant $C(a,f)$ such that, for any $u \in D(A)$, $v \in Au$,

$$\bigl(v - f, J_r(u - a)\bigr) \geq C(a,f). \tag{$*$}$$

Theorem 1.1 ([1]) *Let X be a real Banach space with a strictly convex dual X^*. Let $J_r : X \to X^*$ be the duality mapping on X satisfying Condition* (I). *Let A, $C_1 : X \to 2^X$ be accretive mappings such that*:

(i) *either both A and C_1 satisfy Condition $(*)$, or $D(A) \subset D(C_1)$ and C_1 satisfies Condition $(*)$,*

(ii) *$A + C_1$ is m-accretive and boundedly inversely compact.*

Let $C_2 : X \to X$ be a bounded continuous mapping such that, for any $y \in X$, there is a constant $C(y)$ satisfying $(C_2(u + y), J_r u) \geq -C(y)$ for any $u \in X$. Then:

(a) $\overline{[R(A) + R(C_1)]} \subset \overline{R(A + C_1 + C_2)}$;

(b) $\operatorname{int}[R(A) + R(C_1)] \subset \operatorname{int} R(A + C_1 + C_2)$.

Lemma 1.5 ([13]) *Let Ω be a bounded conical domain in $\mathbb{R}^N$. If $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow C_B(\Omega)$; if $0 < mp < N$ and $q = \frac{Np}{N-mp}$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$; if $mp = N$ and $p > 1$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $1 \le q < +\infty$.*

Lemma 1.6 ([13]) *Let Ω be a bounded conical domain in $\mathbb{R}^N$. If $mp > N$, then $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$; if $0 < mp \le N$ and $q_0 = \frac{Np}{N-mp}$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$, where $1 \le q < q_0$.*

2 Main results

In this paper, unless otherwise stated, we shall assume that $N \ge 1$, $\frac{2N}{N+1} < p < +\infty$, $1 \le q \le \min\{p, p'\}$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, $m + s + 1 = p$ and $m \ge 0$.

In (1.4), Ω is a bounded conical domain of a Euclidean space $\mathbb{R}^N$ with its boundary $\Gamma \in C^1$ [3], T is a positive constant, ε is a non-negative constant, and ϑ denotes the exterior normal derivative of Γ. $0 \le C(x,t) \in L^p(0,T;W^{1,p}(\Omega))$.

Suppose that $\alpha = \partial j$, where $j : \mathbb{R} \to \mathbb{R}$ is a proper, convex, and lower-semi continuous function, $0 \in \partial j(0)$, and $\beta_x \equiv \partial\varphi_x$, where $\varphi_x = \varphi(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is a proper, convex, and lower-semi continuous function. For each $x \in \Gamma$, $0 \in \beta_x(0)$, and for each $t \in \mathbb{R}$, the function $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in \mathbb{R}$ is measurable for $\lambda > 0$. Suppose $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function satisfying the Carathéodory conditions and $u(x,t) \in L^2(0,T;L^2(\Omega)) \to g(x,u(x,t)) \in L^2(0,T;L^2(\Omega))$ is well defined. Moreover, suppose there exists non-negative function $T(x) \in L^2(\Omega)$ satisfying $g(x,s)s \ge 0$, for all $|s| \ge T(x)$ and $x \in \Omega$. We shall assume that Green's formula is available.

Now, we present our discussion in the sequel.

Lemma 2.1 ([9]) *For $\frac{2N}{N+1} < p < +\infty$ and $N \ge 1$, we have, for $u(x,t) \in L^p(0,T;W^{1,p}(\Omega))$,*

$$\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \le k_1\left(\int_0^T \int_\Omega |\nabla u|^p \,dx\,dt\right)^{\frac{1}{p}} + k_2,$$

where k_1 and k_2 are positive constants.

Lemma 2.2 *For $p \ge 2$, define the mapping $B^{(1)} : L^p(0,T,W^{1,p}(\Omega)) \to L^{p'}(0,T,(W^{1,p}(\Omega))^*)$ by*

$$\left(v, B^{(1)}u\right) = \int_0^T \int_\Omega \left\langle \left(C(x,t) + |\nabla u|^2\right)^{\frac{s}{2}} |\nabla u|^{m-1}\nabla u, \nabla v\right\rangle dx + \varepsilon \int_\Omega \left|u(x)\right|^{q-2} u(x)v(x)\,dx\,dt$$

for any $u, v \in L^p(0,T,W^{1,p}(\Omega))$. Then $B^{(1)}$ is everywhere defined, monotone, hemicontinuous, and coercive.

(Here, $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the Euclidean inner product and Euclidean norm in $\mathbb{R}^N$, respectively.)

Proof Step 1. $B^{(1)}$ is everywhere defined.

Case 1. Suppose $s \ge 0$. For $u, w \in L^p(0,T;W^{1,p}(\Omega))$, we find

$$\begin{aligned}\left|\left(w, B^{(1)}u\right)\right| \le{}& 2^{\frac{s}{2}} \int_0^T \int_\Omega C(x,t)^{\frac{s}{2}} |\nabla u|^m |\nabla w| \,dx\,dt \\ &+ 2^{\frac{s}{2}} \int_0^T \int_\Omega |\nabla u|^{s+m} |\nabla w| \,dx\,dt + \varepsilon \int_0^T \int_\Omega |u|^{q-1}|w| \,dx\,dt\end{aligned}$$

$$
\begin{aligned}
&\le 2^{\frac{s}{2}}\left(\int_0^T\int_\Omega C(x,t)^{\frac{s}{2}p'}|\nabla u|^{mp'}\,dx\,dt\right)^{\frac{1}{p'}}\|w\|_{L^p(0,T;W^{1,p}(\Omega))} \\
&\quad + 2^{\frac{s}{2}}\|u\|^{p/p'}_{L^p(0,T;W^{1,p}(\Omega))}\|w\|_{L^p(0,T;W^{1,p}(\Omega))} + \varepsilon\|w\|_{L^q(0,T;L^q(\Omega))}\|u\|^{q/q'}_{L^q(0,T;L^q(\Omega))} \\
&\le 2^{\frac{s}{2}}\left\|C(x,t)\right\|^s_{L^p(0,T;L^p(\Omega))}\|u\|^m_{L^p(0,T;W^{1,p}(\Omega))}\|w\|_{L^p(0,T;W^{1,p}(\Omega))} \\
&\quad + 2^{\frac{s}{2}}\|u\|^{p/p'}_{L^p(0,T;W^{1,p}(\Omega))}\|w\|_{L^p(0,T;W^{1,p}(\Omega))} + \varepsilon\|w\|_{L^q(0,T;L^q(\Omega))}\|u\|^{q/q'}_{L^q(0,T;L^q(\Omega))}.
\end{aligned}
$$

From Lemma 1.5, we have $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow L^q(\Omega)$, then for $v \in W^{1,p}(\Omega)$, we have $\|v\|_{L^q(\Omega)} \le k_3\|v\|_{W^{1,p}(\Omega)}$, where k_3 is a positive constant. Hence,

$$
\begin{aligned}
\left|\left(w,B^{(1)}u\right)\right| &\le 2^{\frac{s}{2}}\left\|C(x,t)\right\|^s_{L^p(0,T;W^{1,p}(\Omega))}\|u\|^m_{L^p(0,T;W^{1,p}(\Omega))}\|w\|_{L^p(0,T;W^{1,p}(\Omega))} \\
&\quad + 2^{\frac{s}{2}}\|u\|^{p/p'}_{L^p(0,T;W^{1,p}(\Omega))}\|w\|_{L^p(0,T;W^{1,p}(\Omega))} \\
&\quad + \varepsilon k_4\|u\|^{q/q'}_{L^p(0,T;W^{1,p}(\Omega))}\|w\|_{L^p(0,T;W^{1,p}(\Omega))},
\end{aligned}
$$

where k_4 is a positive constant, which implies that $B^{(1)}$ is everywhere defined.

Case 2. Suppose $s < 0$. For $u, v \in L^p(0,T;W^{1,p}(\Omega))$, we have

$$
\begin{aligned}
\left|\left(w,B^{(1)}u\right)\right| &\le \int_0^T\int_\Omega \frac{|\nabla u|^m|\nabla w|}{(C(x,t)+|\nabla u|^2)^{-\frac{s}{2}}}\,dx\,dt + \varepsilon\|w\|_{L^q(0,T;L^q(\Omega))}\|u\|^{q/q'}_{L^q(0,T;L^q(\Omega))} \\
&\le \int_0^T\int_\Omega \frac{|\nabla u|^m|\nabla w|}{|\nabla u|^{-s}}\,dx\,dt + \varepsilon\|w\|_{L^q(0,T;L^q(\Omega))}\|u\|^{q/q'}_{L^q(0,T;L^q(\Omega))} \\
&\le \|u\|^{p/p'}_{L^p(0,T;W^{1,p}(\Omega))}\|w\|_{L^p(0,T;W^{1,p}(\Omega))} + \varepsilon k_4\|u\|^{q/q'}_{L^p(0,T;W^{1,p}(\Omega))}\|w\|_{L^p(0,T;W^{1,p}(\Omega))},
\end{aligned}
$$

which implies that $B^{(1)}$ is everywhere defined.

Step 2. $B^{(1)}$ is strictly monotone.

For $u, v \in L^p(0,T;W^{1,p}(\Omega))$, we have

$$
\begin{aligned}
&\left(u-v,B^{(1)}u-B^{(1)}v\right) \\
&\quad \ge \int_0^T\int_\Omega \Big[\left(C(x,t)+|\nabla u|^2\right)^{\frac{s}{2}}|\nabla u|^{m+1} - \left(C(x,t)+|\nabla v|^2\right)^{\frac{s}{2}}|\nabla v|^m|\nabla u| \\
&\qquad - \left(C(x,t)+|\nabla u|^2\right)^{\frac{s}{2}}|\nabla u|^m|\nabla v| + \left(C(x,t)+|\nabla v|^2\right)^{\frac{s}{2}}|\nabla v|^{m+1}\Big]\,dx\,dt \\
&\qquad + \varepsilon\int_0^T\int_\Omega \left(|u|^{q-1}-|v|^{q-1}\right)\left(|u|-|v|\right)dx\,dt \\
&\quad = \int_0^T\int_\Omega \Big[\left(C(x,t)+|\nabla u|^2\right)^{\frac{s}{2}}|\nabla u|^m - \left(C(x,t)+|\nabla v|^2\right)^{\frac{s}{2}}|\nabla v|^m\Big]\left(|\nabla u|-|\nabla v|\right)dx\,dt \\
&\qquad + \varepsilon\int_0^T\int_\Omega \left(|u|^{q-1}-|v|^{q-1}\right)\left(|u|-|v|\right)dx\,dt.
\end{aligned}
$$

Let $h(x) = (C+x^2)^{\frac{s}{2}}x^m$, for $x \ge 0$. Then $h'(x) = x^{m-1}(C+x^2)^{\frac{s}{2}-1}[mC+(p-1)x^2] \ge 0$, for $x \ge 0$, which implies that $B^{(1)}$ is strictly monotone.

Step 3. $B^{(1)}$ is hemi-continuous.

It suffices to show that for any $u,v,w \in L^p(0,T;W^{1,p}(\Omega))$ and $t \in [0,1]$, $(w, B^{(1)}(u+tv) - B^{(1)}u) \to 0$ as $t \to 0$. Since

$$\begin{aligned}
0 &\le \lim_{t\to 0}\bigl|\bigl(w, B^{(1)}(u+tv) - B^{(1)}u\bigr)\bigr| \\
&\le \biggl|\int_0^T \int_\Omega \lim_{t\to 0}\bigl\langle \bigl(C(x,t) + |\nabla u + t\nabla v|^2\bigr)^{\frac{s}{2}} |\nabla u + t\nabla v|^{m-1}(\nabla u + t\nabla v) \\
&\quad - \bigl(C(x,t) + |\nabla u|^2\bigr)^{\frac{s}{2}} |\nabla u|^{m-1}\nabla u, \nabla w\bigr\rangle \,dx\,dt\biggr| \\
&\quad + \varepsilon \int_0^T \int_\Omega \lim_{t\to 0}\bigl||u+tv|^{q-2}(u+tv) - |u|^{q-2}u\bigr||w|\,dx\,dt,
\end{aligned}$$

by Lebesque's dominated convergence theorem, we find

$$\lim_{t\to 0}\bigl(w, B^{(1)}(u+tv) - B^{(1)}u\bigr) = 0.$$

Hence, $B^{(1)}$ is hemi-continuous.

Step 4. $B^{(1)}$ is coercive.

Case 1. Suppose $s \ge 0$. For $u \in L^p(0,T;W^{1,p}(\Omega))$, let $\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \to +\infty$. Using Lemma 2.1, we find

$$\begin{aligned}
\frac{(u, B^{(1)}u)}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} &= \frac{\int_0^T \int_\Omega (C(x,t) + |\nabla u|^2)^{\frac{s}{2}} |\nabla u|^{m+1}\,dx\,dt}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} + \varepsilon \frac{\int_0^T \int_\Omega |u|^q\,dx\,dt}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \\
&\ge \frac{1}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \biggl(\int_0^T \int_\Omega |\nabla u|^p\,dx\,dt + \varepsilon \int_0^T \int_\Omega |u|^q\,dx\,dt\biggr) \\
&> \frac{1}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \int_0^T \int_\Omega |\nabla u|^p\,dx\,dt \to +\infty,
\end{aligned}$$

which implies that $B^{(1)}$ is coercive.

Case 2. Suppose $s < 0$. For $u \in L^p(0,T;W^{1,p}(\Omega))$, let $\|u\|_{L^p(0,T;W^{1,p}(\Omega))} \to +\infty$. Using Lemma 2.1 again, we find

$$\begin{aligned}
&\frac{(u, B^{(1)}u)}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \\
&\quad = \frac{\int_0^T \int_\Omega (C(x,t) + |\nabla u|^2)^{\frac{s}{2}} |\nabla u|^{p-s}\,dx\,dt}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} + \varepsilon \frac{\int_0^T \int_\Omega |u|^q\,dx\,dt}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \\
&\quad \ge \frac{1}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \\
&\qquad \times \biggl(\int_0^T \int_\Omega 2^{\frac{s}{2}}\bigl[\max\bigl\{C(x,t), |\nabla u|^2\bigr\}\bigr]^{\frac{s}{2}} |\nabla u|^{p-s}\,dx\,dt + \varepsilon \int_0^T \int_\Omega |u|^q\,dx\,dt\biggr) \\
&\quad > \frac{1}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \int_0^T \int_\Omega 2^{\frac{s}{2}} |\nabla u|^p\,dx\,dt \to +\infty,
\end{aligned}$$

which implies that $B^{(1)}$ is coercive.

This completes the proof. □

Lemma 2.3 *For* $\frac{2N}{N+1} < p \leq 2$, *define the mapping* $B^{(2)} : L^{p'}(0,T;W^{1,p}(\Omega)) \to L^p(0,T;(W^{1,p}(\Omega))^*)$ *by*

$$\begin{aligned}(v, B^{(2)}u) &= \int_0^T \int_\Omega \langle (C(x,t) + |\nabla u|^2)^{\frac{s}{2}} |\nabla u|^{m-1} \nabla u, \nabla v \rangle \, dx \\ &\quad + \varepsilon \int_\Omega |u(x)|^{q-2} u(x) v(x) \, dx \, dt\end{aligned}$$

for any $u, v \in L^{p'}(0,T,W^{1,p}(\Omega))$. *Then* $B^{(2)}$ *is everywhere defined, monotone, hemicontinuous, and coercive.*

Proof The proof is similar to that of Lemma 2.2. □

Definition 2.1 ([9]) For $p \geq 2$, $\Phi^{(1)} : L^p(0,T;W^{1,p}(\Omega)) \to \mathbb{R}$ defined by

$$\Phi^{(1)}(u) = \int_0^T \int_\Gamma \varphi_x(u|_\Gamma(x,t)) \, d\Gamma(x) \, dt,$$

for any $u \in L^p(0,T;W^{1,p}(\Omega))$, is proper, convex, and lower-semi continuous on $L^p(0,T;W^{1,p}(\Omega))$.

Further, Lemma 1.2 implies that $\partial\Phi^{(1)} : L^p(0,T;W^{1,p}(\Omega)) \to L^{p'}(0,T;(W^{1,p}(\Omega))^*)$ is maximal monotone.

Lemma 2.4 ([9]) *For* $u, v \in L^p(0,T;W^{1,p}(\Omega))$, *where* $p \geq 2$,

$$(v, \partial\Phi^{(1)}(u)) = \int_0^T \int_\Gamma \beta_x(u|_\Gamma(x,t)) v|_\Gamma(x,t) \, d\Gamma(x) \, dt,$$

and $0 \in \partial\Phi^{(1)}(0)$, *also* $(\varphi, \partial\Phi^{(1)}(u)) = 0$, *for any* $\varphi \in C_0^\infty(\Omega \times (0,T))$ *and* $u \in L^p(0,T;W^{1,p}(\Omega))$. *Moreover, if* $w(x,t) \in \partial\Phi^{(1)}(u(x,t))$, *then* $w(x,t) = \beta_x(u(x,t))$, *a.e. on* $\Gamma \times (0,T)$.

Definition 2.2 ([9]) For $\frac{2N}{N+1} < p \leq 2$, $\Phi^{(2)} : L^{p'}(0,T;W^{1,p}(\Omega)) \to \mathbb{R}$ defined by

$$\Phi^{(2)}(u) = \int_0^T \int_\Gamma \varphi_x(u|_\Gamma(x,t)) \, d\Gamma(x) \, dt,$$

for any $u \in L^{p'}(0,T;W^{1,p}(\Omega))$, is proper, convex, and lower-semi continuous on $L^{p'}(0,T;W^{1,p}(\Omega))$.

Further, Lemma 1.2 implies that $\partial\Phi^{(2)} : L^{p'}(0,T;W^{1,p}(\Omega)) \to L^p(0,T;(W^{1,p}(\Omega))^*)$ is maximal monotone.

Lemma 2.5 ([9]) *For* $u, v \in L^{p'}(0,T;W^{1,p}(\Omega))$, *where* $\frac{2N}{N+1} < p \leq 2$,

$$(v, \partial\Phi^{(2)}(u)) = \int_0^T \int_\Gamma \beta_x(u|_\Gamma(x,t)) v|_\Gamma(x,t) \, d\Gamma(x) \, dt$$

and $0 \in \partial\Phi^{(2)}(0)$, *also* $(\varphi, \partial\Phi^{(2)}(u)) = 0$, *for any* $\varphi \in C_0^\infty(\Omega \times (0,T))$ *and* $u \in L^p(0,T;W^{1,p}(\Omega))$. *Moreover, if* $w(x,t) \in \partial\Phi^{(2)}(u(x,t))$, *then* $w(x,t) = \beta_x(u(x,t))$, *a.e. on* $\Gamma \times (0,T)$.

Definition 2.3 For $p \geq 2$, define a mapping $A^{(1)} : L^2(0,T;L^2(\Omega)) \to 2^{L^2(0,T;L^2(\Omega))}$ by

$$A^{(1)}u = \{w(x) \in L^2(0,T;L^2(\Omega)) \mid w(x) \in B^{(1)}u + \partial\Phi^{(1)}(u)\}$$

for $u \in D(A^{(1)}) = \{u \in L^2(0,T;L^2(\Omega)) \mid$ there exists a $w(x) \in L^2(0,T;L^2(\Omega))$ such that $w(x) \in B^{(1)}u + \partial\Phi^{(1)}(u)\}$.

Lemma 2.6 *The mapping* $A^{(1)} : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ *defined in Definition* 2.3 *is m-accretive.*

Proof It is easy to check that $A^{(1)}$ is accretive in view of Lemmas 2.2 and 2.4.

Next, we shall show that $R(I+A^{(1)}) = L^2(0,T;L^2(\Omega))$, which ensures that A is m-accretive. Since $p \geq 2$, we define $F^{(1)} : L^p(0,T;W^{1,p}(\Omega)) \to L^{p'}(0,T;(W^{1,p}(\Omega))^*)$ by

$$F^{(1)}u = u,$$
$$(v, F^{(1)}u)_{L^p(0,T;W^{1,p}(\Omega))\times L^{p'}(0,T;(W^{1,p}(\Omega))^*)} = (v,u)_{L^2(0,T;L^2(\Omega))},$$

where $(\cdot,\cdot)_{L^2(0,T;L^2(\Omega))}$ denotes the inner product of $L^2(0,T;L^2(\Omega))$.Then $F^{(1)}$ is everywhere defined, monotone, and hemi-continuous, which implies that $F^{(1)}$ is maximal monotone in view of Lemma 1.1. Combining with the facts of Lemmas 1.3, 2.2, and 1.1, we have $R(B^{(1)} + F^{(1)} + \partial\Phi^{(1)}) = L^{p'}(0,T;(W^{1,p}(\Omega))^*)$. For $f \in L^2(0,T;L^2(\Omega)) \subset L^{p'}(0,T;(W^{1,p}(\Omega))^*)$, there exists $u \in L^p(0,T;W^{1,p}(\Omega)) \subset L^2(0,T;L^2(\Omega))$ such that

$$f = B^{(1)}u + F^{(1)}u + \partial\Phi^{(1)}(u) = A^{(1)}u + u,$$

which implies that $R(I + A^{(1)}) = L^2(0,T;L^2(\Omega))$.

This completes the proof. □

Definition 2.4 For $\frac{2N}{N+1} < p \leq 2$, define a mapping $A^{(2)} : L^2(0,T;L^2(\Omega)) \to 2^{L^2(0,T;L^2(\Omega))}$ by

$$A^{(2)}u = \{w(x) \in L^2(0,T;L^2(\Omega)) \mid w(x) \in B^{(2)}u + \partial\Phi^{(2)}(u)\}$$

for $u \in D(A^{(2)}) = \{u \in L^2(0,T;L^2(\Omega)) \mid$ there exists a $w(x) \in L^2(0,T;L^2(\Omega))$ such that $w(x) \in B^{(2)}u + \partial\Phi^{(2)}(u)\}$.

Lemma 2.7 *The mapping* $A^{(2)} : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ *defined in Definition* 2.4 *is m-accretive.*

Proof Since $\frac{2N}{N+1} < p \leq 2$, we have $p' \geq 2$. Similar to the proof of Lemma 2.6, define $F^{(2)} : L^{p'}(0,T;W^{1,p}(\Omega)) \to L^p(0,T;(W^{1,p}(\Omega))^*)$ by

$$F^{(2)}u = u,$$
$$(v, F^{(2)}u)_{L^{p'}(0,T;W^{1,p}(\Omega))\times L^p(0,T;(W^{1,p}(\Omega))^*)} = (v,u)_{L^2(0,T;L^2(\Omega))}.$$

Then $R(B^{(2)} + F^{(2)} + \partial\Phi^{(2)}) = L^p(0,T;(W^{1,p}(\Omega))^*)$. So, for $f \in L^2(0,T;L^2(\Omega)) \subset L^p(0,T;(W^{1,p}(\Omega))^*)$, there exists $u \in L^{p'}(0,T;W^{1,p}(\Omega)) \subset L^2(0,T;L^2(\Omega))$ such that

$$f = B^{(2)}u + F^{(2)}u + \partial\Phi^{(2)}u = A^{(2)}u + u,$$

which implies that $R(I + A^{(2)}) = L^2(0,T;L^2(\Omega))$.

This completes the proof. □

Lemma 2.8 *Define* $S : L^2(0,T;L^2(\Omega)) \to (-\infty,+\infty]$ *by*

$$Su(x,t) = \int_0^T \int_\Omega j\left(\frac{\partial u}{\partial t}\right)dx\,dt, \quad u(x,t) \in D(S),$$

where $D(S) = \{u(x,t) \in H^2(0,T;L^2(\Omega)) : j(\frac{\partial u}{\partial t}) \in L^1(0,T;\Omega), u(x,0) = u(x,T), \alpha(\frac{\partial u}{\partial t}(x,0)) = \alpha(\frac{\partial u}{\partial t}(x,T))\} \subset L^2(0,T;L^2(\Omega))$. *Then the mapping* S *is proper, convex, and lower-semi continuous.*

Proof Since j is proper and convex, it is not difficult to show that S is proper and convex. It remains to show that S is lower-semi continuous on $H^2(0,T;L^2(\Omega))$.

For this purpose, let $\{u_n\}$ be such that $u_n \to u$ in $H^2(0,T;L^2(\Omega))$ as $n \to \infty$. Then there exists s subsequence of $\{u_n\}$, which, for simplicity we still denote by $\{u_n\}$, such that $\frac{\partial u_n(x,t)}{\partial t} \to \frac{\partial u(x,t)}{\partial t}$ for a.e. $(x,t) \in \Omega \times (0,T)$. Since j is lower-semi continuous, we have

$$j\left(\frac{\partial u(x,t)}{\partial t}\right) \le \liminf_{n\to\infty} j\left(\frac{\partial u_n(x,t)}{\partial t}\right),$$

a.e. on $\Omega \times (0,T)$. Using Fatou's lemma, it follows that

$$\int_0^T \int_\Omega j\left(\frac{\partial u(x,t)}{\partial t}\right)dx\,dt \le \liminf_{n\to\infty} \int_0^T \int_\Omega j\left(\frac{\partial u_n(x,t)}{\partial t}\right)dx\,dt.$$

Therefore, $Su \le \liminf_{n\to\infty} S(u_n)$, whenever $u_n \to u$ in $H^2(0,T;L^2(\Omega))$. The proof is complete. □

Lemma 2.9 *Let* S *be the same as that in Lemma* 2.8. *If* $w(x,t) \in \partial S(u(x,t))$, *then* $w(x,t) = -\frac{\partial}{\partial t}(\alpha(\frac{\partial u}{\partial t}))$ *a.e. in* $\Omega \times (0,T)$.

Proof Let $w(x,t) = \frac{\partial \overline{w}(x,t)}{\partial t}$. In view of the definition of subdifferential, we have for $w(x,t) \in \partial S(u(x,t))$,

$$\begin{aligned}\int_0^T \int_\Omega \left[j\left(\frac{\partial u}{\partial t}\right) - j\left(\frac{\partial v}{\partial t}\right)\right]dx\,dt &\le \int_0^T \int_\Omega w(x,t)\big[u(x,t) - v(x,t)\big]\,dx\,dt \\ &= \int_0^T \int_\Omega \frac{\partial \overline{w}(x,t)}{\partial t}\big[u(x,t) - v(x,t)\big]\,dx\,dt \\ &= -\int_0^T \int_\Omega \overline{w}(x,t)\left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}\right)dx\,dt. \end{aligned} \tag{2.1}$$

Let E be any measurable subset of Ω such that for $t \in (0, T)$,

$$\widetilde{w}(x,t) = \begin{cases} v(x,t), & x \in E, \\ u(x,t), & x \in E^C, \end{cases}$$

where E^C is the complement of E in Ω. Taking $v(x,t) = \widetilde{w}(x,t)$ in (2.1), we have

$$\int_0^T \int_E \left[j\left(\frac{\partial u}{\partial t}\right) - j\left(\frac{\partial v}{\partial t}\right) + \overline{w}(x,t)\left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}\right)\right] dx\,dt \leq 0.$$

For any measurable subset E of Ω, we have

$$j\left(\frac{\partial u}{\partial t}\right) - j\left(\frac{\partial v}{\partial t}\right) \leq -\overline{w}(x,t)\left(\frac{\partial u}{\partial t} - \frac{\partial v}{\partial t}\right), \quad \text{a.e. } (x,t) \in \Omega \times (0,T).$$

Thus, $\overline{w}(x,t) = -\partial j(\frac{\partial u}{\partial t}) = -\alpha(\frac{\partial u}{\partial t})$, a.e. $(x,t) \in \Omega \times (0,T)$. Then $w(x,t) = -\frac{\partial}{\partial t}(\alpha(\frac{\partial u}{\partial t}))$, a.e. $(x,t) \in \Omega \times (0,T)$. This completes the proof. □

Remark 2.1 Lemmas 2.8 and 2.9 will play an important role in the discussion of the term $-\frac{\partial}{\partial t}(\alpha(\frac{\partial u}{\partial t}))$ in (1.4).

Definition 2.5 Define a mapping $C^{(1)} : L^2(0,T;L^2(\Omega)) \to 2^{L^2(0,T;L^2(\Omega))}$ by

$$C^{(1)}u(x,t) = A^{(1)}u(x,t) + \partial S\big(u(x,t)\big),$$

for $u(x,t) \in D(S)$. Define a mapping $C^{(2)} : L^2(0,T;L^2(\Omega)) \to 2^{L^2(0,T;L^2(\Omega))}$ by

$$C^{(2)}u(x,t) = A^{(2)}u(x,t) + \partial S\big(u(x,t)\big),$$

for $u(x,t) \in D(S)$.

Proposition 2.1

(i) *The mapping* $C^{(1)} : L^2(0,T;L^2(\Omega)) \to 2^{L^2(0,T;L^2(\Omega))}$ *is m-accretive and has a compact resolvent.*

(ii) *The mapping* $C^{(2)} : L^2(0,T;L^2(\Omega)) \to 2^{L^2(0,T;L^2(\Omega))}$ *is also m-accretive and has a compact resolvent.*

Proof (i) Since $L^2(0,T;L^2(\Omega))$ is a Hilbert space, m-accretive mappings and maximal monotone operators are the same. Thus, Lemmas 1.2, 1.3, 2.6, and 2.8 imply that $C^{(1)}$ is m-accretive.

To show that $C^{(1)}$ has a compact resolvent, it suffices to show that if $u(x,t) + \lambda C^{(1)}u(x,t) = f(x,t)$ ($\lambda > 0$) with $\{f(x,t)\}$ being bounded in $L^2(0,T;L^2(\Omega))$, then $\{u(x,t)\}$ is relatively compact in $L^2(0,T;L^2(\Omega))$.

To proceed, define $H^{(1)} : L^p(0,T;W^{1,p}(\Omega)) \to L^{p'}(0,T;(W^{1,p}(\Omega))^*)$ by

$$H^{(1)}u(x,t) = F^{(1)}u(x,t) + \lambda B^{(1)}u(x,t) + \lambda\partial\Phi^{(1)}u(x,t),$$

where $F^{(1)}$ is the same as that in the proof of Lemma 2.6. Then $H^{(1)}$ is maximal monotone and coercive in view of Lemmas 1.1-1.3, 2.2, 2.6 and Definition 2.1. Since $L^p(0,T;W^{1,p}(\Omega)) \hookrightarrow\hookrightarrow L^2(0,T;L^2(\Omega))$, we have

$$\begin{aligned}
\frac{(u,H^{(1)}u)}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} &= \frac{(u,F^{(1)}u(x,t)) + \lambda(u,B^{(1)}u(x,t)) + \lambda(u,\partial\Phi^{(1)}u(x,t))}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \\
&= \frac{(u,u) + \lambda(u,C^{(1)}u(x,t)) - \lambda(u,\partial S(u(x,t)))}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \\
&= \frac{(u,f(x,t)) - \lambda(u,\partial S(u(x,t)))}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \\
&\le \frac{(u,f(x,t))}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \le \frac{\|u\|_{L^2(0,T;L^2(\Omega))}\|f(x,t)\|_{L^2(0,T;L^2(\Omega))}}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \\
&\le \frac{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}\|f(x,t)\|_{L^2(0,T;L^2(\Omega))}}{\|u\|_{L^p(0,T;W^{1,p}(\Omega))}} \\
&= \|f(x,t)\|_{L^2(0,T;L^2(\Omega))},
\end{aligned}$$

which implies that $\{u(x,t)\}$ is bounded in $L^p(0,T;W^{1,p}(\Omega))$ and so it is relatively compact in $L^2(0,T;L^2(\Omega))$.

The proof of (ii) is similar. This completes the proof. □

Proposition 2.2 *If $f(x,t) \in L^2(0,T;L^2(\Omega))$ satisfies $f(x,t) = C^{(1)}u(x,t)$ or $f(x,t) = C^{(2)}u(x,t)$, then the following hold*:

(a) $-\frac{\partial}{\partial t}(\alpha(\frac{\partial u}{\partial t})) - \operatorname{div}[(C(x,t)+|\nabla u|^2)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u] + \varepsilon|u|^{q-2}u = f(x,t)$, $(x,t) \in \Omega \times (0,T)$;

(b) $-\langle\vartheta, (C(x,t)+|\nabla u|^2)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u\rangle \in \beta_x(u(x,t))$, $(x,t) \in \Gamma \times (0,T)$;

(c) $\alpha(\frac{\partial u}{\partial t}(x,0)) = \alpha(\frac{\partial u}{\partial t}(x,T))$, $u(x,0) = u(x,T)$, $x \in \Omega$.

Proof It suffices to show that if $f(x,t) = C^{(1)}u(x,t)$, then the results are true. The proof for the case $f(x,t) = C^{(2)}u(x,t)$ is similar.

Now, for any $\psi \in C_0^\infty(\Omega \times (0,T))$, Lemma 2.4 implies that $(\psi, \partial\Phi^{(1)}(u)) = 0$. Thus, $f(x,t) = C^{(1)}u(x,t)$ implies that

$$\begin{aligned}
&-\int_0^T \int_\Omega \frac{\partial}{\partial t}\left(\alpha\left(\frac{\partial u}{\partial t}\right)\right)\psi\,dx\,dt \\
&\quad+ \int_0^T \int_\Omega \left\langle \left(C(x,t)+|\nabla u|^2\right)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u, \nabla\psi\right\rangle dx\,dt \\
&\quad+ \varepsilon \int_0^T \int_\Omega |u|^{q-2}u\psi\,dx\,dt = \int_0^T \int_\Omega f(x,t)\psi\,dx\,dt.
\end{aligned}$$

Then

$$\frac{\partial}{\partial t}\left(\alpha\left(\frac{\partial u}{\partial t}\right)\right) - \operatorname{div}\left[\left(C(x,t)+|\nabla u|^2\right)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u\right] + \varepsilon|u|^{q-2}u = f(x,t),$$
$$(x,t) \in \Omega \times (0,T).$$

Hence, (a) is true.

Since $f(x,t) = C^{(1)}u(x,t)$, we have $\partial S(u(x,t)) + B^{(1)}u(x,t) + \partial\Phi^{(1)}(u(x,t)) = f(x,t)$. For $\widehat{u} \in L^2(0,T;L^2(\Omega))$, we have $(\widehat{u}-u, \partial S(u(x,t)) + B^{(1)}u + \partial\Phi^{(1)}(u(x,t)) - f(x,t)) = 0$. In view of the definition of the subdifferential, we get

$$\left(\widehat{u}-u, -\frac{\partial}{\partial t}\left(\alpha\left(\frac{\partial u}{\partial t}\right)\right)\right) + \left(\widehat{u}-u, B^{(1)}u\right) + \Phi^{(1)}(\widehat{u}) - \Phi^{(1)}(u) - (\widehat{u}-u, f) \geq 0.$$

Using Green's formula, we find

$$\begin{aligned}
&-\int_0^T\int_\Omega \frac{\partial}{\partial t}\left(\alpha\left(\frac{\partial u}{\partial t}\right)\right)(\widehat{u}-u)\,dx\,dt \\
&\quad -\int_0^T\int_\Omega \operatorname{div}\left[\left(C(x,t)+|\nabla u|^2\right)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u\right](\widehat{u}-u)\,dx\,dt \\
&\quad +\int_0^T\int_\Gamma \left\langle\vartheta, \left(C(x,t)+|\nabla u|^2\right)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u\right\rangle(\widehat{u}-u)|_\Gamma\,d\Gamma(x)\,dt \\
&\quad +\varepsilon\int_0^T\int_\Omega |u|^{q-2}u(\widehat{u}-u)\,dx\,dt + \Phi^{(1)}(\widehat{u}) - \Phi^{(1)}(u) \\
&\geq \int_0^T\int_\Omega f(x,t)(\widehat{u}-u)\,dx\,dt.
\end{aligned}$$

The result of (a) implies that

$$\Phi^{(1)}(\widehat{u}) - \Phi^{(1)}(u) \geq -\int_0^T\int_\Gamma \left\langle\vartheta, \left(C(x,t)+|\nabla u|^2\right)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u\right\rangle(\widehat{u}-u)|_\Gamma\,d\Gamma(x)\,dt.$$

Then $-\langle\vartheta, (C(x,t)+|\nabla u|^2)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u\rangle \in \partial\Phi^{(1)}(u)$.

Using Lemma 2.4, we have $-\langle\vartheta, (C(x,t)+|\nabla u|^2)^{\frac{s}{2}}|\nabla u|^{m-1}\nabla u\rangle \in \beta_x(u)$, $(x,t) \in \Gamma\times(0,T)$. Hence, (b) is true.

It can be seen from the definition of S that (c) is true. This completes the proof. □

Lemma 2.10 ([9]) *If $\beta_x \equiv 0$ for any $x \in \Gamma$, then $\partial\Phi^{(1)}(u) \equiv 0$ for $u \in L^p(0,T;W^{1,p}(\Omega))$, when $p \geq 2$; and $\partial\Phi^{(2)}(u) \equiv 0$ for $u \in L^p(0,T;W^{1,p}(\Omega))$, when $\frac{2N}{N+1} < p \leq 2$, where $N \geq 1$.*

Proposition 2.3

(i) *For $p \geq 2$, if $\beta_x \equiv 0$ for any $x \in \Gamma$, then*

$$\left\{f \in L^2\left(0,T;L^2(\Omega)\right) \,\middle|\, \int_0^T\int_\Omega f\,dx\,dt = 0\right\} \subset R\left(C^{(1)}\right).$$

(ii) *For $\frac{2N}{N+1} < p \leq 2$, if $\beta_x \equiv 0$ for any $x \in \Gamma$, then*

$$\left\{f \in L^2\left(0,T;L^2(\Omega)\right) \,\middle|\, \int_0^T\int_\Omega f\,dx\,dt = 0\right\} \subset R\left(C^{(2)}\right).$$

Proof (i) In view of Lemmas 2.2, 2.8, 1.2, and 1.1, we have $R(B^{(1)} + \partial\Phi^{(1)} + \partial S) = L^{p'}(0,T;(W^{1,p}(\Omega))^*)$. Note that, for any $f \in L^2(0,T;L^2(\Omega))$ with $\int_0^T\int_\Omega f\,dx\,dt = 0$, the linear function $u \in L^p(0,T;W^{1,p}(\Omega)) \to \int_0^T\int_\Omega fu\,dx\,dt$ is an element of $L^{p'}(0,T;(W^{1,p}(\Omega))^*)$. So there exists an $u \in L^p(0,T;W^{1,p}(\Omega))$ (we actually have $u \in L^2(0,T;L^2(\Omega))$), such that

$$\begin{aligned}
&\int_0^T \int_\Omega fv\,dx\,dt \\
&\quad = \int_0^T \int_\Omega -\frac{\partial}{\partial t}\left(\alpha\left(\frac{\partial u}{\partial t}\right)\right) v(x,t)\,dx\,dt \\
&\qquad + \int_0^T \int_\Omega \bigl\langle \bigl(C(x,t)+|\nabla u|^2\bigr)^{\frac{s}{2}} |\nabla u|^{m-1}\nabla u, \nabla v\bigr\rangle\,dx\,dt + \varepsilon \int_0^T \int_\Omega |u|^{q-2}uv\,dx\,dt,
\end{aligned}$$

for any $v \in L^p(0,T;W^{1,p}(\Omega))$. In view of Lemma 2.10, we have $f = C^{(1)}u$.

The proof of (ii) is similar. This completes the proof. □

Definition 2.6 ([1]) For $t \in \mathbb{R}$ and $x \in \Gamma$, let $\beta_x^0(t) \in \beta_x(t)$ be the element with the least absolute value if $\beta_x(t) \neq \emptyset$, and $\beta_x^0(t) = \pm\infty$ if $\beta_x(t) = \emptyset$, where $t > 0$ or < 0, respectively. Finally, let $\beta_\pm(x) = \lim_{t\to\pm\infty} \beta_x^0(t)$ (in the extended sense) for $x \in \Gamma$. Note that $\beta_\pm(x)$ defines measurable functions on Γ in view of our assumptions on β_x.

Proposition 2.4 *Let $f \in L^2(0,T;L^2(\Omega))$ satisfy*

$$T\int_\Gamma \beta_-(x)\,d\Gamma(x) < \int_0^T \int_\Omega f\,dx\,dt < T\int_\Gamma \beta_+(x)\,d\Gamma(x). \tag{2.2}$$

Then $f \in \operatorname{int} R(C^{(1)})$ and $f \in \operatorname{int} R(C^{(2)})$.

Proof Let $f \in L^2(0,T;L^2(\Omega))$ and satisfy (2.2). In view of Proposition 2.1, there exists $u_n \in L^2(0,T;L^2(\Omega))$ such that for each $n \geq 1$, $f = \frac{1}{n}u_n + C^{(i)}u_n$, $i = 1,2$. As in Proposition 3.4 of [1], we only need to prove that $\|u_n\|_{L^2(0,T;L^2(\Omega))} \leq const$, for all $n \geq 1$.

Indeed, suppose to the contrary that $1 \leq \|u_n\|_{L^2(0,T;L^2(\Omega))} \to \infty$, as $n \to \infty$. Let

$$v_n = \frac{u_n}{\|u_n\|_{L^2(0,T;L^2(\Omega))}}.$$

Taking the inner product of the equation $f = \frac{1}{n}u_n + C^{(i)}u_n$ with u_n, we get, for $i = 1,2$,

$$(u_n,f) = \left(u_n, \frac{1}{n}u_n\right) + \bigl(u_n, B^{(i)}u_n\bigr) + \bigl(u_n, \partial\Phi^{(i)}(u_n)\bigr) + \bigl(u_n, \partial S(u_n)\bigr). \tag{2.3}$$

From (2.3) we have $(u_n, B^{(i)}u_n) \leq (u_n,f)$, *i.e.*,

$$\int_0^T \int_\Omega \bigl\langle \bigl(C(x,t)+|\nabla u_n|^2\bigr)^{\frac{s}{2}} |\nabla u_n|^{m-1}\nabla u_n, \nabla u_n\bigr\rangle\,dx\,dt + \varepsilon \int_0^T \int_\Omega |u_n|^q\,dx\,dt \leq (u_n,f),$$

so $\int_0^T \int_\Omega |\nabla u_n|^p\,dx\,dt \leq (u_n,f)$. Dividing by $\|u_n\|^p_{L^2(0,T;L^2(\Omega))}$ then gives

$$\int_0^T \int_\Omega |\nabla v_n|^p\,dx\,dt \leq \frac{(u_n,f)}{\|u_n\|^p_{L^2(0,T;L^2(\Omega))}} \leq \frac{\|f\|_{L^2(0,T;L^2(\Omega))}}{\|u_n\|^{p-1}_{L^2(0,T;L^2(\Omega))}} \to 0,$$

as $n \to +\infty$. So $v_n \to k_5$ (a constant) in $L^p(0,T;W^{1,p}(\Omega))$, as $n \to +\infty$.

Next, we shall show that k_5 is in $L^2(0,T;L^2(\Omega))$ and $k_5 \not\equiv 0$.

If $p \geq 2$, we can easily see that k_5 is in $L^2(0,T;L^2(\Omega))$ and $k_5 \not\equiv 0$ since $\|v_n\|_{L^2(0,T;L^2(\Omega))} = 1$.

If $\frac{2N}{N+1} < p < 2$, from the above proof, we see that $\{v_n\}$ is bounded in $L^p(0,T;W^{1,p}(\Omega))$. Since $L^p(0,T;W^{1,p}(\Omega)) \hookrightarrow\hookrightarrow L^2(0,T;L^2(\Omega))$, we know that $\{v_n\}$ is relatively compact in $L^2(0,T;L^2(\Omega))$. Then there exists a subsequence of $\{v_n\}$, for simplicity we denote it by $\{v_n\}$ again, such that $v_n \to k_6$ in $L^2(0,T;L^2(\Omega))$. Then $k_6 \not\equiv 0$ since $\|v_n\|_{L^2(0,T;L^2(\Omega))} = 1$ and $k_5 = k_6$ a.e. in $\Omega \times (0,T)$. So $k_5 \not\equiv 0$ in $L^2(0,T;L^2(\Omega))$.

Assume now that $k_5 > 0$, we see from (2.3) that $(u_n, \partial\Phi^{(i)}(u_n)) \leq (u_n, f)$, $i = 1,2$. It follows that

$$\begin{aligned}
&\frac{1}{2}\int_0^T \int_\Gamma \beta_x\left(\frac{1}{2}u_n|_\Gamma(x,t)\right)u_n|_\Gamma(x,t)\,d\Gamma(x)\,dt \\
&\quad\leq \int_0^T \int_\Gamma \left[\varphi_x\big(u_n|_\Gamma(x,t)\big) - \varphi_x\left(\frac{1}{2}u_n|_\Gamma(x,t)\right)\right]d\Gamma(x)\,dt \\
&\quad\leq \frac{1}{2}\big(u_n, \partial\Phi^{(i)}(u_n)\big) \leq \frac{1}{2}(u_n, f).
\end{aligned}$$

Dividing the above inequality by $\|u_n\|_{L^2(0,T;L^2(\Omega))}$, we get

$$\int_0^T \int_\Gamma \beta_x\left(\frac{1}{2}u_n|_\Gamma(x,t)\right)v_n|_\Gamma(x,t)\,d\Gamma(x)\,dt \leq (v_n, f).$$

Choose a subsequence of $\{u_n\}$, which is also denoted by $\{u_n\}$, such that $u_n|_\Gamma(x,t) \to +\infty$ a.e. on $\Gamma \times (0,T)$ and let $n \to +\infty$. We can see from the above that

$$T\int_\Gamma \beta_+(x)\,d\Gamma(x) = \int_0^T \int_\Gamma \beta_+(x)\,d\Gamma(x)\,dt \leq \int_0^T \int_\Omega f(x,t)\,dx\,dt,$$

which is a contradiction to (2.2).

Similarly, if $k_5 < 0$, it also leads to a contradiction. Thus, $f \in \operatorname{int} R(C^{(1)})$ and $f \in \operatorname{int} R(C^{(2)})$. This completes the proof. □

Definition 2.7 ([1]) Define $g_+(x) = \liminf_{t\to+\infty} g(x,t)$ and $g_-(x) = \limsup_{t\to-\infty} g(x,t)$. Further, define a function $g_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$g_1(x,t) = \begin{cases} (\inf_{s\geq t} g(x,s)) \wedge (t - T(x)), & t \geq T(x), \\ 0, & t \in [-T(x), T(x)], \\ (\sup_{s\leq t} g(x,s)) \vee (t + T(x)), & t \leq -T(x). \end{cases}$$

Then, for any $x \in \Omega$, $g_1(x,t)$ is increasing in t and $\lim_{t\to\pm\infty} g_1(x,t) = g_\pm(x)$. Moreover, $g_1 : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory's conditions and the functions $g_\pm(x)$ are measurable on Ω. Further, if $g_2(x,t) = g(x,t) - g_1(x,t)$, then $g_2(x,t)t \geq 0$ for $|t| \geq T(x)$ and $x \in \Omega$.

Similar to Propositions 1.1 and 3.5 in [1], we have the following result.

Proposition 2.5 *Define the mapping* $H_1 : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ *by* $(H_1u)(x,t) = g_1(x,u(x,t))$ *for* $u \in L^2(0,T;L^2(\Omega))$ *and* $(x,t) \in \Omega \times (0,T)$. *Then* H_1 *is a bounded, continuous, m-accretive mapping and satisfies Condition* $(*)$. *Also define* $H_2 : L^2(0,T;L^2(\Omega)) \to$

$L^2(0,T;L^2(\Omega))$ *by* $(H_2u)(x,t) = g_2(x,u(x,t))$, *where* $g_2(x,t) = g(x,t) - g_1(x,t)$. *Then* H_2 *satisfies the inequality*

$$\big(H_2(u+y),u\big) \geq -C(y),$$

for any $u, y \in L^2(0,T;L^2(\Omega))$, *where* $C(y)$ *is a constant depending on* y.

Theorem 2.1 *Let* $f(x,t) \in L^2(0,T;L^2(\Omega))$ *be such that*

$$\begin{aligned} T\int_\Gamma \beta_-(x)\,d\Gamma(x) + \int_0^T \int_\Omega g_-(x)\,dx\,dt &< \int_0^T \int_\Omega f(x,t)\,dx\,dt \\ &< T\int_\Gamma \beta_+(x)\,d\Gamma(x) + \int_0^T \int_\Omega g_+(x)\,dx\,dt. \end{aligned}$$

Then equation (1.4) *has a solution in* $L^2(0,T;L^2(\Omega))$.

Proof For $i = 1,2$, let $C^{(i)} : L^2(0,T;L^2(\Omega)) \to 2^{L^2(0,T;L^2(\Omega))}$ be the m-accretive mapping defined in Definition 2.5 and $H_i : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ be the mappings defined in Proposition 2.5. In view of Proposition 2.2, it suffices to show that $f \in R(C^{(i)} + H_1 + H_2)$ which, in view of Theorem 1.1, would be implied by $f \in \operatorname{int}[R(C^{(i)}) + R(H_1)]$.

To use Theorem 1.1, we only need to check that $C^{(i)} + H_1$ is boundedly inversely compact, for $i = 1,2$.

In fact, it suffices to show that if $w = C^{(i)}u + H_1u$ with $\{w\}$ and $\{u\}$ being bounded in $L^2(0,T;L^2(\Omega))$, then $\{u\}$ is relatively compact in $L^2(0,T;L^2(\Omega))$, $i = 1,2$.

Case 1. Suppose $p \geq 2$.

We note that

$$\begin{aligned} \int_0^T \int_\Omega |\nabla u|^p\,dx\,dt &\leq \big(u, B^{(1)}u\big) \\ &= \big(u, C^{(1)}u\big) - \big(u, \partial\Phi^{(1)}(u)\big) - \big(u, \partial S(u)\big) \\ &\leq \big(u, C^{(1)}u\big) + (u, H_1u) \\ &\leq \|u\|_{L^2(0,T;L^2(\Omega))}\|w\|_{L^2(0,T;L^2(\Omega))} \leq k_7. \end{aligned}$$

Since $p \geq 2$, then $\{u\}$ is bounded in $L^2(0,T;H^1(\Omega))$. Thus, $\{u\}$ is relatively compact in $L^2(0,T;L^2(\Omega))$ in view of the fact that $L^2(0,T;H^1(\Omega)) \hookrightarrow\hookrightarrow L^2(0,T;L^2(\Omega))$.

Case 2. $\frac{2N}{N+1} < p < 2$. The relative compactness of $\{u\}$ in $L^2(0,T;L^2(\Omega))$ follows from the result that $C^{(2)}$ has a compact resolvent in Proposition 2.1.

Similar to the proof of the main result in [1], we see that the other conditions of Theorem 1.1 are also satisfied. Furthermore, we can also get the result that $f \in \operatorname{int}[R(C^{(1)}) + R(H_1)]$ by using Propositions 2.3 and 2.4 and discussing in two cases as in [1]. Therefore, the conclusion follows.

The proof is complete. □

Remark 2.2 If, in (1.4), the function $\alpha \equiv I$ (the identity mapping), then (1.4) reduces to the following:

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - \operatorname{div}[(C(x,t) + |\nabla u|^2)^{\frac{s}{2}} |\nabla u|^{m-1} \nabla u] \\ \quad + \varepsilon |u|^{q-2} u + g(x, u(x,t)) = f(x,t), & (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (C(x,t) + |\nabla u|^2)^{\frac{s}{2}} |\nabla u|^{m-1} \nabla u \rangle \in \beta_x(u(x,t)), & (x,t) \in \Gamma \times (0,T), \\ \frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,T), & x \in \Omega, \\ u(x,0) = u(x,T), & x \in \Omega. \end{cases} \tag{2.4}$$

If, in (2.4), $m = 1$ and $s = p - 2$, then (2.4) becomes

$$\begin{cases} -\frac{\partial^2 u}{\partial t^2} - \operatorname{div}[(C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u] \\ \quad + \varepsilon |u|^{q-2} u + g(x, u(x,t)) = f(x,t), & (x,t) \in \Omega \times (0,T), \\ -\langle \vartheta, (C(x,t) + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u \rangle \in \beta_x(u(x,t)), & (x,t) \in \Gamma \times (0,T), \\ \frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x,T), & x \in \Omega, \\ u(x,0) = u(x,T), & x \in \Omega. \end{cases} \tag{2.5}$$

If, in addition, $C(x,t) \equiv 0$, then (2.5) reduces to the case of hyperbolic p-Laplacian problems.

Competing interests
None of the authors have any competing interests in the manuscript.

Authors' contributions
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