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The stability of the solutions of an equation related to the p-Laplacian with degeneracy on the boundary

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Abstract

The equation related to the p-Laplacian

$$u_t = \operatorname{div}(\rho^\alpha |\nabla u|^{p-2}\nabla u) + \sum_{i=1}^N \frac{\partial b_i(u)}{\partial x_i}, \quad (x,t) \in \Omega \times (0,T),$$

is considered, where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance function from the boundary. If $\alpha < p - 1$, the weak solution belongs to H^γ for some $\gamma > 1$, the Dirichlet boundary condition can be imposed as usual, the stability of the solutions is proved. If $\alpha \geq p - 1$, the weak solution lacks the regularity to define the trace on the boundary. It is surprising that we can still prove the stability of the solutions without any boundary condition. In other words, when $\alpha \geq p - 1$, the phenomenon that the solutions of the equation may be free from any limitations of the boundary condition is revealed.

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1 Introduction and the main results

Consider an equation related to the p-Laplacian,

$$u_t = \operatorname{div}\left(\rho^\alpha |\nabla u|^{p-2}\nabla u\right) + \sum_{i=1}^N \frac{\partial b_i(u)}{\partial x_i}, \quad (x,t) \in Q_T, \tag{1.1}$$

with the initial value

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{1.2}$$

Here Ω is a bounded domain in R^N with appropriately smooth boundary, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$, $p > 1$, $\alpha > 0$, $Q_T = \Omega \times (0,T)$. Can we impose the homogeneous boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.3}$$

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as usual? Recently, Jiří *et al.* [1] studied the equation

$$u_t = \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) + q(x)u^{\gamma}, \quad (x,t) \in Q_T, \tag{1.4}$$

with $0 < \gamma < 1$, and they showed that the uniqueness of the solutions of equation (1.4) is not true. The author [2] studied the equation

$$u_t = \mathrm{div}\big(\rho^{\alpha}|\nabla u|^{p-2}\nabla u\big) + f(u,x,t) \tag{1.5}$$

and showed that the stability of the solutions can be obtained without any boundary value condition. Comparing (1.5) with (1.4), one can see that the degeneracy of the coefficient ρ^{α} can eliminate the effect from the source term $f(u,x,t)$. Thus, it is naturally to ask whether the coefficient ρ^{α} can eliminate the effect from the convection term $\frac{\partial b_i(u)}{\partial x_i}$ in equation (1.1).

Yin and Wang [3] studied the following equation:

$$\frac{\partial u}{\partial t} - \mathrm{div}\big(a(x)|\nabla u|^{p-2}\nabla u\big) - b_i(x)D_i u + c(x,t)u = f(x,t), \tag{1.6}$$

where $D_i = \frac{\partial}{\partial x_i}$, $a \in C(\Omega)$, and $a(x) > 0$ in Ω. Yin and Wang classified the boundary into three parts: the nondegenerate boundary, the weakly degenerate boundary, and the strongly degenerate boundary, by means of a reasonable integral description. The boundary value condition should be supplemented definitely on the nondegenerate boundary and the weakly degenerate boundary although the equation is degenerate on this portion of the degenerate boundary. On the strongly degenerate boundary, they formulated a new approach to prescribe the boundary value condition rather than define the Fichera function as treating the linear case. Moreover, they formulated the boundary value condition on this strongly degenerate boundary in a much weaker sense since the regularity of the solution is much weaker near this boundary. Stated succinctly, instead of the usual boundary value condition (1.3), only the partial boundary condition

$$u(x,t) = 0, \quad (x,t) \in \Sigma_p \times (0,T), \tag{1.7}$$

is imposed in [3], where $\Sigma_p \subseteq \partial\Omega$.

In our paper, we will study how the degeneracy of ρ^{α} affects the well-posedness of the solutions of equation (1.1).

Definition 1.1 A function $u(x,t)$ is said to be a weak solution of equation (1.1) with the initial value (1.2), if

$$u \in L^{\infty}(Q_T), \quad \rho^{\alpha}|\nabla u|^p \in L^1(Q_T), \tag{1.8}$$

and, for any function $\varphi \in C_0^{\infty}(Q_T)$,

$$\iint_{Q_T}\left(-u\varphi_t + \rho^{\alpha}|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \sum_{i=1}^{N} b_i(u)\cdot\varphi_{x_i}\right)dx\,dt = 0.$$

The initial value is satisfied in the sense of that

$$\lim_{t\to 0}\int_\Omega \left|u(x,t)-u_0(x)\right|dx=0. \tag{1.9}$$

Definition 1.2 Let $p>1$. The function $u(x,t)$ is said to be the weak solution of equation (1.1) with the initial value (1.2) and the usual boundary condition (1.3), if u satisfies Definition 1.1, and the usual boundary condition (1.3) is satisfied in the sense of trace.

The existence of the solutions of equation (1.1) with the initial value (1.2) can be obtained similar to [4, 5]. The main aim of the paper is to study the stability of the solutions.

Theorem 1.3 *Let $p>1$, $\alpha<p-1$, $b_i(s)$ be a Lipschitz function. If u, v are two solutions of equation* (1.1) *with the same homogeneous value condition* (1.3) *and with the initial values $u_0(x), v_0(x)$, respectively, then*

$$\int_\Omega \left|u(x,t)-v(x,t)\right|dx\le \int_\Omega |u_0-v_0|,\quad \forall t\in[0,T). \tag{1.10}$$

Theorem 1.4 *Let $p\ge 2$, $\alpha\ge p-1$, $b_i(s)$ be bounded when s is bounded. If u, v are two solutions of equation* (1.1) *with the initial values $u_0(x), v_0(x)$, respectively, then*

$$\int_\Omega \left|u(x,t)-v(x,t)\right|^2dx\le \int_\Omega \left|u_0(x)-v_0(x)\right|^2dx. \tag{1.11}$$

Theorem 1.5 *Let $p>1$, $\alpha\ge p-1$, $b_i(s)$ satisfy*

$$\left|b_i(s_1)-b_i(s_2)\right|\le c|s_1-s_2|^{\frac{\alpha+2(p-1)}{p}}. \tag{1.12}$$

If u, v are two solutions of equation (1.1) *with the initial values $u_0(x), v_0(x)$, respectively, such that*

$$\left|u(x,t)\right|\le c\rho(x),\qquad \left|v(x,t)\right|\le c\rho(x), \tag{1.13}$$

then the stability (1.11) *is true.*

Theorem 1.6 *Let $p>1$, $\alpha\ge p-1$, $b_i(s)$ be a C^1 function which satisfies*

$$\left|b_i'(s)\right|\le c|s|^r. \tag{1.14}$$

Assume the following:

(i) *When* $1<p<2$, $\alpha\le 2+p(r-1)$, $r\ge 1$.
(ii) *When* $p=2$, $\alpha\le pr$.
(iii) *When* $p>2$, $r\ge\frac{1}{2}$, $\alpha<\frac{(2r+1)p-2}{2}$.

Let u, v be two solutions of equation (1.1) *with the initial values $u_0(x), v_0(x)$, respectively, such that* (1.13) *is true. Then the stability* (1.11) *is true.*

Since the solution u lacks the regularity when $\alpha\ge p-1$, we cannot define the trace of u on the boundary. How to construct a suitable test function to get the stability (1.11) is a full challenge. By Theorem 1.4, the remaining important problem is that if $p > 1, \alpha \geq p-1$, the same conclusion is true or not. Theorems 1.5 and 1.6 partially solve the problem. Certainly, the conditions of (1.12)-(1.13) in Theorem 1.5, and the conditions (i)-(iii) in Theorem 1.6, may be not the best.

2 The case of $\alpha < p - 1$

Lemma 2.1 *If* $\alpha < p-1$, *let* u *be the solution of equation* (1.1) *with the initial value* (1.2), *then* $u \in H^{\gamma}(\Omega)$ *for some* $\gamma > 1$ *the trace of* u *on the boundary* $\partial\Omega$ *can be defined in the traditional way.*

If $b_i \equiv 0$, the lemma had been proved in Theorem 1.1 in [6]. For the general case, the lemma had been proved in [7] recently. By the lemma, the initial-boundary value problem (1.1)-(1.2)-(1.3) is reasonable.

Proof of Theorem 1.3 Let u and v be two weak solutions with the different initial values $u(x,0), v(x,0)$, respectively. From the definition of the weak solution, we have $\rho^{\alpha}|\nabla u|^{p}, \rho^{\alpha}|\nabla v|^{p} \in L^{1}(Q)$, and for all $\varphi \in C_0^{\infty}(Q_T)$,

$$\iint_{Q_T} \varphi \frac{\partial(u-v)}{\partial t}\,dx\,dt = -\iint_{Q_T} \rho^{\alpha}\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot\nabla\varphi\,dx\,dt - \sum_{i=1}^{N}\iint_{Q_T}\left(b_i(u)-b_i(v)\right)\cdot\varphi_{x_i}\,dx\,dt. \tag{2.1}$$

For any given positive integer n, let $g_n(s)$ be an odd function, and when $s > 0$,

$$g_n(s) = \begin{cases} 1, & s > \frac{1}{n}, \\ n^2 s^2 \mathrm{e}^{1-n^2s^2}, & s \leq \frac{1}{n}. \end{cases} \tag{2.2}$$

By the usual boundary condition (1.3),

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T).$$

By a process of limit, we can choose $g_n(u-v)$ as the test function in (2.1), then

$$\begin{aligned} &\iint_{Q_T} g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt \\ &+ \iint_{Q_T}\rho^{\alpha}\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot\nabla(u-v)g_n'\,dx\,dt \\ &+ \sum_{i=1}^{N}\iint_{Q_T}\left(b_i(u)-b_i(v)\right)\cdot(u-v)_{x_i}g_n'\,dx\,dt = 0. \end{aligned} \tag{2.3}$$

Thus

$$\int_{\Omega} g_n(u-v)\frac{\partial(u-v)}{\partial t}dx = \frac{d}{dt}\|u-v\|_1,$$

$$\iint_{Q_T}\rho^{\alpha}\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot\nabla(u-v)g_n'\,dx\,dt \geq 0,$$

$$\lim_{n\to\infty}\sum_{i=1}^{N}\iint_{Q_T}\big(b_i(u)-b_i(v)\big)g_n'(u-v)(u-v)_{x_i}\,dx\,dt=0. \tag{2.4}$$

Here, (2.4) is established by

$$\left|g_n'(s)\right|\le\frac{c}{s},\quad |s|\le\frac{1}{n},$$

according to the definition of $g_n(s)$, and by the following facts:

$$\begin{aligned}
&\left|\iint_{Q_T\cap\{|u-v|<\frac{1}{n}\}}\big[b_i(u)-b_i(v)\big]g_n(u-v)_{x_i}\,dx\,dt\right|\\
&\quad=\left|\iint_{Q_T\cap\{|u-v|<\frac{1}{n}\}}\big[b_i(u)-b_i(v)\big]g_n'(u-v)(u-v)_{x_i}\,dx\,dt\right|\\
&\quad\le c\iint_{Q_T\cap\{|u-v|<\frac{1}{n}\}}\left|\frac{b_i(u)-b_i(v)}{u-v}\right|\left|(u-v)_{x_i}\right|\,dx\,dt\\
&\quad=c\iint_{Q_T\cap\{|u-v|<\frac{1}{n}\}}\left|\rho^{-\frac{\alpha}{p}}\frac{b_i(u)-b_i(v)}{u-v}\right|\left|\rho^{\frac{\alpha}{p}}(u-v)_{x_i}\right|\,dx\,dt\\
&\quad\le c\left[\iint_{Q_T\cap\{|u-v|<\frac{1}{n}\}}\left(\left|\rho^{-\frac{\alpha}{p}}\frac{b_i(u)-b_i(v)}{u-v}\right|\right)^{\frac{p}{p-1}}\,dx\,dt\right]^{\frac{p-1}{p}}\\
&\qquad\cdot\left(\iint_{Q\cap\{|u-v|<\frac{1}{n}\}}\left|\rho^{\alpha}\nabla(u-v)\right|^{p}\,dx\,dt\right)^{\frac{1}{p}}.
\end{aligned}\tag{2.5}$$

Since $\alpha<p-1$, and $b_i(u)$ is a Lipschitz function,

$$\iint_{Q_T\cap\{|u-v|<\frac{1}{n}\}}\left(\left|\rho^{-\frac{\alpha}{p}}\frac{b_i(u)-b_i(v)}{u-v}\right|\right)^{\frac{p}{p-1}}\,dx\,dt\le c\iint_{Q_T}\rho^{-\frac{\alpha}{p-1}}\,dx\,dt\le c. \tag{2.6}$$

In (2.5), let $n\to\infty$. If $\{x\in\Omega:|u-v|=0\}$ is a set with 0 measure, then

$$\lim_{n\to\infty}\iint_{Q_T\cap\{|u-v|<\frac{1}{n}\}}\rho^{\frac{\alpha}{p-1}}\,dx\,dt=\iint_{Q_T\cap\{|u-v|=0\}}\rho^{\frac{\alpha}{p-1}}\,dx\,dt=0. \tag{2.7}$$

If the set $\{x\in\Omega:|u-v|=0\}$ has a positive measure, then

$$\lim_{n\to\infty}\iint_{Q_T\cap\{|u-v|<\frac{1}{n}\}}\rho^{\alpha}\left|\nabla(u-v)\right|^{p}\,dx\,dt=\iint_{Q_T\cap\{|u-v|=0\}}\rho^{\alpha}\left|\nabla(u-v)\right|^{p}\,dx\,dt=0. \tag{2.8}$$

Therefore, in both cases, (2.5) goes to 0 as $n\to\infty$.

Now, let $n\to\infty$ in (2.3). Then

$$\frac{d}{dt}\|u-v\|_1\le 0.$$

It implies that

$$\int_{\Omega}\left|u(x,t)-v(x,t)\right|dx\le\int_{\Omega}|u_0-v_0|\,dx,\quad\forall t\in[0,T).$$

Theorem 1.3 is proved. □

Remark 2.2 If $b_i(s)$ is a $C^1(R)$, the conclusion of Theorem 1.3 had been proved by the author and Yuan in [7] recently, the method used here is very similar as that in [7]. Since [7] is written in Chinese, we give the details here.

3 The stability of the case $\alpha \geq p-1$

When $\alpha \geq p-1$, let u be a weak solution of equation (1.1) with the initial value (1.2). Generally, we cannot define the trace of u on the boundary, how to prove the uniqueness of the solutions seems very difficult. Theorem 1.4 solves the problem when $p \geq 2$, in the following, we give its proof.

Proof of Theorem 1.4 Denote $\Omega_\lambda = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \lambda\}$. Let

$$\xi_\lambda = \left[\operatorname{dist}(x, \Omega \setminus \Omega_\lambda)\right]^{\frac{\alpha}{p}} = d_\lambda^{\frac{\alpha}{p}}. \tag{3.1}$$

From the definition of the weak solution, we have

$$\iint_{Q_T} \varphi \frac{\partial(u-v)}{\partial t}\,dx\,dt = -\iint_{Q_T} \rho^\alpha \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla\varphi\,dx\,dt - \sum_{i=1}^N \iint_{Q_T} \left[b_i(u) - b_i(v)\right]\varphi_{x_i}\,dx\,dt, \tag{3.2}$$

for any $\varphi \in C_0^\infty(Q_T)$. For any fixed $\tau, s \in [0,T]$, we may choose $\chi_{[\tau,s]}(u_\varepsilon - v_\varepsilon)\xi_\lambda$ as a test function in the above equality, where $\chi_{[\tau,s]}$ is the characteristic function on $[\tau,s]$, u_ε and v_ε are the mollified functions of the solutions u and v, respectively. Thus, letting $Q_{\tau s} = \Omega \times [\tau, s]$,

$$\begin{aligned}
&\iint_{Q_{\tau s}} (u_\varepsilon - v_\varepsilon)\xi_\lambda \frac{\partial(u-v)}{\partial t}\,dx\,dt \\
&\quad = -\iint_{Q_{\tau s}} \rho^\alpha \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla\left[(u_\varepsilon - v_\varepsilon)\xi_\lambda\right]dx\,dt \\
&\qquad - \sum_{i=1}^N \iint_{Q_{\tau s}} \left[b_i(u) - b_i(v)\right]\left[(u_\varepsilon - v_\varepsilon)\xi_\lambda\right]_{x_i}dx\,dt.
\end{aligned} \tag{3.3}$$

For any give $\lambda > 0$, denoting $Q_{T\lambda} = \Omega_\lambda \times (0,T)$, we know that $\nabla u \in L^p(Q_{T\lambda})$, $\nabla v \in L^p(Q_{T\lambda})$. Thus according to the definition of the mollified functions u_ε and v_ε, we have

$$u_\varepsilon \in L^\infty(Q_T), \qquad v_\varepsilon \in L^\infty(Q_T), \tag{3.4}$$

$$\|\nabla u_\varepsilon\|_{p,\Omega_\lambda} \leq \|\nabla u\|_{p,\Omega_\lambda}, \qquad \|\nabla v_\varepsilon\|_{p,\Omega_\lambda} \leq \|\nabla v\|_{p,\Omega_\lambda}. \tag{3.5}$$

By the Young inequality,

$$\begin{aligned}
&\left|\left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla(u_\varepsilon - v_\varepsilon)\right| \\
&\quad \leq \frac{p-1}{p}\left||\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right|^{\frac{p}{p-1}} + \frac{1}{p}\left|\nabla(u_\varepsilon - v_\varepsilon)\right| \\
&\quad \leq c\left(|\nabla u|^p + |\nabla v|^p\right) + c\left(|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p\right).
\end{aligned}$$

Since on Ω_λ, $\rho^\alpha \geq \lambda^\alpha$, we have

$$\begin{aligned}
&\bigl|\rho^\alpha\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\nabla\bigl[(u_\varepsilon - v_\varepsilon)\bigr|\\
&\quad\leq c(\lambda)\bigl(|\nabla u|^p + |\nabla v|^p\bigr) + c\bigl(|\nabla u_\varepsilon|^p + |\nabla v_\varepsilon|^p\bigr)\\
&\quad\leq 2c(\lambda)\bigl(|\nabla u|^p + |\nabla v|^p\bigr),
\end{aligned}$$

by the Lebesgue control convergence theorem,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}}\rho^\alpha\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\nabla\bigl[(u_\varepsilon - v_\varepsilon)\xi_\lambda\bigr]\,dx\,dt\\
&\quad=\iint_{Q_{\tau s}}\rho^\alpha\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\nabla\bigl[(u-v)\xi_\lambda\bigr]\,dx\,dt\\
&\quad=\iint_{Q_{\tau s}}\rho^\alpha\xi_\lambda\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\nabla(u-v)\,dx\,dt\\
&\qquad+\iint_{Q_{\tau s}}\rho^\alpha\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)(u-v)\nabla\xi_\lambda\,dx\,dt. \qquad (3.6)
\end{aligned}$$

The first term on the right hand side of (3.6),

$$\iint_{Q_{\tau s}}\rho^\alpha\xi_\lambda\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\nabla(u-v)\,dx\,dt \geq 0. \qquad (3.7)$$

The last term on the right hand side of (3.6), since, when $x\in\Omega_\lambda$, $d_\lambda(x) > 0$, obeys

$$\begin{aligned}
&\biggl|\iint_{Q_{\tau s}}(u-v)\rho^\alpha\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\nabla\xi_\lambda\,dx\,dt\biggr|\\
&\quad\leq\iint_{Q_{\tau s}}|u-v|\rho^\alpha\bigl(|\nabla u|^{p-1} + |\nabla v|^{p-1}\bigr)|\nabla\xi_\lambda|\,dx\,dt\\
&\quad\leq c\biggl(\int_\tau^s\int_{\Omega_\lambda}\rho^\alpha\bigl(|\nabla u|^p + |\nabla v|^p\bigr)\,dx\,dt\biggr)^{\frac{p-1}{p}}\cdot\biggl(\int_\tau^s\int_{\Omega_\lambda}\rho^\alpha|\nabla\xi_\lambda|^p|u-v|^p\,dx\,dt\biggr)^{\frac{1}{p}}\\
&\quad\leq c\biggl(\int_\tau^s\int_{\Omega}\rho^\alpha\bigl(|\nabla u|^p + |\nabla v|^p\bigr)\,dx\,dt\biggr)^{\frac{p-1}{p}}\\
&\qquad\cdot\biggl(\int_\tau^s\int_{\Omega_\lambda}\rho^\alpha d_\lambda^{p(\frac{\alpha}{p}-1)}|\nabla d_\lambda|^p|u-v|^p\,dx\,dt\biggr)^{\frac{1}{p}}\\
&\quad\leq c\biggl(\int_\tau^s\int_{\Omega_\lambda}\rho^\alpha d_\lambda^{p(\frac{\alpha}{p}-1)}|u-v|^p\,dx\,dt\biggr)^{\frac{1}{p}}. \qquad (3.8)
\end{aligned}$$

Here, we have used the fact that $|\nabla d_\lambda| = 1$ is true almost everywhere. Now, since $p\geq 2$ and $\alpha\geq p-1$ imply that $2\alpha - p\geq 0$, we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\biggl|\iint_{Q_{\tau s}}(u-v)\rho^\alpha\bigl(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\bigr)\nabla\xi_\lambda\,dx\,dt\biggr|\\
&\quad\leq\lim_{\lambda\to 0}c\biggl(\int_\tau^s\int_{\Omega_\lambda}\rho^\alpha(\rho-\lambda)^{p(\frac{\alpha}{p}-1)}|u-v|^p\,dx\,dt\biggr)^{\frac{1}{p}}
\end{aligned}$$

$$= c\left(\int_\tau^s \int_\Omega \rho^{\alpha+p(\frac{\alpha}{p}-1)}|u-v|^p\,dx\,dt\right)^{\frac{1}{p}}$$
$$\leq c\left(\int_\tau^s \int_\Omega \rho^{2\alpha-p}|u-v|^p\,dx\,dt\right)^{\frac{1}{p}}$$
$$\leq c\left(\int_\tau^s \int_\Omega |u-v|^2\,dx\,dt\right)^{\frac{1}{p}}. \tag{3.9}$$

Using the Hölder inequality

$$\int_{\Omega_\lambda} |\nabla u_\varepsilon|\,dx \leq \left(\int_{\Omega_\lambda} |\nabla u_\varepsilon|^p\,dx\right)^{\frac{1}{p}}$$
$$\leq \left(\int_{\Omega_\lambda} |\nabla u|^p\,dx\right)^{\frac{1}{p}} \leq c(\lambda),$$

$|\nabla u_\varepsilon| \in L^1(\Omega_\lambda)$. Then by the definition of the mollified function u_ε, one has

$$|\nabla u_\varepsilon|_1 \leq |\nabla u|_1.$$

By the Lebesgue control convergence theorem, we have

$$\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}} \bigl[b_i(u)-b_i(v)\bigr]\bigl[(u_\varepsilon-v_\varepsilon)\xi_\lambda\bigr]_{x_i}\,dx\,dt$$
$$= \iint_{Q_{\tau s}} \bigl[b_i(u)-b_i(v)\bigr]\bigl[(u-v)\xi_\lambda\bigr]_{x_i}\,dx\,dt$$
$$= \iint_{Q_{\tau s}} \bigl[b_i(u)-b_i(v)\bigr](u-v)\xi_{\lambda x_i}\,dx\,dt + \iint_{Q_s} \bigl[b_i(u)-b_i(v)\bigr](u-v)_{x_i}\xi_\lambda\,dx\,dt. \tag{3.10}$$

Since $\alpha \geq p-1$, $\frac{\alpha}{p}-1 \geq \frac{-1}{p}$. For any $1 > \delta > 0$, by $p \geq 2$, $\frac{2-\delta}{p} < 1$ and

$$\int_\Omega \rho^{-\frac{2-\delta}{p}}\,dx \leq c,$$

due to Ω being appropriately smooth. By these two facts, we have

$$\lim_{\lambda\to 0}\iint_{Q_{\tau s}} \bigl[b_i(u)-b_i(v)\bigr](u-v)\xi_{\lambda x_i}\,dx\,dt$$
$$\leq c\lim_{\lambda\to 0}\int_\tau^s \int_{\Omega_\lambda} \bigl[b_i(u)-b_i(v)\bigr](u-v)d_\lambda^{\frac{\alpha}{p}-1}|d_{\lambda x_i}|\,dx$$
$$\leq c\lim_{\lambda\to 0}\int_\tau^s \int_\Omega |u-v|^2 d_\lambda^{\frac{\alpha}{p}-1}\,dx$$
$$\leq c\left(\int_\tau^s \int_\Omega |u-v|^2\rho^{-\frac{1}{p}}\,dx\right)$$
$$\leq c\left(\int_\tau^s \int_\Omega |u-v|^{2\frac{2-\delta}{1-\delta}}\,dx\right)^{\frac{1-\delta}{2-\delta}}\left(\int_\tau^s \int_\Omega \rho^{-\frac{2-\delta}{p}}\,dx\right)^{\frac{1}{2-\delta}}$$
$$\leq c\left(\int_\tau^s \int_\Omega |u-v|^2\,dx\right)^{\frac{1-\delta}{2-\delta}} \tag{3.11}$$

and

$$\begin{aligned}
&\lim_{\lambda\to 0}\iint_{Q_{\tau s}}\bigl[b_i(u)-b_i(v)\bigr](u-v)_{x_i}\xi_\lambda\,dx\,dt\\
&\quad=\lim_{\lambda\to 0}\iint_{Q_{\tau s}}\bigl[b_i(u)-b_i(v)\bigr](u-v)_{x_i}d_\lambda^{\frac{\alpha}{p}}\,dx\,dt\\
&\quad\leq\biggl(\int_\tau^s\int_\Omega\bigl(\bigl|b_i(u)-b_i(v)\bigr|\bigr)^{p'}\,dx\,dt\biggr)^{\frac{1}{p'}}\biggl(\int_\tau^s\int_\Omega\rho^\alpha\bigl(|\nabla u|^p+|\nabla v|^p\bigr)\,dx\,dt\biggr)^{\frac{1}{p}}\\
&\quad\leq c\biggl(\int_\tau^s\int_\Omega\bigl(\bigl|b_i(u)-b_i(v)\bigr|\bigr)^{p'}\,dx\,dt\biggr)^{\frac{1}{p'}}\\
&\quad\leq c\biggl(\int_\tau^s\int_\Omega|u-v|\,dx\,dt\biggr)^{\frac{1}{p'}}\leq c\biggl(\int_\tau^s\int_\Omega|u-v|^2\,dx\,dt\biggr)^{\frac{1}{2p'}}.
\end{aligned}\tag{3.12}$$

By (3.6)-(3.11), after letting $\varepsilon\to 0$, we let $\lambda\to 0$ in (3.3). Then

$$\begin{aligned}
&\int_\Omega\bigl[u(x,s)-v(x,s)\bigr]^2\,dx-\int_\Omega\bigl[u(x,\tau)-v(x,\tau)\bigr]^2\,dx\\
&\quad\leq c\biggl(\int_0^s\int_\Omega\bigl|u(x,t)-v(x,t)\bigr|^2\,dx\,dt\biggr)^q,
\end{aligned}\tag{3.13}$$

where $q<1$. Let $\kappa(s)=\int_\Omega[u(x,s)-v(x,s)]^2\,dx$. Then

$$\frac{\kappa(s)-\kappa(\tau)}{s-\tau}\leq c\frac{(\int_\tau^s\kappa(t)\,dt)^q}{s-\tau}.$$

By the L'Hospital rule, we have

$$\begin{aligned}
\kappa'(\tau)&\leq c\lim_{s\to\tau}\frac{\kappa(s)}{(\int_\tau^s\kappa(t)\,dt)^{1-q}}\\
&=c\lim_{s\to\tau}\frac{\kappa'(s)}{\kappa(s)}\biggl(\int_\tau^s\kappa(t)\,dt\biggr)^q=0.
\end{aligned}\tag{3.14}$$

Thus, by the arbitrary of τ, we have

$$\int_\Omega\bigl|u(x,s)-v(x,s)\bigr|^2\,dx\leq\int_\Omega|u_0-v_0|^2\,dx.\tag{3.15}$$

The proof is complete. □

4 Proof of Theorems 1.5 and 1.6

Instead of the condition $p\geq 2$, if we only assume that $p>1$, then we also can obtain the uniqueness of the solutions in some cases.

Proof of Theorems 1.5 *and* 1.6 Denote $\Omega_\lambda=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>\lambda\}$ as before, let $\xi_\lambda\in C_0^\infty(\Omega_\lambda)$ such that $\xi_\lambda=1$ on $\Omega_{2\lambda}$, $0\leq\xi_\lambda\leq 1$, and

$$|\nabla\xi_\lambda|\leq\frac{c}{\lambda}.\tag{4.1}$$

From the definition of the weak solution, we have

$$\iint_{Q_T} \varphi \frac{\partial(u-v)}{\partial t}\,dx\,dt = -\iint_{Q_T} \rho^\alpha \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\nabla\varphi\,dx\,dt \\ - \sum_{i=1}^{N} \iint_{Q_T} \big[b_i(u) - b_i(v)\big]\varphi_{x_i}\,dx\,dt, \tag{4.2}$$

for any $\varphi \in C_0^\infty(Q_T)$. For any fixed $\tau, s \in [0,T]$, we may choose $\chi_{[\tau,s]}(u_\varepsilon - v_\varepsilon)\xi_\lambda$ as a test function in the above equality, where $\chi_{[\tau,s]}$ is the characteristic function on $[\tau,s]$, u_ε and v_ε are the mollified functions of the solutions u and v, respectively. Thus, letting $Q_{\tau s} = \Omega \times [\tau, s]$,

$$\iint_{Q_{\tau s}} (u_\varepsilon - v_\varepsilon)\xi_\lambda \frac{\partial(u-v)}{\partial t}\,dx\,dt \\ = -\iint_{Q_{\tau s}} \rho^\alpha \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\nabla\big[(u_\varepsilon - v_\varepsilon)\xi_\lambda\big]\,dx\,dt \\ - \sum_{i=1}^{N} \iint_{Q_{\tau s}} \big[b_i(u) - b_i(v)\big]\big[(u_\varepsilon - v_\varepsilon)\xi_\lambda\big]_{x_i}\,dx\,dt. \tag{4.3}$$

For any give $\lambda > 0$, denoting $Q_{T\lambda} = \Omega_\lambda \times (0,T)$, we know that $\nabla u \in L^p(Q_{T\lambda})$, $\nabla v \in L^p(Q_{T\lambda})$. Thus according to the definition of the mollified functions u_ε and v_ε, similarly, we have (3.4)-(3.6). The first term on the right hand side of (3.6),

$$\iint_{Q_{\tau s}} \rho^\alpha \xi_\lambda \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\nabla(u-v)\,dx\,dt \ge 0. \tag{4.4}$$

The last term on the right hand side of (3.6), since $\alpha \ge p-1$,

$$\left|\iint_{Q_{\tau s}} (u-v)\rho^\alpha \big(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big)\nabla\xi_\lambda\,dx\,dt\right| \\ \le \iint_{Q_{\tau s}} |u-v|\rho^\alpha \big(|\nabla u|^{p-1} + |\nabla v|^{p-1}\big)|\nabla\xi_\lambda|\,dx\,dt \\ \le c\left(\int_\tau^s \int_{\Omega_\lambda\setminus\Omega_{2\lambda}} \rho^\alpha \big(|\nabla u|^p + |\nabla v|^p\big)\,dx\,dt\right)^{\frac{p-1}{p}} \\ \cdot \left(\int_\tau^s \int_{\Omega_\lambda\setminus\Omega_{2\lambda}} \rho^\alpha |\nabla\xi_\lambda|^p\,dx\,dt\right)^{\frac{1}{p}} \\ \le c\left(\int_\tau^s \int_{\Omega_\lambda\setminus\Omega_{2\lambda}} \rho^\alpha \big(|\nabla u|^p + |\nabla v|^p\big)\,dx\,dt\right)^{\frac{p-1}{p}} \cdot \left(\int_\tau^s \int_{\Omega_\lambda\setminus\Omega_{2\lambda}} \lambda^{\alpha-p}\,dx\,dt\right)^{\frac{1}{p}} \\ \le c\lambda^{\frac{(\alpha+1-p)}{p}} \left(\int_\tau^s \int_{\Omega_\lambda\setminus\Omega_{2\lambda}} \rho^\alpha \big(|\nabla u|^p + |\nabla v|^p\big)\,dx\,dt\right)^{\frac{p-1}{p}} \\ \le c\left(\int_\tau^s \int_{\Omega_\lambda\setminus\Omega_{2\lambda}} \rho^\alpha \big(|\nabla u|^p + |\nabla v|^p\big)\,dx\,dt\right)^{\frac{p-1}{p}} \to 0, \quad \text{as } \lambda \to 0. \tag{4.5}$$

Also by the Lebesgue control convergence theorem,

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\iint_{Q_{\tau s}}\left[b_i(u)-b_i(v)\right]\left[(u_\varepsilon-v_\varepsilon)\xi_\lambda\right]_{x_i}\,dx\,dt \\
&\quad=\iint_{Q_{\tau s}}\left[b_i(u)-b_i(v)\right]\left[(u-v)\xi_\lambda\right]_{x_i}\,dx\,dt \\
&\quad=\iint_{Q_{\tau s}}\left[b_i(u)-b_i(v)\right](u-v)\xi_{\lambda x_i}\,dx\,dt+\iint_{Q_s}\left[b_i(u)-b_i(v)\right](u-v)_{x_i}\xi_\lambda\,dx\,dt. \qquad (4.6)
\end{aligned}$$

By that $|u(x,t)|\leq c\rho(x)$, $|v(x,t)|\leq c\rho(x)$,

$$\lim_{\lambda\to 0}\iint_{Q_{\tau s}}\left[b_i(u)-b_i(v)\right](u-v)\xi_{\lambda x_i}\,dx\,dt\leq c\lim_{\lambda\to 0}\int_\tau^s\int_{\Omega_\lambda\setminus\Omega_{2\lambda}}dx=0. \qquad (4.7)$$

By the Hölder inequality

$$\begin{aligned}
&\lim_{\lambda\to 0}\iint_{Q_{\tau s}}\left[b_i(u)-b_i(v)\right](u-v)_{x_i}\xi_\lambda\,dx\,dt \\
&\quad=\lim_{\lambda\to 0}\iint_{Q_{\tau s}}\rho^{-\frac{\alpha}{p}}\xi_\lambda\left[b_i(u)-b_i(v)\right]\rho^{\frac{\alpha}{p}}(u-v)_{x_i}\,dx\,dt \\
&\quad\leq\left(\int_\tau^s\int_\Omega\left(\rho^{-\frac{\alpha}{p}}\left|b_i(u)-b_i(v)\right|\right)^{p'}dx\,dt\right)^{\frac{1}{p'}}\left(\int_\tau^s\int_\Omega\rho^\alpha\left(|\nabla u|^p+|\nabla v|^p\right)dx\,dt\right)^{\frac{1}{p}} \\
&\quad\leq c\left(\int_\tau^s\int_\Omega\left(\rho^{-\frac{\alpha}{p}}\left|b_i(u)-b_i(v)\right|\right)^{p'}dx\,dt\right)^{\frac{1}{p'}}. \qquad (4.8)
\end{aligned}$$

(I) If $p>1$,

$$\left|b_i(u)-b_i(v)\right|\leq c|u-v|^{\frac{2}{p'}}|u-v|^{\frac{\alpha}{p}}=c|u-v|^{\frac{\alpha+2(p-1)}{p}},$$

by (4.8),

$$\lim_{\lambda\to 0}\iint_{Q_{\tau s}}\left[b_i(u)-b_i(v)\right](u-v)_{x_i}\xi_\lambda\,dx\,dt\leq c\left(\int_0^s\int_\Omega|u-v|^2\,dx\,dt\right)^{\frac{1}{p'}}. \qquad (4.9)$$

(II)

(i) If $p<2$, $|b_i'(\zeta)|\leq|\zeta|^r$, $r\geq 1$,

$$\begin{aligned}
\iint_{Q_s}\left(\rho^{-\frac{\alpha}{p}}\left|b_i(u)-b_i(v)\right|\right)^{p'}dx\,dt&\leq\iint_{Q_{\tau s}}\rho^{-\frac{\alpha}{p-1}}\left|b_i'(\zeta)\right|^{\frac{p}{p-1}}|u-v|^{\frac{p}{p-1}-2}|u-v|^2\,dx\,dt \\
&\leq\iint_{Q_{\tau s}}\rho^{-\frac{\alpha}{p-1}+\frac{p(1+r)}{p-1}-2}|u-v|^2\,dx\,dt \\
&\leq c\iint_{Q_{\tau s}}|u-v|^2\,dx\,dt. \qquad (4.10)
\end{aligned}$$

Here, the last inequality is based on the assumption $\alpha\leq p(r-1)+2$, which implies that

$$-\frac{\alpha}{p-1}+\frac{p(1+r)}{p-1}-2\geq 0.$$

Then

$$\lim_{\lambda\to 0}\iint_{Q_{\tau s}}\big[b_i(u)-b_i(v)\big](u-v)_{x_i}\xi_\lambda\,dx\,dt\le c\left(\int_0^s\int_\Omega|u-v|^2\,dx\,dt\right)^{\frac{1}{p'}}. \tag{4.11}$$

(ii) If $p=2$, $|b_i'(\zeta)|\le|\zeta|^r, r\ge\frac{\alpha}{p}$,

$$\big|b_i(u)-b_i(v)\big|^2\le\big|b_i'(\zeta)\big||u-v|^2,$$

then

$$\lim_{\lambda\to 0}\iint_{Q_{\tau s}}\big[b_i(u)-b_i(v)\big](u-v)_{x_i}\xi_\lambda\,dx\,dt\le c\left(\int_0^s\int_\Omega|u-v|^2\,dx\,dt\right)^{\frac{1}{2}}. \tag{4.12}$$

(iii) If $p>2$, by $|b_i'(\zeta)|\le|\zeta|^s$, $s\ge\frac{1}{2}$,

$$\begin{aligned}
&\iint_{Q_{\tau s}}\left(\rho^{-\frac{\alpha}{p}}\big|b_i(u)-b_i(v)\big|\right)^{p'}dx\,dt\\
&\quad\le\iint_{Q_s}\rho^{-\frac{\alpha}{p-1}}\big|b_i'(\zeta)\big|^{\frac{p}{p-1}}|u-v|^{\frac{p}{p-1}}\,dx\,dt\\
&\quad\le\left(\iint_{Q_{\tau s}}\left[\rho^{-\frac{\alpha}{p-1}}\big|b_i'(\zeta)\big|^{\frac{p}{p-1}}\right]^{\frac{2(p-1)}{p-2}}dx\,dt\right)^{\frac{(p-2)}{2(p-1)}}\left(\iint_{Q_{\tau s}}|u-v|^2\,dx\,dt\right)^{\frac{\frac{p}{p-1}}{2}}\\
&\quad\le\left(\iint_{Q_{\tau s}}\rho^{\frac{2(sp-\alpha)}{p-2}}\,dx\,dt\right)^{\frac{(p-2)}{2(p-1)}}\left(\iint_{Q_{\tau s}}|u-v|^2\,dx\,dt\right)^{\frac{p}{2(p-1)}},
\end{aligned}$$

then, based on the assumption $\alpha<\frac{(2s+1)p-2}{2}$, which implies that

$$\frac{2(sp-\alpha)}{p-2}>-1,$$

we have

$$\left(\iint_{Q_{\tau s}}\left(\rho^{-\frac{\alpha}{p}}\big|b_i(u)-b_i(v)\big|\right)^{p'}dx\,dt\right)^{\frac{1}{p'}}\le c\left(\iint_{Q_{\tau s}}|u-v|^2\,dx\,dt\right)^{\frac{1}{2}}, \tag{4.13}$$

and we have (4.12) too.

By (4.4)-(4.13), after letting $\varepsilon\to 0$, we let $\lambda\to 0$ in (4.3). Then

$$\begin{aligned}
&\int_\Omega\big[u(x,s)-v(x,s)\big]^2\,dx-\int_\Omega\big[u(x,\tau)-v(x,\tau)\big]^2\,dx\\
&\quad\le c\left(\int_0^s\int_\Omega\big|u(x,t)-v(x,t)\big|^2\,dx\,dt\right)^q,
\end{aligned} \tag{4.14}$$

where $q<1$. Let $\kappa(s)=\int_\Omega[u(x,s)-v(x,s)]^2\,dx$. As the proof of Theorem 1.4, we have the conclusions.

At the end of the paper, we should point out that the conditions (1.12)-(1.14) and (i)-(iii) are used to deal with the convection term $\sum_{i=1}^N\frac{\partial b_i(u)}{\partial x_i}$. In other words, if $b_i\equiv 0$, then all these conditions are not necessary, the same conclusions had been obtained in [2]. □

Competing interests
The author declares that they have no competing interests.

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References
1. Jiří, B, Peter, G, Lukáš, K, Peter, T: Nonuniqueness and multi-bump solutions in parabolic problems with the *p*-Laplacian. J. Differ. Equ. **260**, 991-1009 (2016)
2. Zhan, H: On a parabolic equation related to the *p*-Laplacian. Bound. Value Probl. **2016**, 78 (2016). doi:10.1186/s13661-016-0587-6
3. Yin, J, Wang, C: Evolutionary weighted *p*-Laplacian with boundary degeneracy. J. Differ. Equ. **237**, 421-445 (2007)
4. Zhan, H: The solution of convection-diffusion equation. Chin. Ann. Math., Ser. A **34**(2), 235-256 (2013) (in Chinese)
5. Zhan, H: The boundary value condition of an evolutionary $p(x)$-Laplacian equation. Bound. Value Probl. **2015**, 112 (2015). doi:10.1186/s13661-015-0377-6
6. Yin, J, Wang, C: Properties of the boundary flux of a singular diffusion process. Chin. Ann. Math., Ser. B **25**(2), 175-182 (2004)
7. Zhan, H, Yuan, H: Diffusion convection equation with boundary degeneracy. J. Jilin Univ. Sci. Ed. **53**(3), 353-358 (2015) (in Chinese)

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