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Orbital stability of generalized Choquard equation

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Abstract

In this paper, we consider the orbital stability of standing waves for the generalized Choquard equation. By the variational method, we see that the ground state solutions of the generalized Choquard equation have orbital stability.

Keywords: Choquard equation; ground state solution; orbital stability

1 Introduction

In this paper, we consider the following generalized Choquard equation:

$$\begin{cases} i\frac{\partial u}{\partial t} + \Delta u + (\omega(x) * |u|^p)|u|^{p-2}u = 0, & (t,x) \in (0,\infty) \times \mathbb{R}^N; \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $\omega(x) * |u|^p := \int_{\mathbb{R}^N} \omega(x-y)|u(y)|^p \, dy$, the constant $N \geq 3$, p satisfies $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$ and $0 < \alpha < N$; $u(t,x)$ is complex-valued solution with an initial condition $u(0,x) = u_0(x)$ and $u_0(x)$ is a given function in $\mathbb{R}^N$. In addition, we assume that $\omega(x) \in C^1(\mathbb{R}^N \setminus 0, (0,+\infty))$ is positive, even, and homogeneous of degree $-(N-\alpha)$. That is, for each $t > 0$, $\omega(tx) = t^{-(N-\alpha)}\omega(x)$.

Due to its application in mathematical physics [1–3], the Choquard equation (1) has attracted a lot of attention from different points of view. When $p = 2$, equation (1) can be reduced to the well-known Hartree equation. Cazenave [4], and Ginibre and Velo [5] established the local well-posedness and asymptotical behavior of the solutions for the Cauchy problem. Later, Genev and Venkov [6] extended the results to the case $N \geq 3$, $2 \leq p < \frac{N+2}{N-2}$, and $\alpha = 2$. Recently, Feng and Yuan [7] considered the general case $0 < \alpha < N$ and $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$, they obtained the local and global well-posedness of equation (1). Also, they investigated the mass concentration for all the blow-up solutions in the L_2-critical case. For more details, we refer to [8, 9] and the references therein.

The standing wave solution of (1) is of the form $u(t,x) = e^{iat}u(x)$, then equation (1) is reduced to the following stationary equation:

$$-\Delta u + au = \left(\omega(x) * |u|^p\right)|u|^{p-2}u, \quad x \in \mathbb{R}^N. \tag{2}$$

When $p = 2$, the existence of solutions for equation (2) was proved with variational methods by Lieb [10] and Lions [2]. Later, Moroz and Schaftingen [11] extended their results to general p. Moreover, they derived the regularity and the decay asymptotic at infinity of the ground states. For more details as regards equation (2), we refer to [11–14].

Here, we are interested in the orbital stability of standing waves for equation (1). The meaning of 'orbital stability' will be given later. When $p = 2$, the orbital stability of standing wave of equation (1) was first established by Cazenave and Lions [15] and then investigated by Cingolani, Secchi, and Squassina [16]. Later, Chen and Guo [17] studied the existence of blow-up solutions and the strong instability of standing waves. However, there are less results for the orbital stability of standing waves for the generalized Choquard equation (1). The purpose of this paper is to investigate this problem.

This paper is divided into two parts. In Section 1, we list some related notations and definitions. In Section 2, we give the proof of our main result.

We now give the related notations and definitions.

Define the functional

$$E(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2p}\mathbb{D}(u), \tag{3}$$

$$J(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}a\|u\|_2^2 - \frac{1}{2p}\mathbb{D}(u), \tag{4}$$

where $\mathbb{D}(u) = \int_{\mathbb{R}^N\times\mathbb{R}^N}\omega(x-y)|u(x)|^p|u(y)|^p\,dx\,dy$, $\|\cdot\|_2$ is the standard L^2-norm.

Given a positive constant $\rho > 0$, we set

$$M = \{u \in H^1(\mathbb{R}^N,\mathbb{C}) : \|u\|_2^2 = \rho\}, \qquad \mathcal{N} = \{u \in H^1(\mathbb{R}^N,\mathbb{C}) : u \neq 0, J'(u) = 0\}$$

and

$$K_M = \{c \in \mathbb{R}^- : \text{there exists } u \in M, \text{such that } E'(u) = 0 \text{ and } E(u) = c\},$$

$$K_{\mathcal{N}} = \{m \in \mathbb{R} : \text{there exists } v \in \mathcal{N}, \text{such that } J'(v) = 0 \text{ and } J(v) = m\}.$$

In order to obtain the result, we also need the following Pohozaev identity of equation (2) ([11, 18]).

Assume $u \in H^1(\mathbb{R}^N,\mathbb{C})$ to be the solution of (2), then

$$\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,dx + \frac{Na}{2}\int_{\mathbb{R}^N}u^2\,dx = \frac{N+\alpha}{2p}\int_{\mathbb{R}^N\times\mathbb{R}^N}\omega(x-y)|u(x)|^p|u(y)|^p\,dx\,dy. \tag{5}$$

Moreover, we denote by G the set of ground state solutions of (2), *i.e.* solutions to the minimization problem $\Gamma = \min_{u\in\mathcal{N}} J(u)$. According to the result of [11], we get $G \neq \emptyset$. In fact, when $a = 1$, let v be the ground state solution of equation (2) obtained by Moroz and Schaftingen [11]. By the transformation $u(x) = a^{\frac{\alpha-2}{4(p-1)}}v(\frac{x}{\sqrt{a}})$, we see that u is the ground state solution of equation (2), which implies that $G \neq \emptyset$.

Also, we assume the following.

Assumption A $p \in (\frac{N+\alpha}{N}, \frac{N+\alpha}{N-2})$ *is such that, for any initial value* $u_0 \in H^1(\mathbb{R}^N,\mathbb{C})$, *problem* (1) *has a unique global solution* $u \in C^1([0,+\infty), H^1(\mathbb{R}^N,\mathbb{C}))$, *the charge and the energy are conserved in time. That is, for all* $t \in [0,+\infty)$, *one has*

$$\|u\|_2 = \|u_0\|_2, \qquad E(u(t)) = E(u_0). \tag{6}$$

Remark 1 As we mentioned before, our assumption is valid when $p = 2$ ([4], Corollary 6.1.2) and when $2 \leq p < 1 + \frac{2+\alpha}{N}$ ([7]).

In the following, we give the definition of orbital stability.

Definition 1.1 The set G of ground state solutions of (2) is said to be orbitally stable for equation (1), if for each $\epsilon > 0$, there exists a $\delta > 0$, such that, for all $u_0 \in H^1(\mathbb{R}^N, \mathbb{C})$,

$$\inf_{\phi \in G} \|u_0 - \phi\|_{H^1} < \delta$$

implies that

$$\sup_{t \geq 0} \inf_{\psi \in G} \big\| u(t, \cdot) - \psi \big\|_{H^1} < \epsilon,$$

where $u(t, x)$ is the solution of (1) with initial datum u_0.

Our main result is as follows.

Theorem 2 *Assume that* $0 < \alpha < N$ *and* p *satisfies Assumption* A, *then the set* G *of the ground state solutions of equation* (2) *is orbitally stable for* (1).

Remark 3 Our result can be regarded as the extension of the result obtained in [16]. However, the existence of general p leads to a more complicated calculation.

2 Proof of Theorem 2

In this section, we give the proof of Theorem 2. First of all, considering the following two minimization problems:

$$\Lambda = \min_{u \in M} E(u), \tag{7}$$

$$\Gamma = \min_{u \in \mathcal{N}} J(u), \tag{8}$$

we establish the equivalence between the two minimization problems.

Lemma 2.1 *When* $\Lambda < 0$, (7) *and* (8) *are equivalent. Moreover,* $\Lambda = \Psi(\Gamma)$, *where* $\Psi : K_{\mathcal{N}} \to K_M$ *is defined by*

$$\Psi(m) = -\frac{Np - N - \alpha - 2}{2(p-1)} \left(\frac{(p-1)a\rho}{N + 2p - pN + \alpha} \right)^{\frac{Np-\alpha-2p-N}{Np-N-\alpha-2}} m^{\frac{2p-2}{Np-N-\alpha-2}}, \quad m \in K_{\mathcal{N}}.$$

Proof If $u \in M$ is a critical point of E with $E(u) = c < 0$, then there exists a Lagrange multiplier $\gamma \in \mathbb{R}$ such that $E'(u)(u) = -\gamma\rho$. Together with $E(u) = c$, one has $(p-1)\|\nabla u\|_2^2 = 2pc + \gamma\rho$. Since $c < 0$, one has $\gamma > 0$. Moreover, u satisfies

$$-\Delta u + \gamma u = \big(\omega * |u|^p\big)|u|^{p-2}u. \tag{9}$$

Set $v(x) = T^\lambda u(x) := \lambda^\sigma u(\lambda x)$ and choose $\lambda = \sqrt{\frac{a}{\gamma}}$, $\sigma = \frac{2+\alpha}{2p-2}$, then from (9) we see that $v(x)$ is a solution of (2), which implies $v(x) \in \mathcal{N}$.

On the contrary, if $\nu(x)$ is a nontrivial critical point of $J|_{\mathcal{N}}$, we choose $\lambda = (\frac{\rho}{\|\nu\|_2^2})^{\frac{1}{N-2\sigma}}$, then $u = T^{\frac{1}{\lambda}}\nu \in M$ and u is a critical point of $E|_M$.

Indeed, due to the choice of λ, it is easy to see that $\|u\|_2^2 = \rho$.

Meanwhile, by the assumption of ν, one has

$$J'(\nu)\nu = \|\nabla\nu\|_2^2 + a\|\nu\|_2^2 - \mathbb{D}(\nu) = 0. \tag{10}$$

Since from $\sigma = \frac{2+\alpha}{2p-2}$, we get $2\sigma + 2 - N = 2p\sigma - N - \alpha = \frac{2p+N-Np+\alpha}{p-1}$, by (10), and then it follows that with σ we get

$$\begin{aligned}
E'(u)u &= \|\nabla u\|_2^2 - \mathbb{D}(u) \\
&= \lambda^{N-2\sigma-2}\|\nabla\nu\|_2^2 - \lambda^{N-2p\sigma+\alpha}\mathbb{D}(\nu) \\
&= \lambda^{N-2\sigma-2}a\|\nu\|_2^2 \\
&= a\lambda^{N-2\sigma-2}\lambda^{\frac{\rho}{N-2\sigma}} = \frac{a}{\lambda^2}\rho \doteq \gamma\rho.
\end{aligned}$$

So u is a critical point of $E|_M$. Now, we calculate the relationship of m and c.

Let $m = J(\nu)$, $\nu \in \mathcal{N}$, $c = E(u)$, $u \in M$, $\nu = T^{\lambda}u$, in which $\lambda = (\frac{\rho}{\|\nu\|_2^2})^{\frac{1}{N-2\sigma}}$, $\sigma = \frac{2+\alpha}{2p-2}$, then

$$\begin{aligned}
m &= \frac{1}{2}\|\nabla\nu\|_2^2 + \frac{1}{2}a\|\nu\|_2^2 - \frac{1}{2p}\mathbb{D}(\nu) \\
&= \frac{1}{2}\lambda^{2\sigma+2-N}\|\nabla u\|_2^2 + \frac{1}{2}a\lambda^{2\sigma-N}\|u\|_2^2 - \frac{1}{2p}\lambda^{2p\sigma-2N+\alpha}\mathbb{D}(u) \\
&= \lambda^{\frac{2p+N-Np+\alpha}{p-1}}E(u) + \frac{1}{2}a\lambda^{2\sigma-N}\rho \\
&= c\lambda^{\frac{2p+N-Np+\alpha}{p-1}} + \frac{1}{2}a\lambda^{2\sigma-N}\rho.
\end{aligned} \tag{11}$$

On the other hand, since ν is the critical point of J,

$$\mathbb{D}(\nu) = \|\nabla\nu\|_2^2 + a\|\nu\|_2^2, \tag{12}$$

hence

$$m = \left(\frac{1}{2} - \frac{1}{2p}\right)\|\nabla\nu\|_2^2 + \left(\frac{1}{2} - \frac{1}{2p}\right)a\|\nu\|_2^2. \tag{13}$$

By the Pohozaev identity (5), one has

$$\left(\frac{N}{2} - 1\right)\|\nabla\nu\|_2^2 + \frac{Na}{2}\|\nu\|_2^2 = \left(\frac{N}{p} - \frac{N-\alpha}{2p}\right)\mathbb{D}(\nu).$$

Then it follows from (12) that

$$\left(\frac{N}{2} - 1 - \frac{N}{p} + \frac{N-\alpha}{2p}\right)\|\nabla\nu\|_2^2 + \left(\frac{N}{2} - \frac{N+\alpha}{2p}\right)a\|\nu\|_2^2 = 0. \tag{14}$$

Thus, (13) and (14) imply that

$$\begin{aligned} \|v\|_2^2 &= \frac{N+2p-pN+\alpha}{(p-1)a}m, \\ \|\nabla v\|_2^2 &= \frac{-\alpha+Np-N}{p-1}m. \end{aligned} \tag{15}$$

Substituting (15) into (11), we have

$$\begin{aligned} m &= c\left(\frac{(p-1)a\rho}{(N+2p-Np+\alpha)m}\right)^{\frac{1}{N-2\sigma}\frac{2p+N-Np+\alpha}{p-1}} + \frac{1}{2}a\left(\frac{(p-1)a\rho}{(N+2p-Np+\alpha)m}\right)^{-1}\rho \\ &= c\left(\frac{(p-1)a\rho}{(N+2p-Np+\alpha)m}\right)^{\frac{2p+N-Np+\alpha}{Np-N-\alpha-2}} + \frac{N+2p-Np+\alpha}{2(p-1)}m. \end{aligned}$$

Hence

$$\begin{aligned} c &= \frac{Np-\alpha-N-2}{2(p-1)}\left(\frac{(p-1)a\rho}{N+2p-pN+\alpha}\right)^{\frac{Np-\alpha-2p-N}{Np-N-\alpha-2}} m^{\frac{2p-2}{Np-N-\alpha-2}} \\ &:= Am^{\frac{2p-2}{Np-N-\alpha-2}} := \Psi(m). \end{aligned}$$

From the assumption of p, it is easy to see that $c<0$. Therefore, Ψ is a map $\Psi: R^+ \to R^-$ and $m = \Psi^{-1}(c) = (\frac{1}{A}c)^{\frac{Np-N-\alpha-2}{2p-2}}$ is injective.

Next we prove that Ψ^{-1} is surjective. Given m, a critical value for J, that is, $m = J(v)$, v is the solution of (2). Consider $u = T^{\frac{1}{\lambda}}v = \lambda^{-\sigma}v(\lambda^{-1}x)$, where $\lambda = (\frac{\rho}{\|v(x)\|_2^2})^{\frac{1}{N-2\sigma}}$, then $u \in M$ is a critical point of $E|_M$ and $\|\nabla u\|_2^2 - \mathbb{D}(u) + \gamma\rho = 0$ with $\gamma = a\lambda^{-2}$. By using $\lambda = (\frac{\rho}{\|v(x)\|_2^2})^{\frac{1}{N-2\sigma}} = (\frac{(p-1)a\rho}{(N+2p-pN+\alpha)m})^{\frac{1}{-N+2\alpha}}$, one has

$$\begin{aligned} c &:= \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2p}\mathbb{D}(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2p}\left(\|\nabla u\|_2^2 + \gamma\rho\right) \\ &= \frac{p-1}{2p}\|\nabla u\|_2^2 - \frac{\gamma}{2p}\rho \\ &= \frac{p-1}{2p}\lambda^{-2\sigma-2+N}\|\nabla v\|_2^2 - \frac{a\lambda^{-2}}{2p}\rho \\ &= \frac{-\alpha+Np-N-2}{2(p-1)}\left(\frac{(p-1)a\rho}{N+2p-Np+\alpha}\right)^{1-\frac{2}{N-2\sigma}} m^{\frac{2}{N-2\sigma}}. \end{aligned}$$

Hence $m = \Psi^{-1}(c)$. Therefore, we have

$$\begin{aligned} \Lambda &= \min_{u\in M} E(u) = \min_{u\in M} c_u = \min_{v\in\mathcal{N}} \Psi(m_v) \\ &= -\max \frac{N+2+\alpha-Np}{2(p-1)}\left(\frac{(p-1)a\rho}{N+2p-Np+\alpha}\right)^{\frac{Np-\alpha-2p-N}{Np-N-\alpha-2}} m_v^{\frac{2p-2}{Np-N-\alpha-2}} \\ &= -\frac{N+2+\alpha-Np}{2(p-1)}\left(\frac{(p-1)a\rho}{N+2p-Np+\alpha}\right)^{\frac{Np-\alpha-2p-N}{Np-N-\alpha-2}} \left(\min_{v\in\mathcal{N}} m_v\right)^{\frac{2p-2}{Np-N-\alpha-2}} \end{aligned}$$

$$= -\frac{N+2+\alpha-Np}{2(p-1)}\left(\frac{(p-1)a\rho}{N+2p-Np+\alpha}\right)^{\frac{Np-\alpha-2p+N}{Np-N-\alpha-2}}\Gamma^{\frac{2p-2}{Np-N-\alpha-2}}$$
$$= \Psi(\Gamma).$$

If $\hat{u} \in M$ is a minimizer for Λ, then $\hat{w} = T^{\lambda}\hat{u}$ is a critical point of J with $J(\hat{w}) = \Psi^{-1}(\Lambda) = \Gamma$, so that $\hat{w}$ is a minimizer for Γ, that is, $J(\hat{w}) = \min_{\mathcal{N}} J$. □

Corollary 2.2 *Any ground state solution of* (2) *satisfies*

$$\|u\|_2^2 = \rho, \quad \rho = \frac{N+2p-pN+\alpha}{(p-1)a}\Gamma,$$

moreover, for this precise value of the radius ρ, *one has*

$$\min_{u\in M} J(u) = \min_{u\in\mathcal{N}} J(u),$$

where $M = M_\rho$.

Proof The first conclusion is clear by the previous proof (see (15)). Now we prove the second conclusion. Indeed, Lemma 2.1 leads to

$$\begin{aligned}\min_{u\in M} J(u) &= \min_{u\in M} E(u) + \frac{1}{2}a\|u\|_2^2 = \Lambda + \frac{1}{2}a\rho \\ &= -\frac{N+2+\alpha-Np}{2(p-1)}\Gamma + \frac{N+2p+\alpha-Np}{2(p-1)}\Gamma \\ &= \Gamma = \min_{u\in\mathcal{N}} J(u).\end{aligned}$$

□

Now we give the proof Theorem 2.

Proof of Theorem 2 By contradiction. Assume the conclusion is false, then there exist $\epsilon > 0$, a sequence of times $\{t_n\} \subset [0,\infty)$, and initial data $\{u_0^n\} \subset H^1(\mathbb{R}^N, \mathbb{C})$ such that

$$\lim_{n\to\infty} \inf_{\phi\in G} \|u_0^n - \phi\|_{H^1} = 0 \tag{16}$$

and

$$\inf_{\psi\in G} \|u_n(t_n,\cdot) - \psi\|_{H^1} \geq \epsilon_0, \tag{17}$$

where $u_n(t,\cdot)$ is the solution of (1) with initial value u_0^n. By Corollary 2.2, for any $\phi \in G$, one has

$$\|\phi\|_2^2 = \rho_0, \qquad J(\phi) = \min_{u\in M_{\rho_0}} J(u), \quad \rho_0 = \frac{N+2p-pN+\alpha}{(p-1)a}\Gamma. \tag{18}$$

Hence, considering the sequence $\{u_n(t_n,x)\}$ in $H^1(\mathbb{R}^N,\mathbb{C})$, using the conservation of charge and (18), we get

$$\|u_n(t_n,\cdot)\|_2^2 = \|u_0^n\|_2^2 = \rho_0 + o(1), \quad \text{when } n \to \infty.$$

Then there exists a sequence $\{w_n\} \subset R^+$, $w_n \to 1$, as $n \to \infty$, such that

$$\left\| w_n u_n(t_n, \cdot) \right\|_2^2 = \rho_0, \quad \text{for all } n \geq 1. \tag{19}$$

Moreover, by the conservation of energy and the continuity of E, we get

$$\begin{aligned} J\bigl(w_n u_n(t_n, \cdot)\bigr) &= E\bigl(w_n u_n(t_n, \cdot)\bigr) + \frac{1}{2} a \left\| w_n u_n(t_n, \cdot) \right\|_2^2 \\ &= E\bigl(u_n(t_n, \cdot)\bigr) + \frac{1}{2} a \left\| u_n(t_n, \cdot) \right\|_2^2 + o(1) \\ &= E\bigl(u_0^n\bigr) + \frac{1}{2} a \left\| u_0^n \right\|_2^2 + o(1) \\ &= J\bigl(u_0^n\bigr) + o(1) \\ &= \min_{u \in M_{\rho_0}} J(u) + o(1). \end{aligned} \tag{20}$$

By (19) and (20), one sees that $\{w_n u_n(t_n, \cdot)\} \subset H^1(\mathbb{R}^N, \mathbb{C})$ is a minimizing sequence for J over M_{ρ_0}. Following the arguments of [15], one sees that, up to a subsequence, $\{w_n u_n(t_n, \cdot)\}$ converges to w_0 in $H^1(\mathbb{R}^N, \mathbb{C})$, and

$$J(w_0) = \min_{u \in M_{\rho_0}} J(u) = \min_{u \in \mathcal{N}} J(u).$$

This implies that $w_0 \in G$. Obviously, this is a contradiction with (17), since

$$\epsilon_0 \leq \lim_{n \to \infty} \inf_{\psi \in G} \left\| w_n u_n(t_n, \cdot) - \psi \right\|_{H^1} \leq \lim_{n \to \infty} \left\| w_n u_n(t_n, \cdot) - w_0 \right\|_{H^1} = 0.$$

The proof of Theorem 2 is complete. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by XS. XW and WL performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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